ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 4, pp. 881–889

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 4, pp. 881–889
Online ISSN: 1735-8515

COEFFICIENT ESTIMATES FOR A SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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(Communicated by Ali Abkar)

Abstract. In this paper, we introduce and investigate a subclass $H_{\Sigma}^{h,p}(\beta)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. Upper bounds for the second and third coefficients of functions in this subclass are founded. Our results generalize and improve over the existing results in the literature.
Keywords: Analytic functions, bi-univalent functions, coefficient estimates, starlike functions, Koebe one-quarter theorem.
MSC(2010): Primary: 30C45; Secondary: 30C50.

1. Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We also denote by $\mathcal{S}$ the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem [3] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

Article electronically published on August 20, 2016.
Received: 17 April 2014, Accepted: 27 May 2015.
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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by Σ.

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions $f \in \mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Lewin [9] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [8] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Tan [12] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class Σ. Recently there interest to study the bi-univalent functions class Σ (see [4, 5, 13, 14]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{1, 2\})$ for each $f \in \Sigma$ given by [1] is still an open problem.

Recently, Frasin [6] introduced two subclasses of class Σ and obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses as follow.

Definition 1.1 ([6]). A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma(\alpha, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \ \textit{and} \ |arg(f'(z) + \beta z f''(z))| < \frac{\alpha\pi}{2} \ (z \in \mathbb{U}),$$

and

$$|arg(g'(w) + \beta w g''(w))| < \frac{\alpha\pi}{2} \ (w \in \mathbb{U}),$$

where, $\beta > 0,\ 0 < \alpha < 1,\ \ 2(1-\alpha)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and the function g is the extension of f^{-1} to $\mathbb{U}$, which

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

Theorem 1.2 ([6]). *Let $f(z)$ given by (1.1) be in the class $H_\Sigma(\alpha, \beta)$ where $\beta > 0,\ 0 < \alpha < 1,\ \ 2(1-\alpha)\sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\beta(\alpha+\beta+2-\alpha\beta)}},$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}.$$

Definition 1.3 ([6]). A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma(\gamma, \beta)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \mathfrak{Re}(f'(z) + \beta z f''(z)) > \gamma \ (z \in \mathbb{U}),$$

and

$$\mathfrak{Re}(g'(w) + \beta w g''(w)) > \gamma \ (w \in \mathbb{U}),$$

where, $\beta > 0,\ 0 \leq \gamma < 1,\ \ 2(1-\gamma) \sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$, and g is given by (1.2).

Theorem 1.4 ([6]). *Let $f(z)$ given by (1.1) be in the class $H_\Sigma(\gamma, \beta)$ where $\beta > 0,\ 0 \leq \gamma < 1,\ \ 2(1-\gamma) \sum\limits_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m+1} \leq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}},$$

and

$$|a_3| \leq \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}.$$

The purpose of our study is to obtain estimates of coefficients $|a_2|$ and $|a_3|$ for functions in subclasses $H_\Sigma^{h,p}(\beta)$ which improve Theorem 1.2 and Theorem 1.4.

2. Coefficient Estimates

In this section, we introduce and investigate the general subclass $H_\Sigma^{h,p}(\beta)$ where $\beta \geq 0$.

Definition 2.1. Let $h, p : \mathbb{U} \to \mathbb{C}$ be analytic functions and

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \ (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

A univalent function $f \in \mathcal{S}$ given by (1.1) is said to be in the class $H_\Sigma^{h,p}(\beta)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } f'(z) + \beta z f''(z) \in h(\mathbb{U}) \ (z \in \mathbb{U}), \tag{2.1}$$

and

$$g'(w) + \beta w g''(w) \in p(\mathbb{U}) \ (w \in \mathbb{U}), \tag{2.2}$$

where, $\beta \geq 0$ and the function g is given by (1.2).

Remark 2.2. There are many choices of h, p and β which would provide interesting subclasses of class $H_\Sigma^{h,p}(\beta)$. For example,

(1) For $\beta > 0$ and $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ where $0 < \alpha \leq 1$, it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in H_{\Sigma}^{h,p}(\beta)$ then

$$|arg(f'(z) + \beta z f''(z))| < \frac{\alpha\pi}{2} \text{ and } |arg(g'(w) + \beta w g''(w))| < \frac{\alpha\pi}{2}.$$

Therefore in this case, the class $H_{\Sigma}^{h,p}(\beta)$ reduces to class $H_{\Sigma}(\alpha, \beta)$ in Definition 1.1.

(2) For $\beta > 0$ and $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$, the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in H_{\Sigma}^{h,p}(\beta)$, then

$$\mathfrak{Re}(f'(z) + \beta z f''(z)) > \gamma \text{ and } \mathfrak{Re}(g'(w) + \beta w g''(w)) > \gamma.$$

This means that the class $H_{\Sigma}^{h,p}(\beta)$ reduces to class $H_{\Sigma}(\gamma, \beta)$ in Definition 1.3.

(3) For $\beta = 0$ and $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ where $0 < \alpha \leq 1$, the class $H_{\Sigma}^{h,p}(0)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ which is defined by Srivastava et al. [10, Definition 1].

(4) For $\beta = 0$ and $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$, the class $H_{\Sigma}^{h,p}(0)$ reduces to the class $\mathcal{H}_{\Sigma}(\gamma)$ which is defined by Srivastava et al. [10, Definition 2].

Now, we derive the estimates of the coefficients $|a_2|$ and $|a_3|$ for class $H_{\Sigma}^{h,p}(\beta)$.

Theorem 2.3. *If $f \in H_{\Sigma}^{h,p}(\beta)$ where $\beta \geq 0$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}}\right\}, \tag{2.3}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2} + \frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}, \frac{|h''(0)|}{6(1+2\beta)}\right\}. \tag{2.4}$$

Proof. Since $f \in H_{\Sigma}^{h,p}(\beta)$ and $g = f^{-1}$, from relations (2.1) and (2.2) we have,

$$f'(z) + \beta z f''(z) = h(z) \ (z \in \mathbb{U}), \tag{2.5}$$

and

$$g'(w) + \beta w g''(w) = p(w) \ (w \in \mathbb{U}), \tag{2.6}$$

respectively, where functions h and p satisfy the conditions of Definition 2.1. Also, functions h and p have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \tag{2.7}$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + p_3 w^2 + \cdots. \tag{2.8}$$

Now, by substituting (2.7) and (2.8) into (2.5) and (2.6), respectively, and equating the coefficients, we get

$$2(1+\beta)a_2 = h_1, \tag{2.9}$$

$$3(1+2\beta)a_3 = h_2, \tag{2.10}$$

$$-2(1+\beta)a_2 = p_1, \tag{2.11}$$

and

$$6(1+2\beta)a_2^2 - 3(1+2\beta)a_3 = p_2. \tag{2.12}$$

From (2.9) and (2.11), it yields

$$h_1 = -p_1, \tag{2.13}$$

and

$$8(1+\beta)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.14}$$

Adding (2.10) and (2.12), gives

$$6(1+2\beta)a_2^2 = p_2 + h_2. \tag{2.15}$$

Consequently, from (2.14) and (2.15), we have that

$$a_2^2 = \frac{h_1^2 + p_1^2}{8(1+\beta)^2}, \tag{2.16}$$

and

$$a_2^2 = \frac{p_2 + h_2}{6(1+2\beta)}, \tag{2.17}$$

respectively. Therefore, we find from the equations (2.16) and (2.17), that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2},$$

and

$$|a_2| \leq \frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}.$$

Thus, the desired estimate on the coefficient $|a_2|$ as asserted in (2.3).

Next, in order to find the bound of the coefficient $|a_3|$, by subtracting (2.12) from (2.10), we get

$$6(1+2\beta)a_3 - 6(1+2\beta)a_2^2 = h_2 - p_2. \tag{2.18}$$

Upon substituting the value of a_2^2 from (2.16) into (2.18), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{8(1+\beta)^2} + \frac{h_2 - p_2}{6(1+2\beta)},$$

Therefore, we obtain

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2} + \frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}. \tag{2.19}$$

On the other hand, by substituting the value of a_2^2 from (2.17) into (2.18), it follows that

$$a_3 = \frac{p_2 + h_2)}{6(1+2\beta)} + \frac{h_2 - p_2)}{6(1+2\beta)} = \frac{h_2}{3(1+2\beta)}.$$

Hence

$$|a_3| \leq \frac{|h''(0)|}{6(1+2\beta)}. \tag{2.20}$$

The desired estimate of the coefficient $|a_3|$ as asserted in (2.4) will be obtained from (2.19) and (2.20). □

By choosing

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

in Theorem 2.3, we have the following result.

Corollary 2.4. *Let the function f be given by (1.1) in the class $H_{\Sigma}^{h,p}(\beta)$ where $\beta \geq 0$. Then*

$$|a_2| \leq \min\left\{\frac{\alpha}{1+\beta}, \frac{\sqrt{2}\alpha}{\sqrt{3(1+2\beta)}}\right\},$$

and

$$|a_3| \leq \frac{2\alpha^2}{3(1+2\beta)}.$$

Remark 2.5. Corollary 2.4 is an improvement of estimates obtained by Frasin [6] in Theorem 1.2. To see this, for the coefficient $|a_2|$, we have that

(i) If $\beta \geq \frac{1}{2} + \frac{\sqrt{3}}{2}$, then $\min\left\{\frac{\alpha}{1+\beta}, \frac{\sqrt{2}\alpha}{\sqrt{3(1+2\beta)}}\right\} = \frac{\alpha}{1+\beta}$, and

$$\frac{\alpha}{1+\beta} \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\beta(\alpha+\beta+2-\alpha\beta)}}.$$

(ii) If $\beta \leq \frac{1}{2} + \frac{\sqrt{3}}{2}$, then $\min\left\{\frac{\alpha}{1+\beta}, \frac{\sqrt{2}\alpha}{\sqrt{3(1+2\beta)}}\right\} = \frac{\sqrt{2}\alpha}{\sqrt{3(1+2\beta)}}$, and

$$\frac{\sqrt{2}\alpha}{\sqrt{3(1+2\beta)}} \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\beta(\alpha+\beta+2-\alpha\beta)}}.$$

Also for the coefficient $|a_3|$, it can be concluded that

$$\frac{2\alpha^2}{3(1+2\beta)} \leq \frac{\alpha^2}{(1+\beta)^2} + \frac{2\alpha}{3(1+2\beta)}.$$

Therefore the bounds obtained in Corollary 2.4 is a refinement of the estimates obtained in Theorem 1.2.

If we take $\beta = 1$ in Theorem 2.3, then we have the following result.

Corollary 2.6. *Let the function f be given by (1.1) in the class $H_\Sigma^{h,p}(1)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{32}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{36}}\right\}, \tag{2.21}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{32} + \frac{|h''(0)|+|p''(0)|}{36}, \frac{|h''(0)|}{18}\right\}. \tag{2.22}$$

By taking

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha \ (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

in Corollary 2.6, the following result will be concluded.

Corollary 2.7. *Let the function f be given by (1.1) in the class $H_\Sigma^{h,p}(1)$. Then*

$$|a_2| \leq \min\left\{\frac{\alpha}{2}, \frac{\sqrt{2}\alpha}{3}\right\} = \frac{\sqrt{2}\alpha}{3},$$

and

$$|a_3| \leq \frac{2\alpha^2}{9}.$$

By setting $\beta = 0$ in Corollary 2.4, we obtain the following consequence which is an improvement of the estimates obtained by Srivastava et al. in [10, Theorem 1].

Corollary 2.8. *Let the function f be given by (1.1) in the class $H_\Sigma(\alpha)$. Then*

$$|a_2| \leq \sqrt{\frac{2}{3}}\alpha,$$

and

$$|a_3| \leq \frac{2\alpha^2}{3}.$$

By letting

$$h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z} \ (0 \leq \gamma < 1,\ z \in \mathbb{U}),$$

in Theorem 2.3, we deduce the following corollary.

Corollary 2.9. *Let the function f be given by (1.1) in the class $H_{\Sigma}^{h,p}(\beta)$ where $\beta \geq 0$. Then,*

$$|a_2| \leq \min\left\{\frac{1-\gamma}{1+\beta}, \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}}\right\},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{3(1+2\beta)}.$$

Remark 2.10. Corollary 2.9 is an improvement of the estimates obtained by Frasin [6] in Theorem 1.4. To see this,
for the coefficient $|a_2|$, if $\beta > \frac{3\delta-2+\sqrt{3\delta(3\delta-2)}}{2}$ and $\delta > \frac{2}{3}$, then

$$\min\left\{\frac{1-\gamma}{1+\beta}, \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}}\right\} < \sqrt{\frac{2(1-\gamma)}{3(1+2\beta)}}.$$

Also for the coefficient $|a_3|$, we have

$$\frac{2(1-\gamma)}{3(1+2\beta)} < \frac{(1-\gamma)^2}{(1+\beta)^2} + \frac{2(1-\gamma)}{3(1+2\beta)}.$$

Therefore the bounds obtained in Corollary 2.9 is a refinement of the estimates obtained in Theorem 1.4.

If we take $\beta = 1$ and

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

in Theorem 2.3, we have the following result.

Corollary 2.11. *Let the function f be given by (1.1) in the class $H_{\Sigma}^{h,p}(1)$. Then,*

$$|a_2| \leq \min\left\{\frac{1-\gamma}{2}, \frac{\sqrt{2(1-\gamma)}}{3}\right\},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{9}.$$

By setting $\beta = 0$ in Corollary 2.9, we obtain the following result which is an improvement of the estimates obtained by Srivastava et al. [10, Theorem 2].

Corollary 2.12. *Let the function f be given by (1.1) in the class $H_{\Sigma}(\gamma)$. Then*

$$|a_2| \leq \min\left\{1-\gamma, \sqrt{\frac{2(1-\gamma)}{3}}\right\},$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{3}.$$

Acknowledgments

The authors wish to thank the referee, for the careful reading of the paper and for the helpful suggestions and comments.

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Subramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.* **25** (2012), no. 3, 344–351.

[2] D. A. Brannan and J. G. Clunie (Eds.), Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute held at the University of Durham, Academic Press, New York-London, 1980.

[3] P. L. Duren, Univalent Functions, Springer-Verlag, New York, Berlin, 1983.

[4] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Panamer. Math. J.* **22** (2012), no. 4 , 15–26.

[5] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24** (2011), no. 9, 1569–1573.

[6] B. A. Frasin, Coefficient bounds for certain classes of bi-univalent functions, *Hacet. J. Math. Stat.* **43** (2014), no. 3, 383–389.

[7] C. Y. Gao and S. Q. Zhou, Certain subclass of starlike functions, *Appl. Math. Comput.* **187** (2007), no. 1, 176–182.

[8] A. W. Kedzierawski, Some remarks on bi-univalent functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A.* **39** (1985) 77–81.

[9] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18** (1967) 63–68.

[10] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010), no. 10, 1188–1192.

[11] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.* **32** (1969) 100–112.

[12] D. L. Tan, Coefficient estimates for bi-univalent functions, *Chinese Ann. Math. Ser. A.* **5** (1984), no. 5, 559–568.

[13] Q. H. Xu, Y. C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), no. 6, 990–994.

[14] Q. H. Xu, H. G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012), no. 23, 11461–11465.

[15] D. G. Yang and J. L. Liu, A class of analytic functions with missing coefficients, *Abstr. Appl. Anal.* 2011, Article ID 456729, 16 pages.

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