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ON STATISTICAL TYPE CONVERGENCE IN UNIFORM SPACES

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Abstract. The concept of $\mathscr{F}_{st}$-fundamentality is introduced in uniform spaces, generated by some filter $\mathscr{F}$. Its equivalence to the concept of $\mathscr{F}$-convergence in uniform spaces is proved. This convergence generalizes many kinds of convergence, including the well-known statistical convergence.

Keywords: $\mathscr{F}$-convergence, $\mathscr{F}_{st}$- fundamentality, statistical convergence, a uniform space
MSC(2010): Primary: 40A05; Secondary: 54A20; 54E35.

1. Introduction

The idea of statistical convergence (*stat-convergence*) was first proposed by A.Zigmund in his famous monograph [35] where he talked about "almost convergence". The first definition of it was given by H. Fast [10] and H. Steinhaus [29]. Later, this concept has been generalized in many directions. More details on this matter and on applications of this concept can be found in [3–6, 9, 11–13, 21, 22, 24, 26, 28, 30]. It should be noted that the methods of non-convergent sequences have long been known and they include e.g. Cesaro method, Abel method and etc. These methods are used in different areas of mathematics. For the applicability of these methods is very important that the considered space has a linear structure. Therefore, the study of statistical convergence in metric spaces is of special scientific interest. Different aspects of this problem is discussed in [19,20]. Statistical convergence is currently actively used in many areas of mathematics such as summation theory [6,12,13], number theory [3, 9, 27], trigonometric series [35], probability theory [11], measure theory [24], optimization [25], approximation theory [14,15], fuzzy theory [1,7,31] and etc.

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It should be noted that the concept of statistical fundamentality (*stat-fundamentality*) was first introduced by J.A. Fridy [13]. He proved its equivalence to *stat-convergence* with respect to numerical sequences. This problem was raised in [23] concerning uniform space $(X;\mathscr{U})$ with a uniformity $\mathscr{U}$. It is proved that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ *stat-convergent* in $(X;\mathscr{U})$, then it is *stat-fundamental*. In the same paper the validity of converse statement is raised.

In this paper, the concept of $\mathscr{F}_{st}$-*fundamentality* with respect to the concept of $\mathscr{F}$-*convergence* (convergence on the filter) is introduced and in the sequential complete uniform spaces it is proved that the concept of $\mathscr{F}_{st}$-*fundamentality* is equivalent to the $\mathscr{F}$-*convergence*. It should be noted that $\mathscr{F}$-*convergence* generalizes many kinds of convergence, including the well-known statistical convergence. Note that some problems relating to the convergence with respect to ideals or filters have been considered in [17, 18, 32–34].

2. Needful information

We will use the standard notation. $\mathbb{N}$ will be the set of all positive integers; $\chi_A(\cdot)$ will be the characteristic function of A; $|A| = card\,A$ will be the number of elements of A; $A\Delta B = (A\backslash B)\cup(B\backslash A)$ will denote a symmetric difference of sets A and B; 2^M will be the set of all subsets M; $M^c = \mathbb{N}\backslash M$. $\wedge$ will be the quantifier which means "and"; $\Rightarrow$ will be the quantifier which means "follows".

Let us also recall the definition of an ideal and a filter.

A family of sets $I \subset 2^{\mathbb{N}}$ is called an ideal if:

$\alpha) A;B\in I \Rightarrow A\cup B\in I$;

$\beta)\,(A\in I \,\wedge\, B\subset A) \Rightarrow B\in I$.

A family $\mathscr{F}\subset 2^{\mathbb{N}}$ is called a filter on $\mathbb{N}$, if:

i) $\emptyset \notin \mathscr{F}$ *;*

ii) from $A;B\in\mathscr{F} \Rightarrow A\cap B\in\mathscr{F}$*;*

iii) from $A\in\mathscr{F}\wedge(A\subset B)\Rightarrow B\in\mathscr{F}$.

Filter $\mathscr{F}$, satisfying the condition:

iv) If $A_1\supset A_2\supset \ldots \wedge A_n\in\mathscr{F}$, $\forall n\in\mathbb{N}\Rightarrow \exists\{n_m\}_{m\in\mathbb{N}}\subset\mathbb{N};\, n_1<n_2<\ldots : \cup_{m=1}^{\infty}((n_m,n_{m+1}]\cap A_m)\in\mathscr{F}$, *is called a monotone closed filter.*

Filter $\mathscr{F}$ satisfying the following condition is called a right filter.

v) $F^c\,(N\backslash F)\in\mathscr{F}$*, for any finite subset* $F\subset\mathbb{N}$.

An ideal I is called non-trivial if $I\neq\emptyset\wedge I\neq\mathbb{N}$. $I\subset 2^{\mathbb{N}}$ is a non-trivial ideal if and only if $\mathscr{F}=\mathscr{F}(I)=\{\mathbb{N}\backslash A : A\in I\}$ is a filter. A non-trivial ideal $I\subset 2^{\mathbb{N}}$ is called admissible if and only if $I\supset\{\{n\}:n\in\mathbb{N}\}$. More details about filters and convergence on the filters can be found in monograph by N. Bourbaki [2] and also in [17, 18].

Let us recall the definition of uniformity on the set X. $\Delta\equiv\{(x;x):x\in X\}$ is called a diagonal or an identity relation. If $U\subset X\times X$ is a relation, then the inverse of this relation U^{-1} is defined as the set of all pairs $(x;y)$ such that

$(y;x) \in U$, i.e. $U^{-1} \equiv \{(x;y) \in X \times X : (y;x) \in U\}$. Let $U;V \subset X \times X$ be some relation. The composition $U \circ V$ of the relations U and V is defined as the set of all pairs $(x;z)$, that for some $y \in X$ we have $(x;y) \in V$ and $(y;z) \in U$, i.e. $U \circ V \equiv \{(x;z) : \exists y \in X, (x;y) \in V \wedge (y;z) \in U\}$.¿ Let $A \subset X$ be some set and $U \subset X \times X$ be a relation. Accept $U[A] \equiv \{y \in X : \exists x \in A \Rightarrow (x;y) \in U\}$. For $A = \{x\}$ assume $U[A] = U[x]$. Uniformity on the set X is a non-empty family $\mathscr{U} \subset 2^{X \times X}$, satisfying the following conditions.

(a) $\Delta \subset U, \forall U \in \mathscr{U}$;
(b) $U \in \mathscr{U} \Rightarrow U^{-1} \in \mathscr{U}$;
(c) $U \in \mathscr{U} \Rightarrow \exists V \in \mathscr{U} : V \circ V \subset U$;
(d) $U;V \in \mathscr{U} \Rightarrow U \cap V \in \mathscr{U}$;
(e) $U \in \mathscr{U} \wedge (U \subset V \subset X \times X) \Rightarrow V \in \mathscr{U}$.

Pair $(X;\mathscr{U})$ is called a uniform space. Subfamily $\mathscr{B} \subset \mathscr{U}$ of the uniformity $\mathscr{U}$ is called its base if and only if any element of the family $\mathscr{U}$ contains an element of the family $\mathscr{B}$.

Let $(X;\mathscr{U})$ be a uniform space. Topology τ, associated with a uniformity $\mathscr{U}$, is a family of all such sets $T \subset X$, for arbitrary $x \in T$, $\exists U \in \mathscr{U} : U[x] \subset T$.

The space $(X;\mathscr{U})$ with a uniform topology is called Hausdorff if and only if $\cap_{U \in \mathscr{U}} U = \Delta$. Let $(X;\mathscr{U})$ be a uniform space and $\{x_n\}_{n \in \mathbb{N}} \subset X$ be some sequence. $\{x_n\}_{n \in \mathbb{N}}$ is called fundamental if $\forall U \in \mathscr{U}$, $\exists n_0 \in \mathbb{N} : (x_n;x_m) \in U$, $\forall n,m \geq n_0$.

For more details we refer the reader to [8,16].

Let us recall the definition of convergence on filter.

Definition 2.1. Let $(X;\mathscr{U})$ be a uniform space and $\mathscr{F} \subset 2^{\mathbb{N}}$ *be some filter. The sequence* $\{x_n\}_{n \in \mathbb{N}} \subset X$ is called $\mathscr{F}$-convergent to x (shortly $\mathscr{F}\text{-}\lim_{n \to \infty} x_n = x$), if $\forall U \in \mathscr{U}$: $\{n \in \mathbb{N} : (x_n;x) \in U\} \in \mathscr{F}$. *In other words, it means that* $\forall U \in \mathscr{U}$: $\{n \in \mathbb{N} : x_n \in U[x]\} \in \mathscr{F}$.

Definition 2.2. Let $(X;\mathscr{U})$ be a uniform space and $\mathscr{F} \subset 2^{\mathbb{N}}$ *be some filter. The sequence* $\{x_n\}_{n \in \mathbb{N}} \subset X$ *is called* $\mathscr{F}_{st}$- *fundamental in* X*, if* $\forall U \in \mathscr{U}$, $\exists n_0 \in \mathbb{N} : \{n \in \mathbb{N} : x_n \in U[x_{n_0}]\} \in \mathscr{F}$.

Let $(X;\mathscr{U})$ be a Hausdorff uniform space. Consequently, $\{x\} = \cap_{U \in \mathscr{U}} U[x]$. Let $\{x_n\}_{n \in \mathbb{N}} \subset X$ be some sequence. Let us show that if $\exists \mathscr{F}\text{-}\lim_{n \to \infty} x_n$, then it is unique, where $\mathscr{F} \subset 2^{\mathbb{N}}$ is some filter. Assume to the contrary, i.e. $\mathscr{F}\text{-}\lim_{n \to \infty} x_n$ has two values $y_1 \neq y_2$. Then it is clear that $\exists U_k \in \mathscr{U}$: $y_1 \notin U_2[y_2] \wedge y_2 \notin U_1[y_1]$. Put $U = U_1 \cap U_2 \Rightarrow U \in \mathscr{U}$, moreover $y_1 \notin U[y_2] \wedge y_2 \notin U[y_1]$. Take $V \in \mathscr{U}$: $V \circ V \subset U \wedge \left(V = V^{-1}\right)$. The possibility of such a choice V directly follows from the definition of uniformity. It is obvious that $y_1 \notin V[y_2] \wedge y_2 \notin V[y_1]$. Assume $A_k \equiv \{n \in \mathbb{N} : x_n \in V[y_k]\}$, $k = 1,2$. We have $A_k \in \mathscr{F}$, $k = 1,2 \Rightarrow A_1 \cap A_2 \in \mathscr{F}$. On the other hand $A_1 \cap A_2 = \emptyset \notin \mathscr{F}$. Since, if $A_1 \cap A_2 \neq \emptyset \Rightarrow \exists n_0 \in \mathbb{N} : x_{n_0} \in A_1 \cap A_2 \Rightarrow (x_{n_0};y_1) \in V \wedge (x_{n_0};y_2) \in V$. From the symmetry of V it follows $(y_2; x_{n_0}) \in V$, as a result, $(y_1; y_2) \in V \circ V \subset U$. The obtained contradiction proves that $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$ is unique. So the following lemma is proved.

Lemma 2.3. *Let $(X; \mathscr{U})$ be a Hausdorff uniform space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. If $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$, where $\{x_n\}_{n\in\mathbb{N}} \subset X$ is some sequence, then this limit is unique.*

Let $(X; \mathscr{U})$ be a uniform space, $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter and $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, where $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some sequence. Let us show that $\{x_n\}_{n\in\mathbb{N}}$ $\mathscr{F}_{st}$-*fundamental*. Take $\forall U \in \mathscr{U}$ and let $V \in \mathscr{U} : V \circ V \subset U \wedge \left(V = V^{-1}\right)$. Let $n_0 \in \{n \in \mathbb{N} : x_n \in V[x]\}$. It is clear that $\{n \in \mathbb{N} : x_n \in V[x]\} \in \mathscr{F}$. If $x_n \in V[x] \Rightarrow (x_n; x_{n_0}) \in V \circ V \subset U$. Consequently, $\{n \in \mathbb{N} : x_n \in V[x]\} \subset \{n \in \mathbb{N} : x_n \in U[x_{n_0}]\} \Rightarrow \{n \in \mathbb{N} : x_n \in U[x_{n_0}]\} \in \mathscr{F}$. So the following theorem is proved.

Theorem 2.4. *Let $(X; \mathscr{U})$ be a Hausdorff uniform space, $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter and $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$, where $\{x_n\}_{n\in\mathbb{N}} \subset X$ be some sequence. Then $\{x_n\}_{n\in\mathbb{N}}$ is $\mathscr{F}_{st}$-fundamental.*

3. Main results

Under certain assumptions, the converse of Theorem 2.4 is also true. Let $(X; \mathscr{U})$ be a sequentially complete uniform space, i.e. in this space any Cauchy sequence converges to some point of X. We assume that $(X; \mathscr{U})$ has a countable base and it is Hausdorff. Then $\exists U_n \in \mathscr{U}$, $\forall n \in \mathbb{N} : \cap_{n\in\mathbb{N}} U_n = \Delta \wedge (U_n \subset U, \forall n \in \mathbb{N})$. Without loss of generality, we will assume that $U^{(n+1)} \circ U^{(n+1)} \subset U^{(n)} \wedge \left(U^{(n)} = \left(U^{(n)}\right)^{-1}\right)$. Let $\mathscr{F} \subset 2^N$ be some filter and the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ be $\mathscr{F}_{st}$-*fundamental* in X. Then, by definition we have $\exists n_j \in \mathbb{N} : K_j \in \mathscr{F}$, where $K_j \equiv \left\{n \in \mathbb{N} : x_n \in U^{(j)}\left[x_{n_j}\right]\right\}$, $j = 1, 2$. It is clear that $K_{(1)} \equiv K_1 \cap K_2 \in \mathscr{F}$. Let $M_1 \equiv U^{(1)}[x_{n_1}] \cap U^{(2)}[x_{n_2}]$. Obviously, $x_n \in M_1$, $\forall n \in K_{(1)}$. Similarly we obtain that $\exists n_3 \in \mathbb{N} : K_3 \in \mathscr{F}$, where $K_3 \equiv \left\{n \in \mathbb{N} : x_n \in U^{(3)}[x_{n_3}]\right\}$. Assume $K_{(2)} = K_{(1)} \cap K_3$. It is clear that $K_{(2)} \in \mathscr{F}$. Put $M_2 \equiv M_1 \cap U^{(3)}[x_{n_3}]$. Consequently, $M_2 \neq \emptyset$, so, $x_n \in M_2$, $\forall n \in K_{(2)}$. Continuing in the same way, we obtain the sequence of open non-empty sets

$$\{M_n\}_{n\in\mathbb{N}} \subset X : M_1 \supset M_2 \supset ..., M_n \subset U^{(n+1)}\left[x_{k_{n+1}}\right], \ \forall n \in \mathbb{N},$$

such as $K_{(j)} \in \mathscr{F}$: $K_{(j)} \equiv \{k \in \mathbb{N} : x_k \in M_j\}$, $j \in \mathbb{N}$. Take $\forall \tilde{x}_n \in M_n$, $\forall n \in \mathbb{N}$. Let us show that $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is a fundamental sequence. Let $U \in \mathscr{U}$ be an arbitrary uniformity. Then, it is obvious that $\exists n_0 \in \mathbb{N} : U^{(n_0)} \subset U$, $\forall n \geq n_0$. Let $n \geq n_0$ be an arbitrary number. We have $\tilde{x}_{n+p} \in M_{n+p} \subset M_n$, $\forall p \in \mathbb{N}$. Since, $M_n : M_{n+p} \subset U^{(n+1)}\left[x_{k_{n+1}}\right]$, it is clear that $\left(\tilde{x}_n; x_{k_{n+1}}\right) \in U^{(n+1)} \wedge$

$\left(\tilde{x}_{n+p};x_{k_{n+1}}\right)\in U^{(n+1)}\Rightarrow\left(\tilde{x}_n;\tilde{x}_{n+p}\right)\in U^{(n+1)}\circ U^{(n+1)}\subset U^{(n)}$, $\forall p\in\mathbb{N}$. Consequently, $(\tilde{x}_n;\tilde{x}_{n+p})\in U$, $\forall n\geq n_0$, $\forall p\in\mathbb{N}$. From the arbitrariness of U it follows that the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is fundamental in $(X;\mathscr{U})$ and let $\lim\limits_{n\to\infty}\tilde{x}_n=x$. Next we show that $\mathscr{F}\text{-}\lim\limits_{n\to\infty}x_n=x$. Take $\forall U\in\mathscr{U}$. Then $\exists n_0\in\mathbb{N}:U^{(n)}\subset U$, $\forall n\geq n_0$. Since, $M_n\subset U^{(n+1)}\left[x_{k_{n+1}}\right]$, then it is clear that $K_{(n)}\subset\left\{k\in\mathbb{N}:x_k\in U^{(n+1)}\left[x_{k_{n+1}}\right]\right\}\Rightarrow\left\{k\in\mathbb{N}:x_k\in U^{(n+1)}\left[x_{k_{n+1}}\right]\right\}\in\mathscr{F}$, $\forall n\in\mathbb{N}$. Let $n_1\in\mathbb{N}:(\tilde{x}_k;x)\in U^{(n_0+1)}$, $\forall k\geq n_1$. Without loss of generality, we will assume that $n_1\geq n_0+1$. Consequently, $\tilde{x}_{n_1}\in M_{n_1}\subset U^{(n_1+1)}\left[x_{k_{n_1+1}}\right]$, i.e. $\left(\tilde{x}_{n_1};x_{k_{n_1+1}}\right)\in U^{(n_1+1)}$. Put $\left(x_k;x_{k_{n_1+1}}\right)\in U^{(n_1+1)}$. Then $(x_k;\tilde{x}_{n_1})\in U^{(n_1+1)}\circ U^{(n_1+1)}\subset U^{(n_1)}$. Since, $\left(\tilde{x}_{n_1};x_{k_{n_1+1}}\right)\in U^{(n_1+1)}\subset U^{(n_1)}$, then it is clear that $\left(x_k;x_{k_{n_1+1}}\right)\in U^{(n_1)}\circ U^{(n_1)}\subset U^{(n_1-1)}\subset U^{(n_0)}\subset U$. This implies the following inclusion

$$\{n\in\mathbb{N}:x_n\in M_{n_0}\}\subset\{n\in\mathbb{N}:(x_n;x)\in U\}\,.$$

So, $K_{(n_0)}\equiv\{n\in\mathbb{N}:x_n\in M_{n_0}\}\in\mathscr{F}$, from the previous inclusion follows that $\{n\in\mathbb{N}:(x_n;x)\in U\}\in\mathscr{F}$. From the arbitrariness of $U\in\mathscr{U}$ it follows $\mathscr{F}\text{-}\lim\limits_{n\to\infty}x_n=x$. Thus, it is proven.

Theorem 3.1. *Let $(X;\mathscr{U})$ be a Hausdorff, sequentially complete uniform space with a countable base and $\mathscr{F}\subset 2^{\mathbb{N}}$ be some filter. Then, if the sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is $\mathscr{F}_{st}$-fundamental, then $\exists x\in X:\mathscr{F}\text{-}\lim\limits_{n\to\infty}x_n=x$.*

Remark 3.2. From the conditions of the Theorem 3.1 it follows that the space $(X;\mathscr{U})$ is metrizable [8, 16]. The proof is provided without using the concept of metric.

Let us assume that $\mathscr{F}\subset 2^{\mathbb{N}}$ be a monotone closed filter and the sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is *$\mathscr{F}_{st}$-fundamental.* Let the uniform space $(X;\mathscr{U})$ satisfy the condition of the Theorem 3.1. Consider the sequence $\left\{K_{(n)}\right\}_{n\in\mathbb{N}}$, constructed in the proof of this theorem. We have

$$K_{(1)}\supset K_{(2)}\supset ...\wedge K_{(n)}\in\mathscr{F},\forall n\in\mathbb{N}.$$

Then by condition (iv) of filter we have

$$\exists\{n_m:n_1<n_2<...\}\ \ :\cup_{m=1}^{\infty}\left((n_m,n_{m+1}]\cap K_{(m)}\right)\in\mathscr{F}.$$

Assume

$$\mathbb{N}_0\equiv\left\{k\in\mathbb{N}:\ k\in(n_m,n_{m+1}]\cap K^c_{(m)},\,m\in\mathbb{N}\right\}\cup[1,n_1]\,,$$

where $M^c\equiv\mathbb{N}\backslash M$. Define

$$y_k=\left\{\begin{array}{ll} x\,, & k\in\mathbb{N}_0\,,\\ x_k, & if\ otherwise\,,\end{array}\right.$$

where $x = \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$. Now we show that $\lim\limits_{k\to\infty} y_k = x$. Let $U \in \mathscr{U}$ be an arbitrary uniformity. If $k \in \mathbb{N}_0$, then it is clear that $y_k \in U[x]$. If $k \notin \mathbb{N}_0$, then $\exists m \in \mathbb{N} : n_m < k \le n_{m+1} \wedge k \notin K^c_{(m)} \Rightarrow k \in K_{(m)} \Rightarrow x_k \in M_m$, where $M_1 \supset M_2 \supset ...-$ is a sequence from Theorem 3.1. Let $n_0 \in \mathbb{N} : U^{(n_0-1)} \subset U$. Let us take k sufficiently large that $m \ge n_0$. We have $(x_k; x) \in U^{(n_0)}$, so $\left(x_k; x_{k_{n_0+1}}\right) \in U^{(n_0+1)} \wedge \left(x_{k_{n_0+1}}; x\right) \in U^{(n_0+1)}$. Thus, $(y_k; x) \in U^{(n_0)} \subset U$, since, in this case $x_k = y_k$. From the arbitrariness of U it follows $\lim\limits_{k\to\infty} y_k = x$. Let us show that $\tilde{K} \equiv \{k \in \mathbb{N} : \, x_k = y_k\} \in \mathscr{F}$. In fact, it is not difficult to see that

$$\cup_{m=1}^{\infty} \left((n_m, n_{m+1}] \cap K_{(m)} \right) \subset \tilde{K}.$$

So, $\cup_{m=1}^{\infty} \left((n_m, n_{m+1}] \cap K_{(m)} \right) \in \mathscr{F}$, from the condition (iii) of filter we obtain $\tilde{K} \in \mathscr{F}$. Thus, if $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then $\exists \tilde{K} \in \mathscr{F}$: $\lim\limits_{n\to\infty} y_n = x \wedge \left(x_n = y_n \, , \; \forall n \in \tilde{K} \right)$.

Let us assume that $\lim\limits_{n\to\infty} y_n = x \wedge \left(\tilde{K} \equiv \{n \in \mathbb{N} : \, x_n = y_n\} \in \mathscr{F} \right)$. Let $\mathscr{F}$ be a *right* filter. Let $U \in \mathscr{U}$ be any uniformity . Then $\exists n_0 \in \mathbb{N} : (y_n; x) \in U, \; \forall n \ge n_0$. We have

$$\left(\{n \in \mathbb{N} : n \ge n_0\} \cap \tilde{K} \right) \subset \{n \in \mathbb{N} : (x_n; x) \in U\}.$$

It is clear that $\left(\{n \in \mathbb{N} : n \ge n_0\} \cap \tilde{K} \right) \in \mathscr{F}$. Then from the condition (iii) of filter follows that $\{n \in \mathbb{N} : (x_n; x) \in U\} \in \mathscr{F}$. So, we get the validity of the following theorem.

Theorem 3.3. *Let $(X; \mathscr{U})$ be a uniform space satisfying the conditions of Theorem 3.1 and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. Then: 1) if $\mathscr{F}$ is a monotone closed filter and $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then $\exists \{y_n\}_{n\in\mathbb{N}} \subset X : \lim\limits_{n\to\infty} y_n = x \wedge \{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F}$; 2) if $\mathscr{F}$ is a right filter and $\lim\limits_{n\to\infty} y_n = x \wedge (\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$, then $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$.*

The Theorems 3.1 and 3.3 implies the following.

Corollary 3.4. *Let $(X; \mathscr{U})$ be sequentially complete uniform space that satisfies the conditions of Theorem 3.1, $\mathscr{F} \subset 2^{\mathbb{N}}$ be a monotone closed and a right filter. Then the following statements are equivalent:*

1) $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$; 2) $\{x_n\}_{n\in\mathbb{N}}$ is $\mathscr{F}_{st}$-fundamental; 3) $\exists \lim\limits_{n\to\infty} y_n = x \wedge$ *$(\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$.*

The Theorem 3.3 immediately implies the following

Corollary 3.5. *Let $(X; \mathscr{U})$ be a uniform space, satisfying the conditions of Theorem 3.1 and $\mathscr{F} \subset 2^N$ be a right filter. If $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then*

$\exists \{n_k : n_1 < n_2 < ...\} \in \mathscr{F} : \lim\limits_{k\to\infty} x_{n_k} = x$.

4. **Filters**

I. Non-right filter. Let $e_0 \subset \mathbb{N} \wedge e_0 \neq \mathbb{N}$. Assume

$$\mathscr{F}_{e_0} \equiv \{e \subset N : e_0 \subset e\}.$$

It is not difficult to verify that $\mathscr{F}_{e_0}$ is a *non-right* filter.

II. An ordinary convergence. $\mathscr{F} \equiv \{M \subset \mathbb{N} : M^c \equiv \mathbb{N} \backslash M$ is a finite set$\}$. $\mathscr{F}$*-convergence*, generated by this filter, coincides with the ordinary convergence.

III. Statistical convergence. Assume $\mathscr{F}_\delta \equiv \{M \subset \mathbb{N} : \delta(M) = 1\}$. $\mathscr{F}_\delta$ is a filter. It is not difficult to see that $\mathscr{F}_\delta$ is a right filter. Let us show that $\mathscr{F}_\delta$ is a monotone closed filter. Let $K_1 \supset K_2 \supset \ldots \wedge (\delta(K_n) = 1,\ \forall n \in \mathbb{N})$. Obviously, $\delta(K_n^c) = 0,\ \ \forall n \in \mathbb{N}$. Therefore $\exists \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N};\ n_1 < n_2 < \ldots :$

$$\frac{1}{n}|I_n \cap K_m^c| < \frac{1}{m},\ \forall n \geq n_m.$$

Assume $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where $\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge \ (k \in K_m^c)\}$. It is clear that $\delta(\mathbb{N}_0) = \delta\left(\tilde{\mathbb{N}}_0\right)$. Take $\forall n \in \mathbb{N}$. Then $\exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. Without loss of generality, we will assume that $n > n_1$. Let us show that

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c). \tag{4.1}$$

Let $k \in \left(I_n \cap \tilde{\mathbb{N}}_0\right) \Rightarrow \exists m_0 \leq m : n_{m_0} < k \leq n_{m_0+1} \wedge \left(k \in K_{m_0}^c\right) \Rightarrow k \in K_m^c$. Thus, the inclusion (4.1) is true. Consequently

$$\frac{1}{n}\left|I_n \cap \tilde{\mathbb{N}}_0\right| \leq \frac{1}{n}|I_n \cap K_m^c| < \frac{1}{m}. \tag{4.2}$$

From (4.2) it directly follows that $\delta\left(\tilde{\mathbb{N}}_0\right) = 0$, as a result, $\delta(\mathbb{N}_0) = 0 \Rightarrow \delta(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_\delta$. In the sequel, it should be pointed out $\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq\ n_{m+1} \wedge \ (k \in K_m)\}$. Thus, $\mathscr{F}_\delta$ is a monotone closed filter. That the $\mathscr{F}_\delta$ satisfies the condition (v) is obvious. Then, with respect to $\mathscr{F}_\delta$*-convergence* the statement of Corollary 3.4 is true.

Statement 4.1. *Filter $\mathscr{F}_\delta$, generated by statistical density, is a monotone closed and a right filter.*

IV. Logarithmic convergence. Let $M \subset \mathbb{N}$. Assume

$$l_n(M) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_M(k)}{k},$$

where $s_n = \sum_{k=1}^{n}\frac{1}{k}$. If $\exists \lim\limits_{n\to\infty} l_n(M) = l(M)$, then $l(M)$ is called a logarithmic density of the set M. Let $\mathscr{F}_l \equiv \{M \subset \mathbb{N} : l(M) = 1\}$. The following lemma is true.

Lemma 4.2. *If $l(M_k) = 1,\ k = 1, 2 \Rightarrow l(M_1 \cap M_2) = 1$.*

Proof. We have

$$M_1 \cap M_2 = (M_1 \cup M_2) \setminus [(M_2 \backslash M_1) \cup (M_1 \backslash M_2)] \, .$$

Consequently

$$M_1 \cap M_2 \cap I_n = [(M_1 \cup M_2) \cap I_n] \setminus [((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n] \, . \tag{4.3}$$

From

$$((M_2 \backslash M_1) \cap I_n) \subset (M_1^c \cap I_n) \, ,$$

we get

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_2 \backslash M_1} (k) \leq \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1^c} (k) \, . \tag{4.4}$$

It is absolutely clear that, if $l\,(M) = 1$, then $l\,(M^c) = 0$. Then from (4.4) we obtain $l\,(M_2 \backslash M_1) = 0$. Similarly, we have $l\,(M_1 \backslash M_2) = 0$. So

$$((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n = ((M_2 \backslash M_1) \cap I_n) \cup ((M_1 \backslash M_2) \cap I_n) \, ,$$

it is clear that

$$l\,((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) = 0. \tag{4.5}$$

It is easy to see that $l\,(M_1 \cup M_2) = 1$. From (4.3) we get

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1 \cap M_2} (k) = \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{M_1 \cup M_2} (k) - \frac{1}{s_n} \sum_{k=1}^{n} \frac{1}{k} \chi_{(M_2 \backslash M_1) \cup (M_1 \backslash M_2)} (k) \, .$$

Taking into account (4.5) we get $l\,(M_1 \cap M_2) = 1$. Hence the Lemma is proved. □

This lemma implies that $\mathscr{F}_l$ is a filter. If $M \subset \mathbb{N}$ is a finite set, then it is clear that $M^c \in \mathscr{F}_l$, i.e. $\mathscr{F}_l$ satisfies the condition (v), then it is absolutely clear that $l\,(M) = 0$. Let us show that $\mathscr{F}_l$ is a monotone closed filter. Let $K_1 \supset K_2 \supset ... \wedge (l\,(K_n) = 1, \forall n \in \mathbb{N}) \Rightarrow l\,(K_n^c) = 0 \, , \forall n \in \mathbb{N}$. Therefore

$$\exists \, \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}, \, n_1 < n_2 < ... : \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{K_m^c} (k)}{k} < \frac{1}{m} \, , \; \forall n \geq n_m.$$

Similar to the previous example, let $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where

$$\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m \leq k \leq n_{m+1} \wedge (k \in K_m^c)\} \, .$$

It is clear that $l\,(\mathbb{N}_0) = l\left(\tilde{\mathbb{N}}_0\right)$. Let $n \in \mathbb{N} \Rightarrow \exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. As before, we assume that $n > n_1$. It is clear that, (4.1) is true, i.e.

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c) \, .$$

Hence

$$\frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{\tilde{\mathbb{N}}_0} (k)}{k} \leq \frac{1}{s_n} \sum_{k=1}^{n} \frac{\chi_{K_m^c} (k)}{k} < \frac{1}{m} \, , \; \forall n \geq n_m.$$

Consequently, $l\left(\tilde{\mathbb{N}}_0\right)=0\Rightarrow l\left(\mathbb{N}_0\right)=0\Rightarrow l\left(\mathbb{N}_0^c\right)=1\Rightarrow \mathbb{N}_0^c\in\mathscr{F}_l$. It is clear that

$$\mathbb{N}_0^c\equiv\left\{k\in\mathbb{N}:n_m<k\leq n_{m+1}\ \wedge\left(k\in K_m\right)\right\}.$$

It directly follows that $\mathscr{F}_l$ is a right filter. Therefore, if $\exists\delta\left(M\right)\Rightarrow\exists l\left(M\right)\wedge l\left(M\right)=\delta\left(M\right)$. The converse is not generally true.

Statement 4.3. *Filter $\mathscr{F}_l$, generated by logarithmic density, is a monotone closed and a right filter.*

V. Uniform convergence. Let $M\subset\mathbb{N}\wedge\left(t\in Z_+;s\in\mathbb{N}\right)$. Assume

$$M\left(t+1;t+s\right)=\left|n\in M:t+1\leq n\leq t+s\right|.$$

Let

$$\beta_s\left(M\right)=\liminf_{t\to\infty}M\left(t+1;t+s\right),$$
$$\beta^s\left(M\right)=\limsup_{t\to\infty}M\left(t+1;t+s\right).$$

If $\lim\limits_{s\to\infty}\frac{\beta_s(M)}{s}=\lim\limits_{s\to\infty}\frac{\beta^s(M)}{s}=\beta\left(M\right)$, then the quantity $\beta\left(M\right)$ is called the uniform density of the set M. Put $\mathscr{F}_\beta\equiv\left\{M\subset\mathbb{N}:\beta\left(M\right)=1\right\}$. Now we show that $\mathscr{F}_\beta$ is a filter. It is clear that

$$M\left(t+1;t+s\right)+M^c\left(t+1;t+s\right)=\left|\left[t+1,t+s\right]\right|=s.$$

Hence it directly follows that $\beta\left(M\right)=1\Leftrightarrow\beta\left(M^c\right)=0$. $I_\beta\equiv\{M\subset\mathbb{N}:\beta\left(M\right)=0\}$ is a non-trivial ideal [23]. Therefore, $\mathscr{F}_\beta$ is a filter. It is clear that $\mathscr{F}_\beta$ satisfies the condition (v). Next we show that $\mathscr{F}_\beta$ is a monotone closed filter. Let $K_1\supset K_2\supset\ldots\ \wedge\left(\beta\left(K_n\right)=1,\ \forall n\in\mathbb{N}\right)\Rightarrow\beta\left(K_n^c\right)=0,\ \forall n\in\mathbb{N}\Rightarrow\exists\left\{n_k\right\}_{k\in\mathbb{N}}\subset\mathbb{N},\ n_1<n_2<\ldots:$

$$\frac{\beta^s\left(K_m^c\right)}{s}<\frac{1}{m},\ \forall s\geq n_m.$$

As earlier, we set $\mathbb{N}_0=\tilde{\mathbb{N}}_0\cup I_{n_1}$, where $\tilde{\mathbb{N}}_0\equiv\left\{k\in\mathbb{N}:n_m\leq k\leq n_{m+1}\wedge\right.$ $\left.\left(k\in K_m^c\right)\right\}$. It is clear that $\beta\left(\mathbb{N}_0\right)=\beta\left(\tilde{\mathbb{N}}_0\right)$. Let $n>n_1$ be an arbitrary integer. Then $\exists m\in\mathbb{N}:n_m<n\leq n_{m+1}$. It is obvious that the inclusion

$$\left(I_n\cap\tilde{\mathbb{N}}_0\right)\subset\left(I_n\cap K_m^c\right),$$

in this case is also true. From the arbitrariness of $n\in\mathbb{N}$ we have

$$\left(\tilde{\mathbb{N}}_0\cap\left[t+1;t+s\right]\right)\subset\left(K_m^c\cap\left[t+1;t+s\right]\right).$$

Consequently

$$\tilde{\mathbb{N}}_0\left(t+1;t+s\right)\leq K_m^c\left(t+1;t+s\right),$$

and, as a result

$$\beta^s\left(\tilde{\mathbb{N}}_0\right)\leq\beta^s\left(K_m^c\right).$$

Thus

$$\frac{\beta^s\left(\tilde{\mathbb{N}}_0\right)}{s} \leq \frac{\beta^s\left(K_m^c\right)}{s} < \frac{1}{m}, \ \forall s \geq n_m.$$

From this relation it directly follows

$$\beta\left(\tilde{\mathbb{N}}_0\right) = 0 \Rightarrow \beta\left(\mathbb{N}_0\right) = 0 \Rightarrow \beta\left(\mathbb{N}_0^c\right) = 1 \Rightarrow N_0^c \in \mathscr{F}_\beta,$$

where

$$\mathbb{N}_0^c \equiv \left\{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\right\},$$

i.e. $\mathscr{F}_\beta$ is a monotone closed filter.

Statement 4.4. *Filter $\mathscr{F}_\beta$, generated by the uniform convergence, is a monotone closed and a right filter.*

VI. Non-monotone closed filter. Let $A_k \equiv \left\{n\,2^k : n \in \mathbb{N}\right\}$, $\forall k \in \mathbb{N}$. Assume

$$\mathscr{F} \equiv \left\{M \subset 2^{\mathbb{N}} : \exists k \in \mathbb{N} \Rightarrow A_k \subset M\right\}.$$

It is clear that $\emptyset \notin \mathscr{F}$. Put $A; B \in \mathscr{F} \Rightarrow \exists k_1; k_2 \in \mathbb{N} : (A_{k_1} \subset A) \wedge (A_{k_2} \subset B)$. Let $k_0 = \max\{k_1; k_2\}$. It is obvious that $(A \cap B) \supset A_{k_0}$, i.e. the condition (ii) of the filter satisfies. Let $(A \in \mathscr{F}) \wedge (A \subset B)$. Consequently, $\exists k_0 \in \mathbb{N} : A_{k_0} \subset A \Rightarrow A_{k_0} \subset B \Rightarrow B \in \mathscr{F}$. So, $\mathscr{F}$ is a filter. Let us show that $\mathscr{F}$ is a non-monotone closed filter. It is clear that $A_1 \supset A_2 \supset ....$ Let $\exists n_k; n_1 < n_2 < ... : \cup_{m=1}^{\infty}\left((n_m, n_{m+1}] \cap A_m\right) \in \mathscr{F}$. Consequently, $\exists p \in \mathbb{N} : A_p \subset \cup_{m=1}^{\infty}\left((n_m, n_{m+1}] \cap A_m\right)$. Put $k_0 \in \mathbb{N} : 2k_0 + 1 > n_{p+1}$. It is easy to see that $(2k_0 + 1)\,2^p \notin A_k, \forall k > p \Rightarrow (2k_0 + 1)\,2^p \notin \cup_{m=1}^{\infty}\left((n_m, n_{m+1}] \cap A_m\right)$. The obtained contradiction shows that $\mathscr{F}$ is a non-monotone closed filter.

Following [23], number of such examples can be extended.

Remark 4.5. Similar results can be obtained with respect to concepts of I-convergence.

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