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Author(s):

A. Kamal

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COMPOSITION OPERATORS AND NATURAL METRICS IN MEROMORPHIC FUNCTION CLASSES Q_p

A. KAMAL

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Abstract. In this paper, we investigate some results on natural metrics on the μ-normal functions and meromorphic Q_p-classes. Also, these classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, compact composition operators C_ϕ and Lipschitz continuous operators acting from μ-normal functions to the meromorphic Q_p-classes are characterized by conditions depending only on ϕ.

Keywords: Meromorphic classes, composition operators, Lipschitz continuous.

MSC(2010): Primary: $47B38$ Secondary: $30D50$, $30D45$, $46E15$.

1. Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ in the complex plane $\mathbf{C}$, $\partial\mathbf{D}$ it's boundary. Let $\mathcal{H}(\mathbf{D})$ and $M(\mathbf{D})$ denote the classes of holomorphic and meromorphic functions on $\mathbf{D}$, respectively. Any holomorphic function $\phi : M(\mathbf{D}) \to M(\mathbf{D})$ gives rise to a map $C_\phi : M(\mathbf{D}) \to M(\mathbf{D})$ defined by $C_\phi := f \circ \phi$, the composition map induced by ϕ. Also, $dA(z)$ is the normalized area measure on $\mathbf{D}$ so that $A(\mathbf{D}) \equiv 1$. Let the Green's function of $\mathbf{D}$ be defined as $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, for $z, a \in \mathbf{D}$ is the Möbius transformation related to the point $a \in \mathbf{D}$.

If (X, d) is a metric space, we denote the open and closed balls with center x and radius $r > 0$ by $B(x, r) := \{y \in X : d(y, x) < r\}$ and $\bar{B}(x, r) := \{y \in X : d(x, y) \le r\}$, respectively.
A positive continuous function μ on the interval $[0, 1)$ is called normal if there are three const. a, b $(0 \le a < b)$ and r such that

$\frac{\mu(r)}{(1-r)^a}$ is decreasing for $r \in [0, 1)$ and $\lim\limits_{r \to 1^-} \frac{\mu(r)}{(1-r)^a} = 0$;

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$\frac{\mu(r)}{(1-r)^b}$ is increasing for $r \in [0,1)$ and $\lim\limits_{r\to 1^-} \frac{\mu(r)}{(1-r)^b} = \infty$.

The Bloch-type $\mathcal{B}_\mu$ consists of all $f \in \mathcal{H}(\mathbf{D})$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(|z|)f(z) < \infty,$$

where μ is normal. The little Bloch-type space $\mathcal{B}_{\mu,0}$ is a subspace of $\mathcal{B}_\mu$ consisting of those $f \in \mathcal{B}_\mu$ such that

$$\lim_{|z|\to 1} \mu(|z|)f(z) = 0.$$

Now, we give some definitions for different classes of analytic functions which have been recently studied intensively in the theory of holomorphic Banach function spaces. We begin with the following definition:

Definition 1.1. [2] For $0 \le p < \infty$, the spaces Q_p and $Q_{p,0}$ are defined by

$$Q_p = \{f \in \mathcal{H}(\mathbf{D}) : \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f(z)|^2 g^p(z,a)dA(z) < \infty\},$$

$$Q_{p,0} = \{f \in \mathcal{H}(\mathbf{D}) : \lim_{|a|\to 1} \int_{\mathbf{D}} |f(z)|^2 g^p(z,a)dA(z) = 0\}.$$

The purpose of this paper is twofold. First we generalize some known results. Moreover, we compare the results for the classes of analytic functions to the corresponding ones for the classes of meromorphic functions. Secondly, we give a characterizations of Lipschitz continuity and compactness.
Now, we turn our attention to certain classes of meromorphic functions, namely, the class of normal functions $\mathcal{N}$ and $\mathcal{B}^\#$ spaces. For a meromorphic function f, a natural counterpart of the derivative $|f'(z)|$ of the analytic case is *the spherical derivative* $f^\#(z)$ defined by

$$f^\#(z) = \frac{|f'(z)|}{1+|f(z)|^2}, \ \ z \in \mathbf{D}.$$

Definition 1.2. (see [1, 7, 12]) The class of normal functions is defined by

$$\mathcal{N} = \{f \in M(\mathbf{D}) : \|f\|_{\mathcal{N}} = \sup_{z\in\mathbf{D}} (1-|z|^2)f^\#(z) < \infty\}.$$

Moreover, the family $\mathcal{N}_0$ of little normal functions is defined as

$$\mathcal{N}_0 = \{f \in M(\mathbf{D}) : \|f\|_{\mathcal{N}_0} = \lim_{|z|\to 1} (1-|z|^2)f^\#(z) = 0\}.$$

Definition 1.3. (see [5]) For some $r \in (0,1)$ the class of spherical Bloch functions $\mathcal{B}^\#$ is defined by

$$\mathcal{B}^\# = \{f \in M(\mathbf{D}) : \sup_{a\in\mathbf{D}} \int_{\mathbf{D}(a,r)} (f^\#(z))^2 dA(z) < \infty\}.$$

We clearly have $\mathcal{N} \subset \mathcal{B}^{\#}$. Yamashita in [15] has proved that there is an essential difference between $\mathcal{N}$ and $\mathcal{B}^{\#}$. A function $f \in M(\mathbf{D})$ is said to belong to the class of normal functions $\mathcal{N}_\mu$ if

$$\|f\|_{\mathcal{N}_\mu} = |f(0)| + \sup_{z\in\mathbf{D}} \mu(z) f^{\#}(z) < \infty,$$

where μ is normal and radial and $\mu(|z|) = \mu(z)$. The space $\mathcal{N}_\mu$ is a Banach space with the norm $\|.\|_{\mathcal{N}_\mu}$. The class of little normal functions $\mathcal{N}_{\mu,0}$ consists of all $f \in \mathcal{N}_\mu$ such that

$$\lim_{|z|\to 1^-} \mu(|z|) f^{\#}(z) = 0.$$

For $\alpha > 0$, $\mu(|z|) = (1-|z|^2)^\alpha$. When $\alpha = 1$, the class $\mathcal{N}_\mu$ of the α-normal functions reduces to the class $\mathcal{N}$ of normal function.

Meromorphic function classes, like $\mathcal{N}_\mu$-classes of those $f \in \mathcal{H}(\mathbf{D})$ for which $\mathcal{N}_\mu$ are not linear.

The functions class $Q_p^{\#}$ or meromrphic Q_p has been introduced by several mathematicians. For $0 < p < \infty$, the meromorphic class $Q_p^{\#}$ consists of those functions $f \in M(\mathbf{D})$ for which

$$\|f\|^2_{Q_p^{\#}} = \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} (f^{\#}(z))^2 g^p(z,a)\, dA(z) < \infty.$$

Moreover, we say that $f \in Q_p^{\#}$ belongs to the class $Q_{p,0}^{\#}$ if

$$\lim_{|a|\to 1} \int_{\mathbf{D}} (f^{\#}(z))^2 g^p(z,a)\, dA(z) = 0.$$

In [2] Aulaskari et al. has proved that $Q_p^{\#} = \mathcal{N}$ and $Q_{p,0}^{\#} = \mathcal{N}_0$ for all $p > 1$. For $p = 1$ it is well know that $Q_1^{\#} = UBC$ (the functions of uniformly bounded characteristic).

For any holomorphic self-mapping ϕ of $\mathbf{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f) = f \circ \phi$ from $\mathcal{H}(\mathbf{D})$ or $B(\mathbf{D})$ into itself. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see e.g. [4, 6, 8, 10, 11, 16] and others).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \le C\|f\|_X$ for all maps $f \in X$.

Two quantities A and B are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_1 B \le A \le C_2 B$, written as $A \approx B$.

The following lemma follows by standard arguments similar to those outlined in [13]. Hence we omit the proof.

Lemma 1.4. *Assume μ is a normal function on the interval $[0,1)$ and ϕ is a holomorphic mapping from $\mathbf{D}$ into itself. Let $0 < p < \infty$. Then $C_\phi : \mathcal{N}_\mu \to$*

$Q_p^{\#}$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in\mathbf{N}} \in \mathcal{N}_\mu$ which converges to zero uniformly on compact subsets of $\mathbf{D}$ as $n\to\infty$,

$$\lim_{n\to\infty} \|C_\phi f_n\|_{Q_p^{\#}} = 0.$$

Lemma 1.5. *(see [14]) Let $0<\alpha<1$, then there exist two functions $f, g \in \mathcal{B}_\alpha$ such that for some constant ϵ,*

$$(|f'(z)| + |g'(z)|) \geq \frac{\epsilon}{(1-|z|^2)^\alpha} > 0, \qquad \forall\ z\ \in\ \mathbf{D}.$$

In order to prove the main result, we first give some Lemmas.

Lemma 1.6. (see [9]) *There are two functions $f_1, f_2 \in \mathcal{N}$ such that*

$$M_0 := \inf_{z\in\mathbf{D}} (1-|z|^2)[f_1^{\#}(z) + f_2^{\#}(z)] > 0.$$

Lemma 1.7. *Let $\mu : [0,1) \to [0,\infty)$ be a nondecreasing radial weight function and normal on the interval $[0,1)$. Then there exist two functions $f, g \in \mathcal{N}_\mu$ such that for each $z \in \mathbf{D}$,*

$$M_0 := \inf_{z\in\mathbf{D}} \mu(|z|)[f^{\#}(z) + g^{\#}(z)] > 0,$$

where M_0 is a positive constant.

Proof. We will consider two Schwarz triangle functions (see [3]). Take a hyperbolic triangle in the unit disk, with appropriate vertex angles which divide π, say, $\frac{\pi}{m}, \frac{\pi}{n}, \frac{\pi}{p}$ where m, n, p are natural numbers satisfying the inequality $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} < 1$. Then map the interior of the triangle conformally onto the upper half plane so that the vertices map to $0, 1$, and ∞. Then use analytic continuation to get a function f_1 meromorphic in $\mathbf{D}$ with poles at all the points corresponding to $\mathbf{D}$. Such a function f_1 is a normal function, and the expression $\mu(|z|)f_1^{\#}(z)$ is invariant under the Fuchsian group Γ_1 of alternate reflections of the triangle and is zero only at the vertices of the original triangle and their reflected images. Note that f_1 is a normal function because of the invariance of $\mu(|z|)f_1^{\#}(z)$ under Γ_1, since this expression is bounded on any compact subset of the unit disk. Now take a second hyperbolic triangle congruent to the first, but with vertices at points disjoint from the vertices of the original triangle and the images of vertices of the original triangle under Γ_1. Then create another function f_2 for this second triangle by the same method as before, where f_2 and $\mu(|z|)f_2^{\#}(z)$ are both invariant under the Fuchsian group Γ_2 of alternate reflections of the second triangle. Then $\mu(|z|)f_2^{\#}(z)$ and $\mu(|z|)f_2^{\#}$ do not have a common zero, and by the invariance of $\mu(|z|)f_j^{\#}(z)$ under $\Gamma_j, j = 1, 2$, each is bounded away from zero near where the other is zero. These bounds are uniform, since they can be taken on a sufficiently large fixed compact set. Thus, these two functions satisfy the conclusion of the lemma.

2. Natural metrics in $\mathcal{N}_\mu$ and $Q_p^\#$ classes

In this section we introduce natural metrics on the normal classes $\mathcal{N}_\mu$ and the meromorphic Q_p classes $Q_p^\#$.

First, we can find a natural metric in $\mathcal{N}_\mu$ by defining

$$\begin{aligned} d(f,g;\mathcal{N}_\mu) &:= d_{\mathcal{N}_\mu}(f,g) + \|f-g\|_{\mathcal{B}_\mu} + |f(0)-g(0)| \\ &:= \sup_{z\in\mathbf{D}}\left|\frac{f'(z)}{1+|f(z)|^2} - \frac{g'(z)}{1+|g(z)|^2}\right|\mu(|z|) \\ &\quad + \|f-g\|_{\mathcal{B}_\mu} + |f(0)-g(0)|\ , \quad \text{for } f,g\in\mathcal{N}_\mu. \end{aligned}$$

For $f,g\in Q_p^\#$, define their distance by

$$\begin{aligned} d(f,g;Q_p^\#) &:= d_{Q_p^\#}(f,g) + \|f-g\|_{Q_p} + |f(0)-g(0)| \\ &:= \left(\sup_{z\in\mathbf{D}}\int_{\mathbf{D}}\left|\frac{f'(z)}{1+|f(z)|^2} - \frac{g'(z)}{1+|g(z)|^2}\right|^2 g^p(z,a)\,dA(z)\right)^{\frac{1}{2}} \\ &\quad + \|f-g\|_{Q_p} + |f(0)-g(0)|. \end{aligned}$$

Proposition 2.1. *The class $\mathcal{N}_\mu$ equipped with the metric $d(.,.;\mathcal{N}_\mu)$ is a complete metric space. Moreover, $\mathcal{N}_{\mu,0}$ is a closed (and therefore complete) subspace of $\mathcal{N}_\mu$.*

Proof. Let $f,g,h\in\mathcal{N}_\mu$. Then clearly $d(f,g;\mathcal{N}_\mu)\ge 0$, $d(f,f;\mathcal{N}_\mu)=0$, $d(f,g;\mathcal{N}_\mu) = d(g,f;\mathcal{N}_\mu)$ and $d(f,h;\mathcal{N}_\mu)\le d(f,g;\mathcal{N}_\mu)+d(g,h;\mathcal{N}_\mu)$. It also follows from the presence of the usual $\mathcal{B}_\mu$-term that $d(f,g;\mathcal{N}_\mu)=0$ implies $f=g$. Hence, d is a metric on $\mathcal{N}_\mu$.

Let $(f_n)_{n=0}^\infty$ be a Cauchy sequence in the metric space $\mathcal{N}_\mu$, that is, for any $\varepsilon>0$ there is an $N=N(\varepsilon)\in\mathbf{N}$ such that $d(f_n,f_m;\mathcal{N}_\mu)<\varepsilon$, for all $n,m>N$. Since $(f_n)\subset M(\mathbf{D})$, the family (f_n) is uniformly bounded and hence normal in $\mathbf{D}$. Therefore there exists $f\in M(\mathbf{D})$ and a subsequence $(f_{n_j})_{j=1}^\infty$ such that f_{n_j} converges to f uniformly on compact subsets of $\mathbf{D}$. It follows that also f_n converges to f uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Now let $m>N$. Then the uniform convergence yields

$$\begin{aligned} &\left|\frac{f'(z)}{1+|f(z)|^2} - \frac{f_m'(z)}{1+|f_m(z)|^2}\right|\mu(|z|) \\ =\ &\lim_{n\to\infty}\left|\frac{f_n'(z)}{1+|f_n(z)|^2} - \frac{f_m'(z)}{1+|f_m(z)|^2}\right|\mu(|z|) \\ \le\ &\lim_{n\to\infty} d(f_n,f_m;\mathcal{N}_\mu)\le\varepsilon \end{aligned}$$

for all $z\in\mathbf{D}$, and it follows that $\|f\|_{\mathcal{N}_\mu}\le\|f_m\|_{\mathcal{N}_\mu}+\varepsilon$. Thus $f\in\mathcal{N}_\mu$ as desired. Moreover, the above inequality and the compactness of the usual $\mathcal{B}_\mu$ space imply that $(f_n)_{n=1}^\infty$ converges to f with respect to the metric d.

Since $\lim_{n\to\infty} d(f_n, f_m; \mathcal{N}_\mu) \le \varepsilon$, the second part of the assertion follows. $\square$

Proposition 2.2. *The class $Q_p^\#$ equipped with the metric $d(.,.;Q_p^\#)$ is a complete metric space. Moreover, $Q_{p,0}^\#$ is a closed (and therefore complete) subspace of $Q_p^\#$.*

Proof. As in the proof of Proposition 2.1 we find that $d(.,.;Q_p^\#)$ is a metric in $Q_p^\#$. For the completeness, let $(f_n)_{n=0}^\infty$ be a Cauchy sequence in the metric space $Q_p^\#$, that is, for any $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbf{N}$ such that $d(f_n, f_m; Q_p^\#) < \varepsilon$, for all $n, m > N$. Similarly as in the proof of Proposition 2.1, we find an $f \in M(\mathbf{D})$ such that f_n converges to f uniformly on compact subsets of $\mathbf{D}$. Let $m > N$ and $0 < r < 1$. Then Fatou's lemma yields

$$\begin{aligned}
&\int_{\mathbf{D}(0,r)} \left| \frac{f'(z)}{1+|f(z)|^2} - \frac{f_m'(z)}{1+|f_m(z)|^2} \right|^2 g^p(z,a)\,dA(z) \\
= &\int_{\mathbf{D}(0,r)} \lim_{n\to\infty} \left| \frac{f_n'(z)}{1+|f_n(z)|^2} - \frac{f_m'(z)}{1+|f_m(z)|^2} \right|^2 g^p(z,a)\,dA(z) \\
\le &\lim_{n\to\infty} \int_{\mathbf{D}} \left| \frac{f_n'(z)}{1+|f_n(z)|^2} - \frac{f_m'(z)}{1+|f_m(z)|^2} \right|^2 g^p(z,a)\,dA(z) \le \varepsilon^2,
\end{aligned}$$

and by letting $r \to 1^-$, it follows that

$$\begin{aligned}
&\int_{\mathbf{D}} (f^\#(z))^2 g^p(z,a)\,dA(z) \\
\le \ & 2\varepsilon^2 + 2\int_{\mathbf{D}} (f_m^\#(z))^2 g^p(z,a)\,dA(z).
\end{aligned}$$

This yields

$$\|f\|_{Q_p^\#}^2 \le 2\varepsilon^2 + 2\|f_m\|_{Q_p^\#}^2, \tag{2.1}$$

and thus $f \in Q_p^\#$. We also find that $f_n \to f$ with respect to the metric of $Q_p^\#$. $\square$

The second part of the assertion follows by (2.1).

3. Lipschitz continuity and boundedness of C_ϕ

The following characterization of bounded composition operators mapping from normal class into $Q_p^\#$ can be found in [9] Theorem 1

Theorem 3.1. *Let $p \in (0,\infty)$ and let $\phi : \mathbf{D} \to \mathbf{D}$ be a holomorphic function. Then the following are equivalent:*

(i) $C_\phi : \mathcal{N} \to Q_p^\#$ *is bounded;*

(ii) $C_\phi : \mathcal{N}_0 \to Q_p^\#$ *is bounded;*

(iii) $\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} [\frac{|\phi'(z)|}{1-\phi'(z)}]^2 g^p(z,a)dA(z) < \infty.$

We prove the following results:

Theorem 3.2. *Assume ϕ is a holomorphic mapping from $\mathbf{D}$ into itself. Let $0 \le p < \infty$. Then the following are equivalent:*

(i) $C_\phi : \mathcal{B}_\mu \to Q_p$ *is bounded;*

(ii) $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ *is bounded;*

(iii) $C_\phi : \mathcal{N}_{\mu,0} \to Q_p^\#$ *is bounded.*

Proof. It is clear that (ii)⇒(iii) since $\mathcal{N}_\mu \subset \mathcal{N}_{\mu,0}$. It is enough to prove the implications (i)⇒(ii) and (iii)⇒(i).

Suppose $C_\phi : \mathcal{B}_\mu \to Q_p$ is bounded, that is, (i) is satisfied. If $f \in \mathcal{N}_\mu$, then

$$\begin{aligned} & \int_{\mathbf{D}} \big((f\circ\phi)^\#(z)\big)^2 g^p(z,a)dA(z) \\ \le \quad & \|f\|^2_{\mathcal{N}_\mu} \int_{\mathbf{D}} \frac{|\phi'(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)dA(z), \end{aligned}$$

where the last integral is uniformly bounded for all $a \in \mathbf{D}$ by Theorem 3.1, and therefore $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ is bounded. Thus (i) implies (ii).

To prove (iii) implies (i), suppose that (iii) holds, that is $\|C_\phi f\|_{Q_p^\#} \le \|f\|_{\mathcal{N}_\mu}$ for all $f \in \mathcal{N}_{\mu,0}$. For given $f \in \mathcal{N}_\mu$, the function $f_t(z) = f(tz)$, where $0 < t < 1$, belongs to $\mathcal{N}_{\mu,0}$ with the property $\|f_t\|_{\mathcal{N}_\mu} \le \|f\|_{\mathcal{N}_\mu}$. □

Let f, g be the functions from Lemma 1.7, we get

$$\begin{aligned} & \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|t\phi'(z)|^2}{\mu^2(|t\phi(z)|)} g^p(z,a)dA(z) \\ = \quad & 2\|C_\phi\|^2 \big(\|f_t\|^2_{\mathcal{N}_\mu} + \|g_t\|^2_{\mathcal{N}_\mu}\big) \\ \le \quad & 2\|C_\phi\|^2 \big(\|f\|^2_{\mathcal{N}_\mu} + \|g\|^2_{\mathcal{N}_\mu}\big). \end{aligned}$$

This estimate together with the Fatou's lemma and Theorem 3.1 implies (i). The proof is complete.

Theorem 3.3. *Assume μ is a normal function, $\int_0^{|z|} \frac{ds}{\mu(s)} < \infty$ and ϕ is a holomorphic mapping from $\mathbf{D}$ into itself, in which z is a finite integer. Let $0 < p < \infty$. Then the following are equivalent:*

(i) $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ *is bounded;*

(ii) $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ *is Lipschitz continuous;*

(iii) $\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\phi'(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)dA(z) < \infty.$

Proof. It is known that $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ is bounded if and only if (iii) is satisfied by Theorem 3.2. Therefore it suffices to prove that the assertions (ii) and (iii) are equivalent.

Assume first that $C_\phi : \mathcal{N}_\mu \to Q_p^\#$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f \circ \phi, g \circ \phi; Q_p^\#) \leq C d(f, g; \mathcal{N}_\mu), \quad \text{for all } f, g \in \mathcal{N}_\mu.$$

Taking $g = 0$, this implies

$$\|f \circ \phi\|_{Q_p^\#} \leq C\big(\|f\|_{\mathcal{N}_\mu} + \|f\|_{\mathcal{B}_\mu} + |f(0)|\big), \quad \text{for all } f \in \mathcal{N}_\mu. \tag{3.1}$$

and therefore Lemma 1.7 implies the existence of $f, g \in \mathcal{N}_\mu$ such that

$$\big(|f^\#(z)| + |g^\#(z)|\big)\mu(|z|) \geq C > 0, \quad \text{for all } z \in \mathbf{D}. \tag{3.2}$$

Combining (3.1), (3.2) and Lemma 1.5, we obtain

$$\begin{aligned} & \|f\|_{\mathcal{N}_\mu} + \|g\|_{\mathcal{N}_\mu} + \|f\|_{\mathcal{B}_\mu} + \|g\|_{\mathcal{B}_\mu} + |f(0)| + |g(0)| \\ \geq \; & C \int_{\mathbf{D}} \frac{|\phi'(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a) dA(z) \end{aligned}$$

from which the assertion (iii) follows.

Assume now that (iii) is satisfied, we obtain

$$\begin{aligned} |f(z)| &= \left| \int_0^z f'(s) ds + f(0) \right| \\ &\leq \|f\|_{\mathcal{B}_\mu} \int_0^{|z|} \frac{ds}{\mu(s)} + |f(0)|. \end{aligned}$$

Let $\int_0^{|z|} \frac{ds}{\mu(s)} < \infty$, in which z is a finite integer, this yields

$$\begin{aligned} \big|f(\phi(0)) - g(\phi(0))\big| &\leq \|f - g\|_{\mathcal{B}_\mu} \int_0^{|\phi(0)|} \frac{ds}{\mu(s)} + |f(0) - g(0)| \\ &\leq \lambda \|f - g\|_{\mathcal{B}_\mu} + |f(0) - g(0)|, \end{aligned}$$

where $\lambda > 1$. Therefore,

$$\begin{aligned} d(f\circ\phi, g\circ\phi; Q_p^{\#}) &= d_{Q_p^{\#}}(f\circ\phi, g\circ\phi) + \|f\circ\phi - g\circ\phi\|_{Q_p} \\ &\quad + \big|f(\phi(0)) - g(\phi(0))\big| \\ &\leq d_{\mathcal{N}_\mu}(f,g)\left(\sup_{a\in\mathbf{D}}\int_{\mathbf{D}} \frac{|\phi'(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)dA(z)\right)^{\frac{1}{2}} \\ &\quad + \|f-g\|_{\mathcal{B}_\mu}\left(\sup_{a\in\mathbf{D}}\int_{\mathbf{D}} \frac{|\phi'(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)dA(z)\right)^{\frac{1}{2}} \\ &\quad + \|f-g\|_{\mathcal{B}_\mu}\int_0^{|z|}\frac{ds}{\mu(s)} + |f(0)-g(0)| \\ &\leq C' d(f,g;\mathcal{N}_\mu). \end{aligned}$$

Thus $C_\phi : \mathcal{N}_\mu \to Q_p^{\#}$ is Lipschitz continuous and the proof is complete. □

4. Compactness of $C_\phi : \mathcal{N}_\mu \to Q_p^{\#}$

Recall that an operator $C_\phi : \mathcal{N}_\mu \to Q_p^{\#}$ is compact, if it maps any ball in $\mathcal{N}_\mu$ onto a precompact set in $Q_p^{\#}$. The following observation is some times useful.

Proposition 4.1. *Assume μ is a normal function and ϕ is a holomorphic mapping from $\mathbf{D}$ into itself. Let $0 < p < \infty$. If $C_\phi : \mathcal{N}_\mu \to Q_p^{\#}$ is compact, it maps closed balls onto compact sets.*

Proof. If $B \subset \mathcal{N}_\mu$ is a closed ball and $g \in Q_p^{\#}$ belongs to the closure of $C_\phi(B)$, we can find a sequence $(f_n)_{n=1}^{\infty} \subset B$ such that $f_n \circ \phi$ converges to $g \in Q_p^{\#}$ as $n \to \infty$. But $(f_n)_{n=1}^{\infty}$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^{\infty}$ converging uniformly on the compact subsets of $\mathbf{D}$ to an analytic function f. As in earlier arguments of Proposition 2.1, we get a positive estimate which shows that f must belong to the closed ball B. On the other hand, also the sequence $(f_{n_j}\circ\phi)_{j=1}^{\infty}$ converges uniformly on compacta to an analytic function, which is $g \in Q_p^{\#}$. We get $g = f\circ\phi$, i.e. g belongs to $C_\phi(B)$. Thus, this set is closed and also compact. □

Compactness of composition operators can be proved in full analogy with the linear case.

Theorem 4.2. *Assume μ is a normal function and ϕ is a holomorphic mapping from $\mathbf{D}$ into itself. Let $0 < p < \infty$. Then $C_\phi : \mathcal{N}_\mu \to Q_p^{\#}$ is compact if*

$$\lim_{r\to 1^-}\sup_{a\in\mathbf{D}}\int_{|\phi(z)|>r} \frac{|\phi(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)dA(z) = 0.$$

Proof. We first assume that (ii) holds. Let $B := \bar{B}(g,\delta) \subset \mathcal{N}_\mu$, were $g \in \mathcal{N}_\mu$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^{\infty} \subset B$ be any sequence. We show that its image has a convergent subsequence in $Q_p^{\#}$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^\infty \subset B(\mathbf{D})$ is a normal, hence, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbf{D}$ to an analytic function f. By the Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly on the compacta to f'. It follows that also the sequences $(f_{n_j} \circ \phi)_{j=1}^\infty$ and $(f'_{n_j} \circ \phi)_{j=1}^\infty$ converge uniformly on the compact subsets of $\mathbf{D}$ to $f \circ \phi$ and $f' \circ \phi$, respectively. Moreover, $f \in B \subset \mathcal{N}_\mu$ since for any fixed $R, 0 < R < 1$, the uniform convergence yields

$$\begin{aligned}
&\sup_{|z|\le R}\left|\frac{f'(z)}{1+|f(z)|^2} - \frac{g'(z)}{1+|g(z)|^2}\right|\mu(|z|) \\
&\quad + \sup_{|z|\le R} |f'(z) - g'(z)|\mu(|z|) + |f(0) - g(0)| \\
&= \lim_{j\to\infty} \sup_{|z|\le R}\left|\frac{f'_{n_j}(z)}{1+|f_{n_j}(z)|^2} - \frac{g'(z)}{1+|g(z)|^2}\right|\mu(|z|) \\
&\quad + \lim_{j\to\infty}\left(\sup_{|z|\le R} |f'_{n_j}(z) - g'(z)|\mu(|z|) + |f_{n_j}(0) - g(0)|\right) \le s.
\end{aligned}$$

hence, $d(f, g; \mathcal{N}_\mu) \le s$.

Let $\varepsilon > 0$. Since (ii) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a\in\mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)\,dA(z) \le \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbf{N}$ such that

$$|f_{n_j} \circ \phi(0) - f \circ \phi(0)| \le \varepsilon, \qquad \text{for all } j \ge N_1. \tag{4.1}$$

The condition (ii) is known to imply the compactness of $C_\phi : \mathcal{B}_\mu \to Q_p$, hence, possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$|f_{n_j} \circ \phi - f \circ \phi| \le \varepsilon, \qquad \text{for all } j \ge N_2 \in \mathbf{N}. \tag{4.2}$$

Since $(f_{n_j})_{j=1}^\infty \subset B$ and $f \in B$, it follows that

$$\begin{aligned}
&\sup_{a\in\mathbf{D}} \int_{|\phi(z)|>r} \left[(f_{n_j} \circ \phi)^{\#}(z) - (f \circ \phi)^{\#}(z)\right]^2 g^p(z,a)\,dA(z) \\
\le\ &\sup_{a\in\mathbf{D}} \int_{|\phi(z)|>r} \left|\frac{(f_{n_j} \circ \phi)'(z)}{1+|(f_{n_j} \circ \phi)(z)|^2} - \frac{(f \circ \phi)'(z)}{1+|(f \circ \phi)(z)|^2}\right|^2 g^p(z,a)\,dA(z) \\
\le\ &\sup_{a\in\mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)\,dA(z) \\
\le\ &d_{\mathcal{N}_\mu}(f_{n_j}, f) \sup_{a\in\mathbf{D}} \int_{|\phi(z)|>r} \frac{|\phi(z)|^2}{\mu^2(|\phi(z)|)} g^p(z,a)\,dA(z),
\end{aligned}$$

hence,

$$\sup_{a\in\mathbf{D}}\int_{|\phi(z)|>r}\left[(f_{n_j}\circ\phi)^{\#}(z)-(f\circ\phi)^{\#}(z)\right]^2 g^p(z,a)dA(z)\le 2s\,\varepsilon. \tag{4.3}$$

On the other hand, by the uniform convergence on compacta of $\mathbf{D}$, we can find an $N_3\in\mathbf{N}$ such that for all $j\ge N_3$,

$$\left|\frac{(f_{n_j})'(\phi(z))}{1+|f_{n_j}(\phi(z))|^2}-\frac{f'(\phi(z))}{1+|f(\phi(z))|^2}\right|\le\varepsilon$$

for all z with $|\phi(z)|\le r$. Hence, for such j,

$$\begin{aligned}
&\sup_{a\in\mathbf{D}}\int_{|\phi(z)|\le r}\left[(f_{n_j}\circ\phi)^{\#}(z)-(f\circ\phi)^{\#}(z)\right]^2 g^p(z,a)dA(z)\\
\le\;&\sup_{a\in\mathbf{D}}\int_{|\phi(z)|\le r}\left|\frac{(f_{n_j}\circ\phi)'(z)}{1+|(f_{n_j}\circ\phi)(z)|^2}-\frac{(f\circ\phi)'(z)}{1+|(f\circ\phi)(z)|^2}\right|^2 g^p(z,a)dA(z)\\
\le\;&\varepsilon\left(\sup_{a\in\mathbf{D}}\int_{|\phi(z)|\le r}\frac{|\phi(z)|^2}{\mu^2(|\phi(z)|)}g^p(z,a)dA(z)\right)^{\frac{1}{2}}\le C\varepsilon,
\end{aligned}$$

hence,

$$\sup_{a\in\mathbf{D}}\int_{|\phi(z)|\le r}\left[(f_{n_j}\circ\phi)^{\#}(z)-(f\circ\phi)^{\#}(z)\right]^2 g^p(z,a)dA(z)\le C\,\varepsilon. \tag{4.4}$$

where C is the bound obtained from (iii) of Theorem 3.3. Combining (4.1), (4.2), (4.3) and (4.4) we deduce that $f_{n_j}\to f$ in $Q_p^{\#}$. □

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(Alaa Kamal) Port Said University, Faculty of Science, Department of Mathematics, Port Said 42521, Egypt.

E-mail address: alaa_mohamed1@yahoo.com