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COMMON SOLUTIONS TO PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we propose two iterative methods for finding a common solution of a finite family of equilibrium problems for pseudomonotone bifunctions. The first is a parallel hybrid extragradient-cutting algorithm which is extended from the previously known one for variational inequalities to equilibrium problems. The second is a new cyclic hybrid extragradient-cutting algorithm. In the cyclic algorithm, using the known techniques, we can perform and develop practical numerical experiments.

Keywords: Hybrid method, parallel algorithm, cyclic algorithm, extragradient method, equilibrium problem.
MSC(2010): Primary: 90C33; Secondary: 68W10, 65K10.

1. Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space H and f be a bifunction from $H \times H$ to the set of real numbers $\mathbb{R}$. The equilibrium problem (EP) for the bifunction f on C is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C. \tag{1.1}$$

The solution set of the EP (1.1) is denoted by $EP(f)$. The EP is a generalization of many mathematical problems [9, 19]. In recent years, many algorithms have been proposed for solving the EP, see [1, 9, 14, 18, 19, 21, 25] and the references therein. When the bifunction f is monotone, the most of existing algorithms for solving the EP involve the regularization equilibrium problem (REP), i.e., at the n^{th} iteration step, known x_n, determine the next approximation x_{n+1} as the solution of the problem:

$$\text{Find } x \in C \text{ such that: } f(x, y) + \frac{1}{r_n} \langle y - x, x - x_n \rangle \geq 0, \ \forall y \in C, \tag{1.2}$$

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where $r_n \geq d > 0$. Note that the problem (1.2) is strongly monotone when the bifunction f is monotone. Thus, its solution exists and is unique under certain assumption of the continuty of the bifunction f. Unforturnately, in general, for instance when f is pseudomonotone, the problem (1.2) is not strongly monotone and so the unique solvability of (1.2) is not guaranteed even its solution set can not be convex. In this case, the authors in [1, 21] replaced the REP (1.2) by two strongly convex programs

$$\begin{cases} y_n = \arg\min\left\{\rho f(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \\ x_{n+1} = \arg\min\left\{\rho f(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \end{cases}$$

where $\rho > 0$ satisfies some suitable conditions.

Now let $K_i, i = 1, \ldots, N$ be a finite family of closed and convex subsets of H such that $K = \cap_{i=1}^N K_i \neq \emptyset$ and $f_i : H \times H \to \mathbb{R}, i = 1, \ldots, N$ be pseudomonotone bifunctions. The problem, so called the common solutions to equilibrium problems (CSEP), for the bifunctions f_i is stated as follows: Find $x^* \in K$ such that

$$f_i(x^*, y) \geq 0, \ \forall y \in K_i, \ i = 1, \ldots, N. \tag{1.3}$$

Clearly, the CSEP with $N = 1$ is the EP. The motivation and inspiration for researching the CSEP with $N > 1$ are originated from some simple observations that if $f_i(x, y) = 0$ for all $x, y \in H$ then all inequalities in (1.3) are automatically satisfied. Thus, the CSEP reduces to the following convex feasibility problem (CFP)

$$\text{Find } x^* \in K := \cap_{i=1}^N K_i \neq \emptyset \tag{1.4}$$

which is to find an element in the intersection of a family of convex sets $\{K_i\}_{i=1}^N$ in a Hilbert space H. The CFP has received great attention due to broad applicability in many areas of applied mathematics, most notably, as image recovery from projections, computerized tomography, and radiation therapy treatment planing, see for instance [6, 12]. Besides, if K_i is the fixed point set of the mapping $S_i : H \to H$, then the CFP (1.4) is the common fixed point problem (CFPP), i.e.,

$$\text{Find } x^* \in F := \cap_{i=1}^N F(S_i) \neq \emptyset, \tag{1.5}$$

where $F(S_i)$ is the fixed point set of S_i, $i = 1, \ldots, N$. Also, if $K_i = H$ and $f_i(x, y) = \langle x - S_i x, y - x\rangle$ then it is easy to show that x^* is a fixed point of S_i if and only if it is a solution of the EP for the bifunction f_i on K_i [9]. Thus, the CSEP also becomes the CFPP (1.5). Some parallel algorithms for solving the CFPP can be found in [4, 5, 15].

If $f_i(x, y) = \langle A_i(x), y - x\rangle$, where $A_i : H \to H$ are nonlinear operators, then the CSEP becomes the following common solutions of variational inequalities problem (CSVIP): Find $x^* \in K := \cap_{i=1}^N K_i$ such that

$$\langle A_i(x^*), y - x^*\rangle \geq 0, \ \forall y \in K_i, \ i = 1, \ldots, N \tag{1.6}$$

which was announced in [11]. Moreover, there are many other mathematical models which are special cases of the CSEP such as: common minimizer problems, common saddle point problems, variational inequalities over the intersection of closed convex subsets, common solutions of operator equations, see [3, 4, 5, 9, 11, 15] and the references therein. These problems have been widely studied over the past decades because of their practical applications to image reconstruction, signal processing, biomedical engineering, communication, etc [6, 10, 12, 24].

In this paper, we propose two parallel and cyclic extragradient - cutting algorithms for solving the CSEP for pseudomonotone bifunctions. The former is extended from a previously known algorithm for variational inequalities [11] to equilibrium problems. The authors in [11] studied the CSVIP for Lipschitz continuous and monotone operators. They used the extragradient (or double projection) method which was introduced by Korpelevich [16] in Euclidean space, and by Nadezhkina and Takahashi [20] in Hilbert space to construct iteration sequences. Our first algorithm reduces to the CSVIP under a weaker hypothesis that operators need only the pseudomonotonicity. The latter is a sequential algorithm which seems to be performed more easily than the first and can delvelop practical numerical experiments by using the known techniques of Solodov and Svaiter [23] when the number of subproblems N is large. The cyclic algorithm can be considered as an improvement of the iterative method in [11] and others when the CSEP is reduced to the CSVIP.

The paper is organized as follows: In Section 2, we collect some definitions and primary results for using in the next section. Section 3 deals with our proposed algorithms and proving the convergence theorems.

2. **Preliminaries**

In this section, we recall some definitions and results for further researches. For solving the CSEP (1.3), we assume that each bifunction f_i satisfies the following conditions:

(A1) f_i is pseudomonotone on H, i.e., for all $x, y \in H$,

$$f_i(x,y) \geq 0 \Rightarrow f_i(y,x) \leq 0;$$

(A2) f_i is Lipschitz-type continuous, i.e., there exist two positive constants c_1, c_2 such that

$$f_i(x,y) + f_i(y,z) \geq f_i(x,z) - c_1||x-y||^2 - c_2||y-z||^2, \quad \forall x,y,z \in H;$$

(A3) f_i is weakly continuous on $H \times H$;

(A4) $f_i(x,.)$ is convex and subdifferentiable on H for every fixed $x \in H$.

Note that the condition (A2) is fulfilled for the bifunction

$$f(x,y) = \langle A(x), y-x \rangle ,$$

where A is a Lipschitz continuous operator (proved in Corollary 3.7 below). We have the following result.

Lemma 2.1. [8, Proposition 4.1] *If the bifunction f satisfies the conditions* (A1) − (A4)*, then the solution set $EP(f)$ is closed and convex.*

The metric projection $P_C : H \to C$ is defined by

$$P_C x = \arg\min \{\|y - x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C x$ exists and is unique. It is also known that P_C has the following characteristic properties

Lemma 2.2. *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

(*i*) *P_C is firmly nonexpansive, i.e.,*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H.$$

(*ii*) *For all $x \in C, y \in H$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2. \tag{2.1}$$

(*iii*) *$z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, x - y\rangle \geq 0, \forall y \in C\}.$$

The proof of the following lemma is similar to the proof of Theorem 27.4 in [22] (also see Theorem 3.1 in [13]) which uses Moreau-Rockafellar Theorem in [17] to find the subdifferential of a sum of convex function g and indicator function δ_C to C in a real Hilbert space H.

Lemma 2.3. [22, Theorem 27.4] *Let C be a convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Then, x^* is a solution to the following convex problem*

$$\min \{g(x) : x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

3. Main results

In this section, we propose two algorithms for solving the CSEP (1.3) and analyse the convergence of the iteration sequences generated by the algorithms. In the sequel, without loss of generality, we assume that the bifunctions $f_i, i = 1, \ldots, N$ are Lipschitz-type continuous with the same positive constants c_1 and c_2, i.e.,

$$f_i(x, y) + f_i(y, z) \geq f_i(x, z) - c_1||x - y||^2 - c_2||y - z||^2$$

for all $x, y, z \in H$. Moreover, the solution set $F = \cap_{i=1}^N EP(f_i)$ is nonempty.

Algorithm 3.1. *(The parallel hybrid extragradient-cutting algorithm)*
Initialize. $x_0 \in H, n := 0,\ 0 < \lambda \leq \lambda_k^i \leq \mu < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \gamma_k^i \in [\epsilon, \frac{1}{2}]$ *for some* $\epsilon \in (0, \frac{1}{2}]$, $k = 1, 2, \ldots$ *and* $i = 1, \ldots, N$.
Step 1. *Solve* N *strongly convex problems in parallel,* $i = 1, \ldots, N$

$$y_n^i = \arg\min\left\{\lambda_n^i f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in K_i\right\}.$$

Step 2. *Solve* N *strongly convex problems in parallel,* $i = 1, \ldots, N$

$$z_n^i = \arg\min\left\{\lambda_n^i f_i(y_n^i, y) + \frac{1}{2}||x_n - y||^2 : y \in K_i\right\}.$$

Step 3. *Determine the next approximation* x_{n+1} *as the projection of* x_0 *onto the intersection* $H_n \cap W_n$

$$x_{n+1} = P_{H_n \cap W_n}(x_0),$$

where $H_n = \cap_{i=1}^N H_n^i$ *and*

$$H_n^i = \left\{z \in H : \left\langle x_n - z_n^i, z - x_n - \gamma_n^i(z_n^i - x_n)\right\rangle \leq 0\right\},$$
$$W_n = \{z \in H : \langle x_0 - x_n, x_n - z\rangle \geq 0\}.$$

Step 4. *If* $x_{n+1} = x_n$ *then stop. Otherwise, set* $n := n + 1$ *and go back* ***Step 1.***

In order to prove the convergence of Algorithm 3.1, we need the following lemmas.

Lemma 3.2. [2, Lemma 3.1] (cf. [21, Theorem 3.2]) *Assume that* $x^* \in F$. *Let* $\left\{y_n^i\right\}, \left\{z_n^i\right\}$ *be the sequences determined as in Steps 1 and 2 of Algorithm* 3.1. *Then, there holds the relation*

$$||z_n^i - x^*||^2 \leq ||x_n - x^*||^2 - \left(1 - 2\lambda_n^i c_1\right)||y_n^i - x_n||^2 - \left(1 - 2\lambda_n^i c_2\right)||z_n^i - y_n^i||^2.$$

Lemma 3.3. *If Algorithm* 3.1 *reaches to the iteration step* n, *then* $F \subset H_n \cap W_n$ *and* x_{n+1} *is well-defined.*

Proof. By Lemma 2.1, the solution set F is closed and convex. From the definitions of $H_n^i, W_n, i = 1, \ldots, N$, we see that these sets are closed and convex. Thus, H_n is also closed and convex. We now show that $F \subset H_n \cap W_n$ for all $n \geq 0$. For each $i = 1, \ldots, N$, we put

$$C_n^i = \left\{z \in H : ||z - z_n^i|| \leq ||z - x_n||\right\}.$$

A straightforward calculation leads to

$$C_n^i = \left\{z \in H : \left\langle x_n - z_n^i, z - x_n - \frac{1}{2}(z_n^i - x_n)\right\rangle \leq 0\right\}.$$

By $\gamma_n^i \in [\epsilon, \frac{1}{2}]$, $C_n^i \subset H_n^i$ for all $i = 1, \ldots, N$. So, $C_n := \cap_{i=1}^N C_n^i \subset H_n$. From Lemma 3.2 and $0 < \lambda \leq \lambda_n^i \leq \mu < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, we obtain $||z_n^i - x^*|| \leq$

$||x_n - x^*||$ for all $x^* \in F$ and $i = 1, \ldots, N$. This implies that $F \subset C_n^i$. Therefore, $F \subset C_n$ for all $n \geq 0$. Next, we show that $F \subset C_n \cap W_n$ for all $n \geq 0$ by the induction. Indeed, we have $F \subset C_0 \cap W_0$. Assume that $F \subset C_n \cap W_n$ for some $n \geq 0$. From $x_{n+1} = P_{H_n \cap W_n}(x_0)$ and (2.2), we obtain

$$\langle x_0 - x_{n+1}, x_{n+1} - z \rangle \geq 0, \ \forall z \in H_n \cap W_n.$$

Since $F \subset C_n \cap W_n \subset H_n \cap W_n$,

$$\langle x_0 - x_{n+1}, x_{n+1} - z \rangle \geq 0, \ \forall z \in F.$$

This together with the definition of W_{n+1} implies that $F \subset W_{n+1}$, and so $F \subset C_{n+1} \cap W_{n+1}$. Thus, by the induction we obtain $F \subset C_n \cap W_n$ for all $n \geq 0$. By $C_n \subset H_n$, we get $F \subset H_n \cap W_n$ for all $n \geq 0$. Since F is nonempty, $H_n \cap W_n$ is also nonempty. Therefore, x_{n+1} is well-defined. □

Lemma 3.4. *If Algorithm 3.1 finishes at the iteration step $n < \infty$, then $x_n \in F$.*

Proof. Assume that $x_{n+1} = x_n$. Since $x_{n+1} = P_{H_n \cap W_n}(x_0)$, $x_n = x_{n+1} \in H_n$. This together with the definition of H_n implies that $\gamma_n^i ||x_n - z_n^i|| \leq 0$. From the last inequality and $\gamma_n^i \geq \epsilon > 0$, one gets $x_n = z_n^i$. By Lemma 3.2 and the hypothesis of λ_n^i, we obtain $y_n^i = x_n$. Thus

$$x_n = \arg\min \left\{ \lambda_n^i f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in K_i \right\}.$$

Thus, from [18, Proposition 2.1], one has $x_n \in EP(f_i)$ for all $i = 1, \ldots, N$, or $x_n \in F$. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Let $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ be the sequences generated by Algorithm 3.1. Then, there hold the following relations for all $i = 1, \ldots, N$*

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n^i - x_n|| = \lim_{n\to\infty} ||z_n^i - x_n|| = 0.$$

Proof. From the definition of W_n and the relation (2.2), we have $x_n = P_{W_n}(x_0)$. For each $u \in F \subset W_n$, from (2.1), one obtains

$$||x_n - x_0|| \leq ||u - x_0||. \tag{3.1}$$

Thus, the sequence $\{||x_n - x_0||\}$ is bounded, and so, from Lemma 3.2 the sequences $\{x_n\}$ and $\{z_n^i\}$ are also bounded. Moreover, the projection $x_{n+1} = P_{H_n \cap W_n}(x_0)$ implies $x_{n+1} \in W_n$. Thus, from $x_n = P_{W_n} x_0$ and (2.1), we also see that

$$||x_n - x_0|| \leq ||x_{n+1} - x_0||.$$

So, the sequence $\{||x_n - x_0||\}$ is non-decreasing. Hence, there exists the limit of the sequence $\{||x_n - x_0||\}$. By $x_{n+1} \in W_n$, $x_n = P_{W_n}(x_0)$ and the relation (2.1), we also have

$$||x_{n+1} - x_n||^2 \leq ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2 \tag{3.2}$$

Passing to the limit in the inequality (3.2) as $n \to \infty$, one gets

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.3}$$

Since $x_{n+1} \in H_n$, $x_{n+1} \in H_n^i$ for all $i = 1, \ldots, N$. From the definition of H_n^i, we have

$$\gamma_n^i ||z_n^i - x_n||^2 \le \left\langle x_n - z_n^i, x_n - x_{n+1} \right\rangle .$$

This together with the inequality $| \langle x, y \rangle | \le ||x|| ||y||$ implies that $\gamma_n^i ||z_n^i - x_n|| \le ||x_n - x_{n+1}||$. From $\gamma_n^i \ge \epsilon > 0$ and (3.3), one has

$$\lim_{n\to\infty} ||z_n^i - x_n|| = 0, \ i = 1, \ldots, N. \tag{3.4}$$

From Lemma 3.2 and the triangle inequality, we have

$$\begin{aligned} \left(1 - 2\lambda_n^i c_1\right) ||y_n^i - x_n||^2 &\le ||x_n - x^*||^2 - ||z_n^i - x^*||^2 \\ &\le (||x_n - x^*|| + ||z_n^i - x^*||)(||x_n - x^*|| - ||z_n^i - x^*||) \\ &\le (||x_n - x^*|| + ||z_n^i - x^*||)||x_n - z_n^i||. \end{aligned}$$

The last inequality together with (3.4), the hypothesis of λ_n^i and the boundedness of $\{x_n\}$, $\left\{z_n^i\right\}$ imply that

$$\lim_{n\to\infty} ||y_n^i - x_n|| = 0, \ i = 1, \ldots, N.$$

The proof Lemma 3.4 is complete. □

Theorem 3.6. *Assume that the bifunctions $f_i, i = 1, \ldots, N$ satisfy all conditions* (A1) − (A4). *In addition the solution set F is nonempty. Then, the sequences $\{x_n\}, \left\{y_n^i\right\}, \left\{z_n^i\right\}$ generated by Algorithm 3.1 converge strongly to $P_F(x_0)$.*

Proof. By Lemmas 2.1 and 3.3, we see that the sets F, H_n, W_n are closed and convex for all $n \ge 0$. Besides, by Lemma 3.5 the sequence $\{x_n\}$ is bounded. Assume that p is any weak cluster point of the sequence $\{x_n\}$. Then, there exists a subsequence of $\{x_n\}$ converging weakly to p. For the sake of simplicity, we denote this subsequence again by $\{x_n\}$ and $x_n \rightharpoonup p$ as $n \to \infty$. We now show that $p \in F$. Indeed, from the relation

$$y_n^i = \text{argmin}\{\lambda_n^i f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in K_i\}, \tag{3.5}$$

and Lemma 2.3, one gets

$$0 \in \partial_2 \left\{ \lambda_n^i f_i(x_n, y) + \frac{1}{2}||x_n - y||^2 \right\} (y_n^i) + N_{K_i}(y_n^i). \tag{3.6}$$

Thus, there exist $\bar{w} \in N_{K_i}(y_n^i)$ and $w \in \partial_2 f_i(x_n, y_n^i)$ such that

$$\lambda_n^i w + x_n - y_n^i + \bar{w} = 0. \tag{3.7}$$

From the definition of the normal cone $N_{K_i}(y_n^i)$, we have $\langle \bar{w}, y - y_n^i \rangle \leq 0$ for all $y \in K_i$. Taking into account (3.7), we obtain

$$\lambda_n^i \langle w, y - y_n^i \rangle \geq \langle y_n^i - x_n, y - y_n^i \rangle \tag{3.8}$$

for all $y \in K_i$. Since $w \in \partial_2 f_i(x_n, y_n^i)$,

$$f_i(x_n, y) - f_i(x_n, y_n^i) \geq \langle w, y - y_n^i \rangle, \forall y \in K_i. \tag{3.9}$$

Combining (3.8) and (3.9), one has

$$\lambda_n^i \left(f_i(x_n, y) - f_i(x_n, y_n^i) \right) \geq \langle y_n^i - x_n, y - y_n^i \rangle, \forall y \in K_i. \tag{3.10}$$

From $||y_n^i - x_n|| \to 0$ and $x_n \rightharpoonup p$, we also have $y_n^i \rightharpoonup p$. Passing to the limit in the inequality (3.10) as $n \to \infty$ and employing the assumption (A3) and $\lambda_n^i \geq \lambda > 0$, we conclude that $f_i(p, y) \geq 0$ for all $y \in K_i, i = 1, \ldots, N$. Hence, $p \in F$. Finally, we show that $x_n \to p$. Let $x^\dagger = P_F(x_0)$. Using the inequality (3.1) with $u = x^\dagger$, we get

$$||x_n - x_0|| \leq ||x^\dagger - x_0||.$$

By the weak lower semicontinuity of the norm $||.||$ and $x_n \rightharpoonup p$, we have

$$||p - x_0|| \leq \lim \inf_{n\to\infty} ||x_n - x_0|| \leq \lim \sup_{n\to\infty} ||x_n - x_0|| \leq ||x^\dagger - x_0||.$$

By the definition of $x^\dagger$, $p = x^\dagger$ and so $\lim_{n\to\infty} ||x_n - x_0|| = ||x^\dagger - x_0||$. Thus, $\lim_{n\to\infty} ||x_n|| = ||x^\dagger||$. By the Kadec-Klee property of the Hilbert space H, we have $x_n \to x^\dagger = P_F x_0$ as $n \to \infty$. From Lemma 3.5, one also obtains that $\{y_n^i\}, \{z_n^i\}$ converge strongly $P_F x_0$. This completes the proof of Theorem 3.6. □

Using Theorem 3.6, we get the following result was obtained in [11].

Corollary 3.7. *Let $A_i, i = 1, \ldots, N$ be L - Lipschitz continuous and pseudomonotone mappings from a real Hilbert space H into itself. In addition, the solution set $\bar{F} = \cap_{i=1}^N VI(A_i, K_i)$ is nonempty, where $VI(A_i, K_i)$ stands for the solution set of the variational inequality which is to find $x^* \in K_i$ such that $\langle A_i(x^*), y - x^* \rangle \geq 0$, $\forall y \in K_i$. Let $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ be the sequences generated by the following parallel manner*

$$\begin{cases} x_0 \in H, \\ y_n^i = P_{K_i}(x_n - \lambda_n^i A_i(x_n)), \\ z_n^i = P_{K_i}(x_n - \lambda_n^i A_i(y_n^i)), \\ H_n^i = \left\{ z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i (z_n^i - x_n) \rangle \leq 0 \right\}, \\ H_n = \cap_{i=1}^N H_n^i, \\ W_n = \{ z \in H : \langle x_0 - x_n, x_n - z \rangle \geq 0 \}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, n \geq 0, \end{cases}$$

where $0 < \lambda \leq \lambda_n^i \leq \mu < 1/L$, $0 < \epsilon \leq \gamma_n^i \leq 1/2$ for some $\epsilon \in (0, 1/2]$. Then, the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ converge strongly to $P_{\bar{F}} x_0$.

Proof. For each $i = 1, \ldots, N$, we put $f_i(x,y) = \langle A_i(x), y - x\rangle$. Since A_i is pseudomonotone, f_i is too. So the condition (A1) is satisfied for each f_i. The conditions (A3), (A4) are automatically fulfilled. We now show that f_i satisfies the condition (A2). Indeed, from the L - Lipschitz continuity of A_i, we have

$$\begin{aligned} f_i(x,y) + f_i(y,z) - f_i(x,z) &= \langle A_i(x), y - x\rangle + \langle A_i(y), z - y\rangle \\ &\quad - \langle A_i(x), z - x\rangle \\ &= \langle A_i(x), y - z\rangle + \langle A_i(y), z - y\rangle \\ &= \langle A_i(x) - A_i(y), y - z\rangle \\ &\geq -||A_i(x) - A_i(y)||||y - z|| \\ &\geq -L||x - y||||y - z|| \\ &\geq -\frac{L}{2}||x - y||^2 - \frac{L}{2}||y - z||^2. \end{aligned}$$

This implies that f_i satisfies the condition (A2) with $c_1 = c_2 = L/2$. From Algorithm 3.1 , we have

$$y_n^i = \text{argmin}\{\lambda_n^i \langle A_i(x_n), y - x_n\rangle + \frac{1}{2}||x_n - y||^2 : y \in K_i\},$$
$$z_n^i = \text{argmin}\{\lambda_n^i \left\langle A_i(y_n^i), y - y_n^i\right\rangle + \frac{1}{2}||x_n - y||^2 : y \in K_i\}.$$

A straightforward computation yields

$$y_n^i = \text{argmin}\{\frac{1}{2}||y - (x_n - \lambda_n^i A_i(x_n))||^2 : y \in K_i\} = P_{K_i}(x_n - \lambda_n^i A_i(x_n)),$$
$$z_n^i = \text{argmin}\{\frac{1}{2}||y - (x_n - \lambda_n^i A_i(y_n^i))||^2 : y \in K_i\} = P_{K_i}(x_n - \lambda_n^i A_i(y_n^i)).$$

Applying Theorem 3.6 to Corollary 3.7, we come to the desired result. □

Remark 3.8. In Corollary 3.7, we need only the pseudomonotonicity of the mappings $A_i, i = 1, \ldots, N$ to obtain the convergence of the iteration sequences. However, in order to get the same result, Censor et al [11] required the monotonicity of these mappings which is more strict than the pseudomonotonicity.

In Algorithm 3.1, at the n^{th} step, in order to determine the next approximation x_{n+1} we have to construct $N+1$ subsets $H_n^i, i = 1, \ldots, N$ and W_n and solve the following optimization problem on the intersection of $N+1$ closed convex sets

$$\begin{cases} \min ||z - x_0||^2, \\ \text{such that} \quad z \in H_n^1 \cap \ldots \cap H_n^N \cap W_n. \end{cases}$$

This seems very costly when the number of subproblems N is large. Thus, Algorithm 3.1 can not develop practical numerical experiments. To overcome the complexity of this algorithm, we next propose the following cyclic algorithm for solving the CSEP for pseudomonotone bifunctions $f_i, i = 1, \ldots, N$. We write $[n] = n(mod\ N) + 1$ to stand for the mod function taking values in $\{1, 2, \ldots, N\}$.

Algorithm 3.9. *(The cyclic hybrid extragradient-cutting algorithm)*
Initialize. $x_0 \in H$, *n:=0,* $0 < \lambda \le \lambda_k \le \mu < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}, \gamma_k \in [\epsilon, \frac{1}{2}]$ *for some* $\epsilon \in (0, \frac{1}{2}]$ *and* $k = 1, 2, \ldots$.
Step 1. *Solve the strongly convex problem*

$$y_n = \arg\min\left\{\lambda_n f_{[n]}(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in K_{[n]}\right\}.$$

Step 2. *Solve the strongly convex problem*

$$z_n = \arg\min\left\{\lambda_n f_{[n]}(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in K_{[n]}\right\}.$$

Step 3. *Determine the next approximation* x_{n+1} *as the projection of* x_0 *onto* $H_n \cap W_n$

$$x_{n+1} = P_{H_n \cap W_n}(x_0),$$

where

$$H_n = \{z \in H : \langle x_n - z_n, z - x_n - \gamma_n(z_n - x_n)\rangle \le 0\},$$
$$W_n = \{z \in H : \langle x_0 - x_n, x_n - z\rangle \ge 0\}.$$

Step 4. *Set* $n := n + 1$ *and go back* ***Step 1.***

Using the same technique as in [23, Algorithm 1], we can find the explicit formula of the projection x_{n+1} of x_0 onto the intersection of two subsets H_n and W_n in Step 3 of Algorithm 3.9. Indeed, from the definitions of H_n and W_n, we see that they are either halfspaces or H. Let $v_n = x_n + \gamma_n(z_n - x_n)$, we rewrite the set H_n as follows

$$H_n = \{z \in H : \langle x_n - z_n, z - v_n\rangle \le 0\}.$$

By analyzing similarly as in [23, Algorithm 1], we get the explicit formula of the projection x_{n+1} of x_0 onto $H_n \cap W_n$

$$x_{n+1} := P_{H_n} x_0 = \begin{cases} x_0 & \text{if} \quad z_n = x_n, \\ x_0 - \frac{\langle x_n - z_n, x_0 - v_n\rangle}{||x_n - z_n||^2}(x_n - z_n) & \text{if} \quad z_n \ne x_n \end{cases}$$

if $P_{H_n} x_0 \in W_n$. Otherwise,

$$x_{n+1} = x_0 + t_1(x_n - z_n) + t_2(x_0 - x_n),$$

where t_1, t_2 are solutions of the system of linear equations with two unknowns

$$\begin{cases} t_1||x_n - z_n||^2 + t_2\langle x_n - z_n, x_0 - x_n\rangle = -\langle x_0 - v_n, x_n - z_n\rangle, \\ t_1\langle x_n - z_n, x_0 - x_n\rangle + t_2||x_0 - x_n||^2 = -||x_0 - x_n||^2. \end{cases}$$

Theorem 3.10. *Assume that the bifunctions* $f_i, i = 1, \dots, N$ *satisfy all conditions* (A1) − (A4)*. In addition, the solution set* F *is nonempty. Then, the sequences* $\{x_n\}, \{y_n\}, \{z_n\}$ *generated by Algorithm* 3.9 *converge strongly to* $P_F(x_0)$*.*

Proof. By the same arguments as in the proof of Lemmas 3.2 − 3.5, we see that F, H_n, W_n are closed and convex, and $F \subset H_n \cap W_n$ for all $n \geq 0$. Besides, the sequence $\{x_n\}$ is bounded and there hold the relations

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Assume that p is any weak cluster point of the sequence $\{x_n\}$. For each fixed index $i \in \{1, 2, \dots, N\}$, since the set of indexes i is finite, by [7, Theorem 5.3] there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p$ as $j \to \infty$ and $[n_j] = i$ for all j. By the same arguments as in (3.5) − (3.10), we conclude that $p \in EP(f_i)$. This is true for all $i = 1, \dots, N$. Thus, $p \in F$. The rest of the proof of Theorem 3.10 is the same to that of of Theorem 3.6. □

Corollary 3.11. *Let* $A_i, i = 1, \dots, N$ *be* L *- Lipschitz continuous and pseudomonotone mappings from a real Hilbert space* H *to itself. In addition, the solution set* $\bar{F} = \cap_{i=1}^N VI(A_i, K_i)$ *is nonempty, where* $VI(A_i, K_i)$ *is defined as in Corollary* 3.7*. Let* $\{x_n\}, \{y_n\}, \{z_n\}$ *be the sequences generated by the following cyclic manner*

$$\begin{cases} x_0 \in H, \\ y_n = P_{K_{[n]}}(x_n - \lambda_n A_{[n]}(x_n)), \\ z_n = P_{K_{[n]}}(x_n - \lambda_n A_{[n]}(y_n)), \\ H_n = \{z \in H : \langle x_n - z_n, z - x_n - \gamma_n(z_n - x_n)\rangle \leq 0\}, \\ W_n = \{z \in H : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \end{cases}$$

where $0 < \lambda \leq \lambda_n \leq \mu < 1/L$, $0 < \epsilon \leq \gamma_n \leq 1/2$ *for some* $\epsilon \in (0, 1/2]$*. Then, the sequences* $\{x_n\}, \{y_n\}, \{z_n\}$ *converge strongly to* $P_{\bar{F}} x_0$*.*

Proof. Using Theorem 3.10 and arguing similarly as in the proof of Corollary 3.7, we lead to the desired conclusion. □

Remark 3.12. Corollaries 3.7 and 3.11 with $N = 1$ give us the corresponding result of Nadezhkina and Takahashi in [20, Theorem 4.1].

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