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Author(s):

J. Quan, Z. Y. Wu and G. Q. Li

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SUFFICIENT GLOBAL OPTIMALITY CONDITIONS FOR GENERAL MIXED INTEGER NONLINEAR PROGRAMMING PROBLEMS

J. QUAN*, Z. Y. WU AND G. Q. LI

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ABSTRACT. In this paper, some KKT type sufficient global optimality conditions for general mixed integer nonlinear programming problems with equality and inequality constraints (MINPP) are established. We achieve this by employing a Lagrange function for MINPP. In addition, verifiable sufficient global optimality conditions for general mixed integer quadratic programming problems are derived easily. Numerical examples are also presented.
Keywords: Sufficient global optimality conditions, mixed integer nonlinear programming, mixed integer quadratic programming.
MSC(2010): Primary:41A65, 41A29; Secondary:90C30.

1. Introduction

Consider the following general mixed integer nonlinear programming problem:

$$MINPP: \qquad \begin{array}{l} \min_{x\in R^n} f(x) \\ s.t. \begin{cases} g_i(x) \le 0, i \in I = \{1, 2, \cdots, m\} \\ h_e(x) = 0, e \in E = \{m+1, m+2, \cdots, m+p\} \\ x_l \in [u_l, v_l], l \in M, \\ x_j \in \{p_j, p_j+1, \dots, q_j\}, j \in N \end{cases} \end{array}$$

where $M \cap N = \emptyset, M \cup N = \{1, \dots, n\}$, $u_l, v_l \in R$ and $u_l < v_l$ for any $l \in M$, p_j, q_j are integers and $p_j < q_j$ for all $j \in N$, f, g_i, h_e are twice continuously differentiable functions on an open subset of R^n containing $\prod_{l\in M}[u_l, v_l] \prod_{j\in N}[p_j, q_j]$.

The mixed integer nonlinear programming problems MINPP are applied to a very wide range of areas, such as engineering design, computational biology,

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*Corresponding author.

reliability networks, facility planning and scheduling, combinatorial optimization problems etc. For more information, the interested reader may refer to [6, 7, 9, 10, 11, 19, 24].

It is very difficult to solve the mixed integer nonlinear programming problems due to the nonlinear property and mixed variable of the objective function. Most approaches to solve the mixed integer nonlinear programming problems are branch-and-bound, decomposition and outer approximation method, which can be found in [2, 16, 8, 4, 5, 3]. Especially Tawarmalani and Sahinidis [17] adopted nonlinear convex relaxations via a polyhedral branch-and-cut approach to solve mixed integer nonlinear programming problems. Ruth and Floudas [18] have presented ANTIGONE, algorithms for continuous/integer global optimization of nonlinear equations, a general mixed integer nonlinear global optimization framework. In recent years the global optimality conditions become the focus of many researches.

Global optimality conditions of many special cases of MINPP have been established by many authors. Some global optimality conditions characterizing global minimizer of quadratic minimization problem have been discussed in [1, 12, 13]. Wu [20] presented sufficient global optimality conditions for weakly convex minimization problems by using abstract convex analysis theory. Sufficient conditions for the global optimality of bivalent nonconvex quadratic programs involving quadratic inequality constraints as well as equality constraints were presented in [21] by employing the Lagrange function. Wu and Bai [22] presented some global optimality conditions for mixed quadratic programming problems without constraints, their approach is based on a L-subdifferential and an associated L-normal cone.

Jeyakumar et al. presented global optimality necessary conditions for polynomial problems with box or bivalent constraints using separable polynomial relaxations in [14]. Jeykumar, Srisatkunrajah and Huy in[15] have presented some new Kuhn-Tucker sufficiency global optimality conditions for multi extremal smooth nonlinear programming problems with equality and inequality constraints. They established Kuhn-Tucker sufficiency criteria for global optimality in terms of the Lagrangian of nonlinear programming problem. Wu et al. [23] have established some global optimality conditions for quartic polynomial optimization.

In this paper, we establish some sufficient global optimality conditions for general mixed integer nonlinear programming problems with equality and inequality constraints by employing a Lagrange function, then derive easily verifiable sufficient global optimality conditions for general mixed integer quadratic programming problems with equality and inequality constraints. We also give some numerical examples to show the significance of sufficient global optimality conditions.

2. Sufficient global optimality conditions for MINPP

In this section, we will derive some sufficient global optimality conditions for MINPP at a feasible point $\bar{x}$. We first present some notations and preliminaries that will be used later in the paper. The real line is denoted by R and the n-dimensional Euclidean space is denoted by R^n. For vectors $x, y \in R^n$, $x \geq y$ means that $x_i \geq y_i$, for $i = 1, \ldots, n$. The notation $A \succeq B$ means $A - B$ is a positive semidefinite matrix and $A \preceq 0$ means $-A \succeq 0$. A diagonal matrix with diagonal elements $\alpha_1, \ldots, \alpha_n$ is denoted by $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. M^n is the set of all symmetric $n \times n$ matrices.

For MINPP given in the introduction, we let $U = \{x = (x_1, \ldots, x_n)^T \mid x_l \in [u_l, v_l], l \in M, x_j \in \{p_j, p_j + 1, \cdots, q_j\}, j \in N\}$; $D = \{x \in R^n | g_i(x) \leq 0, i \in I, h_e(x) = 0, e \in E\}$; $S = U \cap D$. For given $\lambda = (\lambda_1, \cdots, \lambda_m)^T \in R^m_+$ and $\mu = (\mu_1, \cdots, \mu_p)^T \in R^p$, let

$$L(x, \lambda, \mu) := f(x) + \sum_{i \in I} \lambda_i g_i(x) + \sum_{e \in E} \mu_e h_e(x).$$

Let $\bar{x} \in S$, for any $l \in M, j \in N$,

$$\widetilde{\widetilde{x}}_l : \; = \; \begin{cases} -1, & \text{if } \bar{x}_l = u_l \\ 1, & \text{if } \bar{x}_l = v_l \\ \mathrm{sign}(\nabla L(\bar{x}, \lambda, \mu))_l, & \text{if } u_l < \bar{x}_l < v_l \end{cases}, \tag{2.1}$$

$$\widetilde{\widetilde{x}}_j : \; = \; \begin{cases} -1, & \text{if } \bar{x}_j = p_j \\ 1, & \text{if } \bar{x}_j = q_j \\ \mathrm{sign}(\nabla L(\bar{x}, \lambda, \mu))_j, & \text{if } p_j < \bar{x}_j < q_j \end{cases}, \tag{2.2}$$

$$b_{\bar{x}_l} : \; = \; \frac{\widetilde{\widetilde{x}}_l (\nabla L(\bar{x}, \lambda, \mu))_l}{v_l - u_l} \tag{2.3}$$

$$b_{\bar{x}_j} : \; = \; \max\{\frac{\widetilde{\widetilde{x}}_j (\nabla L(\bar{x}, \lambda, \mu))_j}{1}, \frac{\widetilde{\widetilde{x}}_j (\nabla L(\bar{x}, \lambda, \mu))_j}{q_j - p_j}\}, \tag{2.4}$$

$$b_{\bar{x}} : \; = \; (b_{\bar{x}_1}, \ldots, b_{\bar{x}_n})^T, \tag{2.5}$$

where $\mathrm{sign}(\nabla L(\bar{x}, \lambda, \mu))_k = \begin{cases} -1, & (\nabla L(\bar{x}, \lambda, \mu))_k < 0 \\ 0, & (\nabla L(\bar{x}, \lambda, \mu))_k = 0 \\ 1, & (\nabla L(\bar{x}, \lambda, \mu))_k > 0, \end{cases}$ $\quad k = 1, 2, \cdots, n$. For $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^T \in S$, let

$$\begin{aligned} S_{\bar{x}} := \{x = (x_1, \ldots, x_n)^T \mid g_i(x) \leq 0, h_e(x) = 0, u_l \leq x_l \leq v_l, \\ x_j = \bar{x}_j, \forall i \in I, e \in E, l \in M, j \in N\}. \end{aligned} \tag{2.6}$$

If $\bar{x}$ is a local minimizer of MINPP, then $\bar{x}$ must be a local minimizer of $f(x)$ on $S_{\bar{x}}$. Moreover, if a certain constraint qualification holds then the following KKT conditions holds:

$$(\exists \lambda \in R_+^m, \mu \in R^p), \sum_{i\in I} \lambda_i g_i(\bar{x}) = 0, \text{and}$$

$$(\nabla L(\bar{x},\lambda,\mu))^T(x-\bar{x}) \geq 0, \forall x \in \prod_{l\in M}[u_l,v_l]\prod_{j\in N}\{\bar{x}_j\}. \tag{2.7}$$

The condition (2.7) can equivalently be written as

$$(\exists \lambda \in R_+^m, \mu \in R^p), \sum_{i\in I} \lambda_i g_i(\bar{x}) = 0, \text{and } \tilde{\tilde{x}}_l(\nabla L(\bar{x},\lambda,\mu))_l \leq 0, \forall l \in M. \tag{2.8}$$

Here we call condition (2.8) as the KKT local necessary conditions for problem MINPP. In the following, we will discuss some KKT sufficient global optimality conditions for problem MINPP. Let $Q = diag(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a diagonal matrix in M^n. For MINPP, define a quadratic function $h := R^n \to R$ by

$$h(x) := \frac{1}{2}x^TQx + (\nabla L(\bar{x},\lambda,\mu) - Q\bar{x})^T x,$$

where $\lambda \in R_+^m, \mu \in R^p$. Let $\tilde{Q} = diag(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n)$, where $\tilde{\alpha}_i = \min\{0, \alpha_i\}$ for $i \in M; \tilde{\alpha}_i = \alpha_i$ for $i \in N$ and let

$$\bar{U} := \prod_{l\in M}[u_l,v_l]\prod_{j\in N}[p_j,q_j]. \tag{2.9}$$

Now we derive sufficient conditions of global optimality for MINPP whenever all functions are twice continuously differentiable functions on an open subset of R^n containing $\bar{U}$.

Theorem 2.1. *(***Global Sufficient Conditions for MINPP***) Let $\bar{x} \in S$, suppose that there exist $\lambda = (\lambda_1, \cdots, \lambda_m)^T \in R_+^m$, $\mu = (\mu_1, \cdots, \mu_p)^T \in R^p$ and a diagonal matrix $Q = diag(\alpha_1, \alpha_2, \cdots, \alpha_n)$, such that*

$$[SC1] \quad \begin{cases} \sum_{i\in I} \lambda_i g_i(\bar{x}) = 0 \\ \text{diag}(b_{\bar{x}}) \preceq \frac{1}{2}\tilde{Q} \\ \nabla^2 L(x,\lambda,\mu) - Q \succeq 0, \forall x \in \bar{U} \end{cases}$$

then $\bar{x}$ is a global minimizer of problem MINPP. Moreover if for each $x \in \bar{U}$, $\nabla^2 L(x,\lambda,\mu) - Q \succ 0$, then $\bar{x}$ is unique.

Proof. Since for each $x \in \bar{U}$, $\nabla^2 L(x,\lambda,\mu) - Q \succeq 0$, we have that $\phi(x) := L(x,\lambda,\mu) - h(x)$ is convex on $\bar{U}$. It is easy to see that $\nabla\phi(\bar{x}) = 0$. So $\bar{x}$ is a minimizer of convex function $\phi(x)$ on convex set $\bar{U}$, i.e., $\phi(x) - \phi(\bar{x}) \geq 0, \forall x \in \bar{U}$. Thus

$$L(x,\lambda,\mu) - L(\bar{x},\lambda,\mu) \geq h(x) - h(\bar{x}), \forall x \in \bar{U}. \tag{2.10}$$

As $\sum_{i\in I} \lambda_i g_i(x) \leq 0, \forall x \in D$, $\sum_{i\in I} \lambda_i g_i(\bar{x}) = 0$, we have

$$\begin{aligned} f(x)-f(\bar{x}) &\geq f(x)+\sum_{i\in I}\lambda_i g_i(x)+\sum_{e\in E}\mu_e h_e(x)-f(\bar{x}) \\ &= f(x)+\sum_{i\in I}\lambda_i g_i(x)+\sum_{e\in E}\mu_e h_e(x)-(f(\bar{x})+\sum_{i\in I}\lambda_i g_i(\bar{x})+\sum_{e\in E}\mu_e h_e(\bar{x})) \\ &= L(x,\lambda,\mu)-L(\bar{x},\lambda,\mu), \forall x\in D. \end{aligned}$$

By (2.10), we have

$$f(x)-f(\bar{x})\geq h(x)-h(\bar{x}), \forall x\in S,$$

where

$$(2.11)\; h(x)-h(\bar{x})=\sum_{k=1}^{n}[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla L(\bar{x},\lambda,\mu))_k(x_k-\bar{x}_k)].$$

In the following, we prove

$$(2.12)\sum_{k=1}^{n}[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla L(\bar{x},\lambda,\mu))_k(x_k-\bar{x}_k)]\geq 0, \text{ for any } x\in U.$$

if and only if for any $k=1,\ldots,n$,

$$(2.13)\quad \frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla L(\bar{x},\lambda,\mu))_k(x_k-\bar{x}_k)\geq 0, \text{ for any } x\in U.$$

In fact, if there exist an $l_0\in M$ and a $y_{l_0}\in[u_l,v_l]$ such that

$$\frac{1}{2}\alpha_{l_0}(y_{l_0}-\bar{x}_{l_0})^2+(\nabla L(\bar{x},\lambda,\mu))_{l_0}(y_{l_0}-\bar{x}_{l_0})<0,$$

then let $x_l=y_{l_0}$ when $l=l_0,\ l\in M$ and $x_l=\bar{x}_l$ when $l\neq l_0,\ l\in M$, let $x_j=\bar{x}_j$ for all $j\in N$, then $x=(x_1,\ldots,x_n)^T\in U$ and we have that

$$\begin{aligned} &\sum_{k=1}^{n}[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla L(\bar{x},\lambda,\mu))_k(x_k-\bar{x}_k)] \\ =&\frac{1}{2}\alpha_{l_0}(y_{l_0}-\bar{x}_{l_0})^2+(\nabla L(\bar{x},\lambda,\mu))_{l_0}(y_{l_0}-\bar{x}_{l_0})<0, \end{aligned}$$

which contradicts (2.12). If there exist a $j_0\in N$ and a $y_{j_0}\in\{p_j,\cdots,q_j\}$ such that

$$\frac{1}{2}\alpha_{j_0}(y_{j_0}-\bar{x}_{j_0})^2+(\nabla L(\bar{x},\lambda,\mu))_{j_0}(y_{j_0}-\bar{x}_{j_0})<0,$$

then let $x_l=\bar{x}_l$ for all $l\in M$, and $x_j=y_{j_0}$ when $j=j_0,\ j\in N$, $x_j=\bar{x}_j$ when $j\neq j_0,\ j\in N$, then $x=(x_1,\ldots,x_n)^T\in U$ and we have that

$$\begin{aligned} &\sum_{k=1}^{n}[\frac{1}{2}\alpha_k(x_k-\bar{x}_k)^2+(\nabla L(\bar{x},\lambda,\mu))_k(x_k-\bar{x}_k)] \\ =&\frac{1}{2}\alpha_{j_0}(y_{j_0}-\bar{x}_{j_0})^2+(\nabla L(\bar{x},\lambda,\mu))_{j_0}(y_{j_0}-\bar{x}_{j_0})<0, \end{aligned}$$

which contradicts (2.12).

Then, we verify that (2.13) is equivalent to condition $\mathrm{diag}(b_{\bar{x}}) \preceq \frac{1}{2}\tilde{Q}$. We consider the following cases:

Case 1. If $\bar{x}_l = u_l$, then (2.13) is equivalent to

$$\frac{1}{2}\alpha_l(x_l - \bar{x}_l) + (\nabla L(\bar{x}, \lambda, \mu))_l \geq 0, \text{ for any } x_l \in (u_l, v_l].$$

If $\alpha_l \geq 0$, then (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_l \geq 0$; If $\alpha_l < 0$, then (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_l \geq -\frac{(v_l - u_l)\alpha_l}{2}$. So when $\bar{x}_l = u_l$, (2.13) is equivalent to

$$\tilde{\bar{x}}_l(\nabla L(\bar{x}, \lambda, \mu))_l \leq \min\{0, \frac{(v_l - u_l)\alpha_l}{2}\}.$$

Case 2. If $\bar{x}_l = v_l$, then (2.13) is equivalent to

$$\frac{1}{2}\alpha_l(x_l - \bar{x}_l) + (\nabla L(\bar{x}, \lambda, \mu))_l \leq 0, \text{ for any } x_l \in [u_l, v_l).$$

If $\alpha_l \geq 0$, then (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_l \leq 0$; If $\alpha_l < 0$, then (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_l \leq \frac{(v_l - u_l)\alpha_i}{2}$. So when $\bar{x}_l = v_l$, (2.13) is equivalent to

$$\tilde{\bar{x}}_l(\nabla L(\bar{x}, \lambda, \mu))_l \leq \min\{0, \frac{(v_l - u_l)\alpha_l}{2}\}.$$

Case 3. If $u_l < \bar{x}_l < v_l$, when $x_l \in [u_l, \bar{x}_l)$, (2.13) is equivalent to

$$\frac{1}{2}\alpha_l(x_l - \bar{x}_l) + (\nabla L(\bar{x}, \lambda, \mu))_l \leq 0,$$

when $x_l \in (\bar{x}_l, v_l]$, (2.13) is equivalent to

$$\frac{1}{2}\alpha_l(x_l - \bar{x}_l) + (\nabla L(\bar{x}, \lambda, \mu))_l \geq 0.$$

So if $u_l < \bar{x}_l < v_l$, (2.13) is equivalent to

$$(\nabla L(\bar{x}, \lambda, \mu))_l = 0, \ \alpha_l \geq 0,$$

and also is equivalent to

$$\tilde{\bar{x}}_l(\nabla L(\bar{x}, \lambda, \mu))_l \leq \min\{0, \frac{(v_l - u_l)\alpha_l}{2}\}.$$

Case 4. If $\bar{x}_j = p_j$, then (2.13) is equivalent to

$$\frac{1}{2}\alpha_j(x_j - \bar{x}_j) + (\nabla L(\bar{x}, \lambda, \mu))_j \geq 0, \text{ for any } x_j \in \{p_j + 1, p_j + 2, \cdots, q_j\}.$$

If $\alpha_j \geq 0$, (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_j \geq -\frac{\alpha_j}{2}$; if $\alpha_j < 0$, (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_j \geq -\frac{(q_j - p_j)\alpha_j}{2}$. So if $\bar{x}_j = p_j$, (2.13) is equivalent to

$$\tilde{\bar{x}}_j(\nabla L(\bar{x}, \lambda, \mu))_j \leq \min\{\frac{\alpha_j}{2}, \frac{(q_j - p_j)\alpha_j}{2}\}.$$

Case 5. If $\bar{x}_j = q_j$, then (2.13) is equivalent to

$$\frac{1}{2}\alpha_j(x_j - \bar{x}_j) + (\nabla L(\bar{x}, \lambda, \mu))_j \leq 0, \text{ for any } x_j \in \{p_j, p_j + 1, \cdots, q_j - 1\}.$$

If $\alpha_j \geq 0$, (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_j \leq \frac{\alpha_j}{2}$; if $\alpha_j < 0$, (2.13) is equivalent to $(\nabla L(\bar{x}, \lambda, \mu))_j \leq \frac{(q_j - p_j)\alpha_j}{2}$. So if $\bar{x}_j = q_j$, (2.13) is equivalent to

$$\tilde{\bar{x}}_j(\nabla L(\bar{x}, \lambda, \mu))_j \leq \min\{\frac{\alpha_j}{2}, \frac{(q_j - p_j)\alpha_j}{2}\}.$$

Case 6. If $\bar{x}_j \in \{p_j + 1, \cdots, q_j - 1\}$, when $x_j \in \{p_j, \cdots, \bar{x}_j - 1\}$, (2.13) is equivalent to $\alpha_j(x_j - \bar{x}_j) + (\nabla L(\bar{x}, \lambda, \mu))_j \leq 0$; when $x_j \in \{\bar{x}_j + 1, \cdots, q_j\}$, (2.13) is equivalent to $\frac{1}{2}\alpha_j(x_j - \bar{x}_j) + (\nabla L(\bar{x}, \lambda, \mu))_j \geq 0$. So (2.13) is equivalent to

$$-\frac{\alpha_j}{2} \leq (\nabla L(\bar{x}, \lambda, \mu))_j \leq \frac{\alpha_j}{2}, \alpha_j \geq 0,$$

and also is equivalent to

$$\tilde{\bar{x}}_j(\nabla L(\bar{x}, \lambda, \mu))_j \leq \min\{\frac{\alpha_j}{2}, \frac{(q_j - p_j)\alpha_j}{2}\}.$$

By the above discussion, we know that $[SC1]$ implies that $f(x) - f(\bar{x}) \geq 0$ for any $x \in S$, i.e., $\bar{x}$ is a global minimizer of problem MINPP.

Moreover if for each $x \in S, \nabla L(x, \lambda, \mu) - Q \succ 0$, then $\phi(x)$ is strictly convex over $\bar{U}$, $\phi(x) - \phi(\bar{x}) > 0, \forall x \in \bar{U} \setminus \{\bar{x}\}$, hence $\bar{x}$ is unique. □

Remark 2.2. Note that if $N = \emptyset$, then our preceding condition $[SC1]$ reduces to the sufficient condition given by Theorem 2.1 in [15], which is the special case of Theorem 2.1 when $N = \emptyset$.

Below, we show that the KKT global sufficient conditions for MINPP.

Theorem 2.3. *(***KKT Global Sufficient Conditions for MINPP***) Let $\bar{x} \in S$, suppose that there exist $\lambda = (\lambda_1, \cdots, \lambda_m)^T \in R^m_+$ and $\mu = (\mu_1, \cdots, \mu_p)^T \in R^p$ such that*

$$[SC2] \quad \begin{cases} \sum_{i \in I} \lambda_i g_i(\bar{x}) = 0 \\ \tilde{\bar{x}}_l(\nabla L(\bar{x}, \lambda, \mu))_l \leq 0, l \in M \\ \operatorname{diag}(b_{\bar{x}}) \preceq \frac{\nabla^2 L(x, \lambda, \mu)}{2}, \forall x \in \bar{U} \end{cases},$$

then $\bar{x}$ is a global minimizer of problem MINPP. Moreover if

$$\operatorname{diag}(b_{\bar{x}}) \prec \frac{\nabla^2 L(x, \lambda, \mu)}{2}, \forall x \in \bar{U},$$

then $\bar{x}$ is unique.

Proof. Let $\alpha_k = 2b_{\bar{x}_k}, k = 1, 2, \cdots, n$. Since $\tilde{\tilde{x}}_l(\nabla L(\bar{x}, \lambda, \mu))_l \leq 0, l \in M$, we know that $b_{\bar{x}_l} \leq 0, \forall l \in M$. Hence, $Q = \tilde{Q}$. Therefore, $[SC2]$ implies that

$$\begin{cases} \sum_{i\in I} \lambda_i g_i(\bar{x}) = 0 \\ \mathrm{diag}(b_{\bar{x}}) = \frac{1}{2}\tilde{Q} \\ Q \preceq \nabla^2 L(x, \lambda, \mu), \forall x \in \bar{U}. \end{cases} \tag{2.14}$$

By Theorem 2.1, we know that $\bar{x}$ is a global minimizer of problem MINPP. □

Remark 2.4. Note that if $N = \emptyset$, then condition $[SC2]$ is just Theorem 2.2 given in [15], which is the special case of Theorem 2.3 when $N = \emptyset$.

We can easily get the following corollaries.

Corollary 2.5. *Let $\bar{x} \in S$, $M = \emptyset$. If there exist $\lambda = (\lambda_1, \cdots, \lambda_m)^T \in R^m_+$ and $\mu = (\mu_1, \cdots, \mu_p)^T \in R^p$ such that*

$$[SC3] \quad \begin{cases} \sum_{i\in I} \lambda_i g_i(\bar{x}) = 0 \\ \mathrm{diag}(b_{\bar{x}}) \preceq \frac{1}{2}\nabla^2 L(x, \lambda, \mu), \forall x \in \bar{U} \end{cases},$$

then $\bar{x}$ is a global minimizer of problem MINPP.

Proof. It can be obtained directly from Theorem 2.3. □

Corollary 2.6. *Let $\bar{x} \in S$. If*

$$[SC4] \quad \begin{cases} \tilde{\tilde{x}}_l(\nabla f(\bar{x}))_l \leq 0, \forall l \in M \\ \mathrm{diag}(b_{\bar{x}}) \preceq \frac{\nabla^2 f(x)}{2}, \forall x \in \bar{U} \end{cases},$$

then $\bar{x}$ is a global minimizer of problem MINPP over U.

Remark 2.7. Note that Corollary 2.1 in [15] is the special case of Corollary 2.6 when $N = \emptyset$.

We now provide a simple mixed integer nonlinear programming example with equality and inequality constraints where the KKT global sufficient conditions can be verified numerically, and it may be used to eliminate local minimizers that are global.

Example 2.8. Consider the following minimization problem:

$$\begin{array}{lll} (EP1) & \min & f(x) := -{x_1}^2 + 6{x_2}^2 - x_1 - {x_2}^3 \\ & s.t. & \begin{cases} x_1 + x_2 - 2 \leq 0 \\ x_1 x_2^2 = 0 \\ x_1 \in [-1, 1] \\ x_2 \in \{-1, 0, 1\}. \end{cases} \end{array}$$

Let $\bar{x} = (\bar{x}_1, \bar{x}_2)^T \in S$, where $\bar{x}_1 \in \{1, -1\}$ and $\bar{x}_2 = 0$. Then we can easily verify that the KKT local necessary conditions for problem $(EP1)$: $\lambda_1(\bar{x}_1 + \bar{x}_2 - 2) = 0$, $\mu_1(\bar{x}_1\bar{x}_2^2) = 0$, $\lambda_1 \geq 0$, $\mu_1 \in R$ and $\tilde{\tilde{x}}_1(-1 - 2\bar{x}_1 + \lambda_1 + \mu_1\bar{x}_2^2) \leq 0$ hold at $\bar{x}$ for $\lambda_1 = 0$ and $\mu_1 = -1$.

Moreover we can check that the KKT global sufficient conditions $[SC1]$ hold at $\bar{x} = (1,0)$ since $b_{\bar{x}_1} = \frac{-3}{2}$, $b_{\bar{x}_2} = 0$ and $\frac{\nabla^2 L(x,\lambda,\mu)}{2} - \mathrm{diag}(b_{\bar{x}}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3-3x_2 \end{pmatrix} \succeq 0, \forall x_2 \in [-1,1]$. Hence, $\bar{x} = (1,0)$ is a global minimizer of problem $(EP1)$. But condition $[SC1]$ does not hold at $(-1,0)$.

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(Jing Quan) Department of Mathematics, Yibin University, Yibin, Sichuan, 644007, China.

E-mail address: quanjingcq@163.com

(Zhiyou Wu) School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China.

E-mail address: zhiyouwu@263.net

(Guoquan Li) School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China.

E-mail address: ligq@cqnu.edu.cn