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MAXIMAL ELEMENTS OF SUB-TOPICAL FUNCTIONS WITH APPLICATIONS TO GLOBAL OPTIMIZATION

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Abstract. We study the support sets of sub-topical functions and investigate their maximal elements in order to establish a necessary and sufficient condition for the global minimum of the difference of two sub-topical functions.
Keywords: Global optimization, abstract convexity, sub-topical functions, Toland-Singer formula, support set, subdifferential.
MSC(2010): Primary: 90C46; Secondary: 26A48.

1. Introduction

Abstract convexity extends the main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets. It is well known that every convex, proper and lower semicontinuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is replaced by an alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of H lead to different classes of envelope functions, which are applied to global optimization problems (see [7–9, 11–13]). Moreover, if a family of functions is abstract convex with respect to a specific choice of H, then we can use some key ideas of convex analysis in order to gain new insights on these functions. On the other hand, some classical concepts such as subdifferential, support set and conjugation theory can be generalized to abstract convex scheme.
Characterizing the maximal elements of a support set plays a crucial role in order to minimize a DAC function (difference of two abstract convex functions); for instance, see [2, 4, 9].

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Recall that a function $p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be sub-topical (see also Definition 2.3) if this function is increasing ($x \geq y \implies p(x) \geq p(y)$) and plus-sub-homogeneous ($p(x + \lambda\mathbf{1}) \leq p(x) + \lambda\mathbf{1}$, for all $x \in \mathbb{R}^n$ and all $\lambda \geq 0$), where $\mathbf{1}$ is the vector of appropriate dimension with all coordinates equal to one. These functions are studied and extended in [1,3,10,14] and have found many applications in various parts of applied mathematics (see, for instance, [5,6]).
Here, we study the problem

$$\min_{x \in X} f(x) := p(x) - q(x),$$

where p and q are sub-topical functions and X is an ordered real Banach space. We establish a necessary and sufficient condition for global minimum of f. We also outline a dual approach to study the global optimization problem for this function. These observations extend the results of [2]. Our approach is based on the Toland-Singer formula ([15]) and abstract convexity.
The layout of the paper is as follows. In Section 2, we give definitions, notations and preliminary results used throughout the paper. In Section 3, we examine and identify the maximal elements of support sets of sub-topical functions. Finally, necessary and sufficient conditions for global minimum of the difference of two sub-topical functions are obtained.

2. Preliminaries

Let X be a Banach space with the norm $\|.\|$ and let C be a closed convex cone in X such that $C \cap (-C) = \{0\}$ and $\operatorname{int} C \neq \emptyset$. We assume that X is equipped with the order relation $\geq$ generated by $C : x \geq y$ (or $y \leq x$) if and only if $x - y \in C$ $(x,\ y \in X)$. Also, we say $x < y$ if $y - x \in \operatorname{int}\, C$. Moreover, we assume that C is a normal cone. Recall that a cone C is called *normal* if there exists a constant $m > 0$ such that $\|x\| \leq m\|y\|$, whenever $0 \leq x \leq y$, and $x,\ y \in X$. Let $\mathbf{1} \in \operatorname{int} C$ and let

$$B = \{x \in X : -\mathbf{1} \leq x \leq \mathbf{1}\}.$$

It is well known and easy to check that B can be considered as the unit ball of a certain norm $\|.\|_1$, which is equivalent to the initial norm $\|.\|$. We assume in the sequel, without loss of generality, that $\|.\| = \|.\|_1$.
Some definitions in abstract convex framework are given next.

Definition 2.1. Let Z be a non-empty subset of X and let L be a set of functions defined on X. A function $f : Z \to [-\infty, +\infty]$ is called abstract convex or L-convex on the set Z if there exists a set $U \subset L$ such that

$$f(z) = \sup\{l(z) :\ l \in U\}, \forall z \in Z.$$

The set L in Definition 2.1 is called the set of all elementary functions. Also we just say f is L-convex if $Z = X$.

Definition 2.2. Let L be a set of elementary functions defined on X. A function h of the form

$$h(x) = l(x) - c, \quad (\forall x \in X) \tag{2.1}$$

with $l \in L,\ c \in \mathbb{R}$ is called L-affine function. The set of all L-affine functions is denoted by H_L.

Recall that the support set of an L-convex function f is defined by

$$supp(f, L) = \{l \in L : l(x) \leq f(x)\ \forall x \in X\},$$

where L is the set of elementary functions. Also, $l \in supp(f, L)$ is called a maximal element if $l(x) \leq k(x)$ $(\forall x \in X)$, for some $k \in supp(f, L)$, then $l = k$.

The L-subdifferential at a point $x_0 \in X$ is defined by

$$\partial_L f(x_0) = \{l \in L :\ f(x) - f(x_0) \geq l(x) - l(x_0)\ \forall x \in X\}.$$

Recall that the Fenchel-Moreau conjugation function of an L-convex function $f : X \to (-\infty, +\infty]$, which is the function $f_L^* : L \to [-\infty, +\infty]$ defined by

$$f_L^*(l) = \sup_{x \in \mathrm{dom}\ f} [l(x) - f(x)]\ , \tag{2.2}$$

where $\mathrm{dom}\ f = \{x \in X : f(x) < +\infty\}$. Also, it is easy to see that $f_L^*(l) = l(x) - f(x)$ if and only if $l \in \partial_L f(x)$.
For the sake of simplicity, when there is no confusion, we will drop L and use f^* instead of f_L^*.

Definition 2.3. [1, 10] A function $f : X \longrightarrow \overline{\mathbb{R}} := [-\infty, +\infty]$ is called *topical* if it is increasing ($x \geq y \implies f(x) \geq f(y)$) and plus-homogeneous ($f(x+\lambda\mathbf{1}) = f(x) + \lambda$, for all $x \in X$ and all $\lambda \in \mathbb{R}$).
A function $f : X \longrightarrow \bar{\mathbb{R}} := [-\infty, +\infty]$ is called *sub-topical* if it is increasing ($x \geq y \implies f(x) \geq f(y)$) and plus-sub-homogeneous ($f(x + \lambda\mathbf{1}) \leq f(x) + \lambda$, for all $x \in X$ and all $\lambda \geq 0$).

The following observations are straightforward consequences of the definition.

Lemma 2.4. *A function $f : X \to \overline{\mathbb{R}}$ is plus-sub-homogeneous if and only if*

$$f(x + \mu\mathbf{1}) \geq f(x) + \mu, \quad \forall x \in X,\ \forall \mu \leq 0.$$

Lemma 2.5. *Let $f : X \to \overline{\mathbb{R}}$ be a sub-topical function. Then, the following assertions hold.*
(i) If $f(x) = +\infty$, for some $x \in X$, then $f \equiv +\infty$.
(ii) If $f(x) = -\infty$, for some $x \in X$, then $f \equiv -\infty$.

Now, we present some examples of sub-topical functions.

Example 2.6. Let $f : X \to \overline{\mathbb{R}}$. Then the followings hold.
(i) If f is topical, then it is sub-topical.
(ii) Assume that f is a proper sub-linear function such that $f(\mathbf{1}) \leq 1$. Then,

$$f(x + \mu\mathbf{1}) \leq f(x) + \mu f(\mathbf{1}) \leq f(x) + \mu, \quad \forall \mu > 0,$$

and hence f is plus-sub-homogeneous.

Consider $\psi : X \times X \times \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(x, y, \alpha) := \sup\{\lambda \ : \ \lambda \leq \alpha, \ \lambda\mathbf{1} \leq x + y\}.$$

This function was introduced and examined in [1]. It follows from (2.1) that the set $\{\lambda : \lambda \leq \alpha, \ \lambda\mathbf{1} \leq x + y\}$ is nonempty and bounded. Clearly, this set is a closed subset of $\mathbb{R}$. So, we deduce that

$$\psi(x, y, \alpha)\mathbf{1} \leq x + y, \quad \forall x, y \in X \text{ and } \forall \alpha \in \mathbb{R}. \tag{2.3}$$

The next fact provides some basic (but essential) properties of this function.

Proposition 2.7. ([1], Proposition 3.1) *Let ψ be defined as above. Then, for all $x, x', y, y' \in X$, $\alpha \in \mathbb{R}$ and $\mu \geq 0$, we have*

$$\psi(x + \mu\mathbf{1}, y, \alpha) \leq \psi(x, y, \alpha) + \mu; \tag{2.4}$$

$$\psi(x, y, \alpha) = \psi(y, x, \alpha); \tag{2.5}$$

$$x \leq x' \Rightarrow \psi(x, y, \alpha) \leq \psi(x', y, \alpha); \tag{2.6}$$

$$y \leq y' \Rightarrow \psi(x, y, \alpha) \leq \psi(x, y', \alpha); \tag{2.7}$$

$$\alpha \leq \alpha' \Rightarrow \psi(x, y, \alpha) \leq \psi(x, y, \alpha'); \tag{2.8}$$

$$\psi(x, y + t\mathbf{1}, \alpha) = \psi(x + t\mathbf{1}, y, \alpha) = \psi(x, y, \alpha - t) + t, \ \forall t \in \mathbb{R}; \tag{2.9}$$

$$\psi(x, -x + \alpha\mathbf{1}, \alpha) = \alpha. \tag{2.10}$$

Let us consider the function $\psi_{y,\alpha} : X \to \mathbb{R}$ defined by $\psi_{y,\alpha}(x) := \psi(x, y, \alpha)$, for all $x, y \in X$, and denote $\Psi := \{\psi_{y,\alpha} : \ y \in X, \alpha \in \mathbb{R}\}$.

Remark 2.8. According to (2.4) and (2.6), the function $\psi_{y,\alpha}$ is sub-topical.

The next results play a crucial role in optimizing the difference of two sub-topical functions. For more details, see [1].

Theorem 2.9. *Let $f : X \to \overline{\mathbb{R}}$ be a function. Then, f is sub-topical if and only if there exists a set $A \subset X \times \mathbb{R}$ such that*

$$f(x) = \sup_{(y,\alpha)\in A} \psi_{y,\alpha}(x).$$

In this case, one can take $A = \{(y, \alpha) \in X \times \mathbb{R} : \ f(-y + \alpha\mathbf{1}) \geq \alpha\}$.

Proposition 2.10. *Let $f : X \to \overline{\mathbb{R}}$ be a sub-topical function. Then,*

$$supp(f, \Psi) = \{\psi_{y,\alpha} \in \Psi : \ f(-y + \alpha\mathbf{1}) \geq \alpha\}.$$

Theorem 2.11. *Let $f : X \to \mathbb{R}$ be a sub-topical function and $x_0 \in X$. Then,*

$$\partial_\Psi f(x_0) = \{\psi_{y,\alpha} \in \Psi : \ f_{-y}(\alpha) = f(x_0) - \psi_{y,\alpha}(x_0)\}.$$

3. Maximal elements of support set

In this section, we shall identify the maximal elements of the support set of sub-topical functions. These results will be applied in Section 4 to optimize functions being the difference of two sub-topical functions.

Proposition 3.1. *Let $\psi_{y_1,\alpha}, \psi_{y_2,\beta} \in \Psi$. Then, $\psi_{y_1,\alpha}(x) \le \psi_{y_2,\beta}(x)$, for all $x \in X$, if and only if $y_1 \le y_2$ and $\alpha \le \beta$. Moreover, if $y_1 < y_2$, then there exists $z \in X$ such that $\psi_{y_1,\alpha}(z) < \psi_{y_2,\alpha}(z)$.*

Proof. Assume that $\psi_{y_1,\alpha}(x) \le \psi_{y_2,\beta}(x), \ \forall x \in X$. Then,

$$\alpha = \psi_{y_1,\alpha}(-y_1 + \alpha\mathbf{1}) \le \psi_{y_2,\beta}(-y_1 + \alpha\mathbf{1}) \le \beta.$$

This fact with (2.3) imply that

$$\alpha\mathbf{1} \le \psi_{y_2,\beta}(-y_1 + \alpha\mathbf{1})\mathbf{1} \le y_2 - y_1 + \alpha\mathbf{1},$$

and so, $y_1 \le y_2$.
The converse follows from (2.7) and (2.8).
Now, consider $y_1 < y_2$ (i.e., $y_2 - y_1 \in \mathrm{int}S$). Since $\mathbf{1} \in intS$, there is a $\lambda > 0$ such that $y_1 + \lambda\mathbf{1} < y_2$. Put $z := -y_1 - \lambda\mathbf{1} + \alpha\mathbf{1}$. Since $\lambda > 0$, one has

$$\psi_{y_1,\alpha}(z) = \alpha - \lambda < \alpha = \psi_{y_2,\alpha}(-y_2 + \alpha\mathbf{1}) \le \psi_{y_2,\alpha}(z).$$

Hence, the proof is complete. □

By the following proposition, a necessary condition for maximality of an element is presented.

Proposition 3.2. *Assume $p : X \to \mathbb{R}$ is a sub-topical function and $\psi_{y,\alpha} \in supp(p, \Psi)$ is a maximal element. Then, $p(-y + \alpha\mathbf{1}) = \alpha$.*

Proof. Since $\psi_{y,\alpha} \in supp(p, \Psi)$, (using Proposition 2.10) $p(-y + \alpha\mathbf{1}) \ge \alpha$. Assume that $p(-y + \alpha\mathbf{1}) > \alpha$. So, there is $\lambda > 0$ such that

$$p(-y - \lambda\mathbf{1} + \alpha\mathbf{1}) \ge p(-y + \alpha\mathbf{1}) - \lambda > \alpha,$$

where the first inequality follows from Lemma 2.4. Put $y' := y + \lambda\mathbf{1}$. Therefore, $p(-y' + \alpha\mathbf{1}) > \alpha$ implies that $\psi_{y',\alpha} \in supp(p, \Psi)$. On the other hand, Proposition 3.1 and the fact that $y < y'$ imply that $\psi_{y,\alpha}(x) \le \psi_{y',\alpha}(x)$, for all $x \in X$ and $\psi_{y,\alpha} \ne \psi_{y',\alpha}$. This is a contradiction with the maximality of $\psi_{y,\alpha}$. □

It is worth noting that the converse of Proposition 3.2 is not valid. Consider the sub-topical function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x + 1$, for all x. Trivially, $\psi_{1,\alpha} \in supp(p, \Psi)$ for all α and $p(-1 + \alpha) = \alpha$, but the maximal element of support set does not exist. To avoid this nonexistence, we consider an extra condition as follows. First, recall that $p : X \to \mathbb{R}$ is called strictly sub-topical if p is plus-sub-homogeneous and strictly increasing (the latter means, if $x \leq y$ and $x \neq y$, then $p(x) < p(y)$).

Proposition 3.3. *Let $p : X \to \mathbb{R}$ be a strictly sub-topical function. Assume that $\varepsilon \in \mathbb{R}$ and y is an element of X. Then, $\psi_{y,\varepsilon}$ is a maximal element of $supp(p, \Psi)$ if and only if $\varepsilon = \max\{\alpha;\ p(-y + \alpha\mathbf{1}) \geq \alpha\}$ and $p(-y + \varepsilon\mathbf{1}) = \varepsilon$.*

Proof. Applying Proposition 3.2 and (2.8), if $\psi_{y,\varepsilon}$ is a maximal element of $supp(p, \Psi)$, then $p(-y + \varepsilon\mathbf{1}) = \varepsilon$ and $\varepsilon = \max\{\alpha;\ p(-y + \alpha\mathbf{1}) \geq \alpha\}$.
Conversely, assume that $\psi_{y',\varepsilon'} \in supp(p, \Psi)$ and $\psi_{y,\varepsilon} \leq \psi_{y',\varepsilon'}$. Using Proposition 3.1, one has $y \leq y'$ and $\varepsilon \leq \varepsilon'$. Since $\psi_{y,\varepsilon'} \leq \psi_{y',\varepsilon'}$ and $\psi_{y',\varepsilon'} \in supp(p, \Psi)$, we have

$$p(-y + \varepsilon'\mathbf{1}) \geq \varepsilon'.$$

This and the fact that $\varepsilon = \sup\{\alpha : p(-y + \alpha\mathbf{1}) \geq \alpha\}$ imply that $\varepsilon' \leq \varepsilon$. Therefore, $\varepsilon = \varepsilon'$.
Moreover, if $y \neq y'$, then by strictly monotonicity of p,

$$p(-y' + \varepsilon'\mathbf{1}) < p(-y + \varepsilon\mathbf{1}) = \varepsilon = \varepsilon'.$$

This implies that $\psi_{y',\varepsilon'} \notin supp(p, \Psi)$, which is impossible. Hence, $y = y'$ and the proof is complete. $\square$

Let $p : X \to \mathbb{R}$ be a sub-topical function. In the sequel of this paper we denote by Ξ_p the following set:

$$\Xi_p := \{y \in X :\ \psi_{y,\alpha} \in supp(p, \Psi), \quad \text{for some } \alpha \in \mathbb{R}\}.$$

Corollary 3.4. *Let $p : X \to \mathbb{R}$ be strictly sub-topical such that $\varepsilon_y = \max\{\alpha : p(-y+\alpha\mathbf{1}) \geq \alpha\} \in \mathbb{R}$, for each $y \in \Xi_p$. Then, for each $\psi_{y,\alpha} \in supp(p, \Psi)$ there exists a maximal element $\psi_{\tilde{y},\tilde{\alpha}}$ of support set such that $\psi_{y,\alpha} \leq \psi_{\tilde{y},\tilde{\alpha}}$. In this case, $\tilde{\alpha} = p(-y + \varepsilon_y\mathbf{1})$ and $\tilde{y} = y + (\tilde{\alpha} - \varepsilon_y)\mathbf{1}$.*

Proof. Let $\psi_{y,\alpha} \in supp(p, \Psi)$. Using Proposition 3.1 and the fact $\psi_{y,\varepsilon_y} \in supp(p, \Psi)$, we have

$$\psi_{\tilde{y},\tilde{\alpha}} = \psi_{y+[p(-y+\varepsilon_y\mathbf{1})-\varepsilon_y]\mathbf{1},p(-y+\varepsilon_y\mathbf{1})} \geq \psi_{y,\varepsilon_y} \geq \psi_{y,\alpha}.$$

Now, we are going to show that $\psi_{\tilde{y},\tilde{\alpha}}$ is maximal. To this end, let $\delta \in \mathbb{R}$ be such that $p(-\tilde{y} + \delta\mathbf{1}) \geq \delta$. Put $\gamma := \delta + \varepsilon_y - p(-y + \varepsilon_y\mathbf{1})$. Since $p(-y + \varepsilon_y\mathbf{1}) \geq \varepsilon_y$, we have

$$p(-y + \gamma\mathbf{1}) = p(-\tilde{y} + \delta\mathbf{1}) \geq \delta \geq \delta + [\varepsilon_y - p(-y + \varepsilon_y\mathbf{1})] = \gamma.$$

This means that $\gamma \in \{\alpha\ :\ p(-y + \alpha\mathbf{1}) \geq \alpha\}$. Therefore,

$$\delta + \varepsilon_y - p(-y + \varepsilon_y\mathbf{1}) = \gamma \leq \varepsilon_y,$$

which implies that $\delta \leq p(-y + \varepsilon_y\mathbf{1}) = \tilde{\alpha}$. Hence,

$$\tilde{\alpha} = \max\{\delta\ :\ p(-\tilde{y} + \delta\mathbf{1}) \geq \delta\}.$$

Since $p(-\tilde{y}+\tilde{\alpha}\mathbf{1})=\tilde{\alpha}$, the result follows from Proposition 3.3. $\square$

4. Necessary and sufficient conditions for the minimum value of the difference of strictly sub-topical functions

Here, we first present a dual optimality condition for the difference of two sub-topical functions, and then we obtain the necessary and sufficient conditions for the minimum value of the difference of two strictly sub-topical functions.
Let $p, q : X \to \mathbb{R}$ be sub-topical functions. Consider the following extremal problem:

$$q(x) - p(x) \to \min \quad \text{subject to } x \in X, \tag{4.1}$$

where $\inf_{x\in X} q(x) - p(x) > -\infty$.
Now, consider the following problem:

$$p^*(\psi_{y,\alpha}) - q^*(\psi_{y,\alpha}) \to \min \quad \text{subject to} \quad \psi_{y,\alpha} \in \ \mathrm{dom} q^*. \tag{4.2}$$

The problem defined by (4.2) is called the dual problem with respect to (4.1). Recall the well-known Singer-Toland formula (see [15]), which is:

$$\inf\{g(x) - h(x) : \ x \in X\} = \inf\{h^*(l) - g^*(l) : \ l \in L\},$$

where $g, h : X \to \mathbb{R}$ are proper H_L-convex functions such that $\inf(g(x) - h(x)) > -\infty$ and H_L is the set of $L-$affine functions defined in Definition 2.2. So, the following observation can be obtained directly from the Toland-Singer formula.

Proposition 4.1. *Let $p, q : X \to \mathbb{R}$ be sub-topical functions such that $inf_{x\in X}(q(x) - p(x)) > -\infty$. Then,*

$$\inf\{q(x) - p(x) : \ x \in X\} = \inf\{p^*(\psi_{y,\alpha}) - q^*(\psi_{y,\alpha}) : \ \psi_{y,\alpha} \in \Psi\}.$$

We now obtain some results on the intersection of subdifferentials in order to characterize the problems defined by (4.1) and (4.2).

Lemma 4.2. *Let $p, q : X \to \mathbb{R}$ be sub-topical functions. Let $x \in X$, $\alpha \in \mathbb{R}$ and $y := -x + \alpha\mathbf{1}$. Then, $\psi_{y,\alpha} \in \partial_\Psi p(x) \cap \partial_\Psi q(x)$. Moreover,*

$$p^*(\psi_{y,\alpha}) - q^*(\psi_{y,\alpha}) = q(x) - p(x).$$

Proof. Applying (2.5) and (2.10), we get

$$\psi_{y,\alpha}(x) = \psi_{x,\alpha}(-x + \alpha\mathbf{1}) = \alpha.$$

This implies that

$$p_{-y}(\alpha) = p(-y + \alpha\mathbf{1}) - \alpha = p(x) - \psi_{y,\alpha}(x).$$

Using this and Theorem 2.11, we have $\psi_{y,\alpha} \in \partial_\Psi p(x)$. The same argument shows that $\psi_{y,\alpha} \in \partial_\Psi q(x)$. The rest of the proof is trivial. $\square$

Theorem 4.3. *Let $p, q : X \to \mathbb{R}$ be sub-topical functions.*
(i) If $x_0 \in X$ is a global minimizer of problem (4.1), then $\psi_{-x_0+\alpha_0\mathbf{1},\alpha_0}$ is a global minimizer of problem (4.2), for all $\alpha_0 \in \mathbb{R}$.
(ii) If $\psi_{y_0,\alpha_0} \in \Psi$ is a global minimizer of problem (4.2), then $x_0 := -y_0 + \alpha_0\mathbf{1}$ is a global minimizer of problem (4.1).

Proof. (i) : Let $\alpha_0 \in \mathbb{R}$ and $y_0 := -x_0+\alpha_0\mathbf{1}$. Using Lemma 4.2 and Proposition 4.1, we have

$$\begin{aligned} p^*(\psi_{y_0,\alpha_0}) - q^*(\psi_{y_0,\alpha_0}) &= q(x_0) - p(x_0) \\ &= \inf_{x\in X} q(x) - p(x) \\ &= \inf_{\psi_{y,\alpha}\in\Psi} p^*(\psi_{y,\alpha}) - q^*(\psi_{y,\alpha}). \end{aligned}$$

Thus, ψ_{y_0,α_0} is a global minimizer of problem (4.2).

(ii): Applying again Proposition 4.1 and the fact that $\psi_{y_0,\alpha_0} \in \partial_\Psi p(x_0) \cap \partial_\Psi q(x_0)$, we have

$$\begin{aligned} q(x_0) - p(x_0) &= p^*(\psi_{y_0,\alpha_0}) - q^*(\psi_{y_0,\alpha_0}) \\ &= \inf_{\psi_{y,\alpha}\in\Psi} p^*(\psi_{y,\alpha}) - q^*(\psi_{y,\alpha}) \\ &= \inf_{x\in X} q(x) - p(x). \end{aligned}$$

Hence, the proof is complete. □

Now, we are going to investigate the necessary and sufficient conditions of the minimum value of problem (4.1).

Proposition 4.4. *Let $p, q : X \to \mathbb{R}$ be sub-topical functions. Assume that $\eta = inf_{x\in X} q(x) - p(x) > -\infty$ and $\gamma \le \eta$. Then, $q(x) \ge p(x) + \gamma$, for all $x \in X$, if and only if $supp(p, L) + \gamma \subset supp(q, L)$.*

Proof. It is easy to see that if $q(x) \ge p(x) + \gamma$, for all $x \in X$, then $supp(p, L) + \gamma \subset supp(q, L)$. For the converse implication, let $x \in X$ be arbitrary. So, by Theorem 2.9 and Proposition 2.10, we have

$$\begin{aligned} p(x) + \gamma &= \sup_{\psi_{y,\alpha}\in supp(p,\Psi)} \psi_{y,\alpha}(x) \ + \gamma \\ &\le \sup_{\psi_{y,\alpha}\in supp(q,\Psi)} \psi_{y,\alpha}(x) \\ &= q(x). \end{aligned}$$

□

The following result plays a crucial role in reaching our goal.

Proposition 4.5. *Let $p, q : X \to \mathbb{R}$ be strictly sub-topical functions. Assume that $\varepsilon_y, \eta_z \in \mathbb{R}$, for all $y \in \Xi_p$ and $z \in \Xi_q$, where $\varepsilon_y = \sup\{\alpha : p(-y+\alpha\mathbf{1}) \geq \alpha\}$ and $\eta_z = \sup\{\beta : q(-z+\beta\mathbf{1}) \geq \beta\}$. Then, the following statements are equivalent.*
(i) $Supp(p, \Psi) \subset supp(q, \Psi)$.
(ii) For each maximal element ψ_{y_1,α_1} of $supp(p, \Psi)$, there exist a maximal element ψ_{y_2,α_2} of $supp(q, \Psi)$ such that $\psi_{y_1,\alpha_1}(x) \leq \psi_{y_2,\alpha_2}(x)$, for all $x \in X$.

Proof. $(i) \Rightarrow (ii)$: Let $supp(p, \Psi) \subset supp(q, \Psi)$. Let ψ_{y_1,α_1} be a maximal element of $supp(p, \Psi)$. So, $\psi_{y_1,\alpha_1} \in supp(q, \Psi)$. By Corollary 3.4, there is a maximal element ψ_{y_2,α_2} in $supp(q, \Psi)$ such that $\psi_{y_1,\alpha_1} \leq \psi_{y_2,\alpha_2}$.
$(ii) \Rightarrow (i)$: Let $\psi_{y,\alpha} \in supp(p, \Psi)$. By Corollary 3.1, there is a maximal element ψ_{y_1,α_1} such that $\psi_{y,\alpha} \leq \psi_{y_1,\alpha_1}$. By the hypothesis, there is a maximal element $\psi_{y_2,\alpha_2} \in supp(q, \Psi)$ such that $\psi_{y_1,\alpha_1} \leq \psi_{y_2,\alpha_2}$. Then, $\psi_{y_2,\alpha_2} \geq \psi_{y,\alpha}$, which implies $\psi_{y,\alpha} \in supp(q, \Psi)$. Hence, $supp(p, \Psi) \subset supp(q, \Psi)$. □

Remark 4.6. Assume that $\tilde{p}(x) = p(x) + m$, for all $x \in X$ and for some $m \in \mathbb{R}$. It is easy to see that $supp(\tilde{p}, \Psi) = supp(p, \Psi) + m$. Also, if $\psi_{y,\alpha}$ is a maximal element of $supp(\tilde{p}, \Psi)$, then $\psi_{y-m\mathbf{1},\alpha-m}$ is a maximal element of $supp(p, \Psi)$ and vice versa.

In the following, we present the necessary and sufficient conditions for the minimum value of the difference of strictly sub-topical functions.

Theorem 4.7. *Let $p, q : X \to \mathbb{R}$ be strictly sub-topical functions and $inf_{x\in X} q(x) - p(x) > -\infty$. Assume that $\varepsilon_y, \eta_z \in \mathbb{R}$, for every $y \in \Xi_p$ and $z \in \Xi_q$, where $\varepsilon_y = \sup\{\alpha : \ p(-y+\alpha\mathbf{1}) \geq \alpha\}$ and $\eta_z = \sup\{\beta : \ q(-z+\beta\mathbf{1}) \geq \beta\}$. Let $m \in \mathbb{R}$. Then, $m \leq inf_{x\in X} q(x) - p(x)$ if and only if for every $y \in \Xi_p$ with $p(-y+\varepsilon_y\mathbf{1}) = \varepsilon_y$, there exists $z \in \Xi_q$ with $q(-z+\eta_z\mathbf{1}) = \eta_z$ such that*

$$y + m\mathbf{1} \leq z \text{ and } \varepsilon_y + m \leq \eta_z.$$

Moreover, if $m \leq \inf_{x\in X} q(x) - p(x)$ and there are $y, z \in X$ such that $y + m\mathbf{1} = z$ and $\varepsilon_y + m = \eta_z$, then, $m = \inf_{x\in X} q(x) - p(x)$ and also $-y + \varepsilon_y\mathbf{1}$ is the global minimizer.

Proof. Let $\tilde{p}(x) = p(x) + m$. Due to Proposition 4.4, $q(x) \geq p(x) + m \quad (\forall x \in X)$ if and only if

$$supp(\tilde{p}, L) \subset supp(q, L). \tag{4.3}$$

Using Remark 4.6, Proposition 3.3 and Proposition 4.5, (4.3) holds if and only if for each $y \in X$ with $p(-y+\varepsilon_y\mathbf{1}) = \varepsilon_y$, there exist $z \in X$ with $q(-z+\eta_z\mathbf{1}) = \eta_z$ such that

$$\psi_{y,\varepsilon_y} + m \leq \psi_{z,\eta_z}.$$

Applying Proposition 3.1 and (2.11), the results follow.
Assume now that there are $y, z \in X$ such that $y + m\mathbf{1} = z$ and $\varepsilon_y + m = \eta_z$.

Using the above, we have

$$p(-y+\varepsilon_y\mathbf{1})+m=q(-y+\varepsilon_y\mathbf{1}).$$

Hence, the proof is complete. $\square$

Remark 4.8. Applying Corollary 3.4 and Proposition 4.5, one has in Theorem 4.7 that $\eta_z := q(-y+\varepsilon_y\mathbf{1})$ and $z = y + [\eta_z - \varepsilon_y]\mathbf{1}$.

Finally, let us give a simple example for Theorem 4.7.

Example 4.9. Suppose that $p, q : \mathbb{R} \to \mathbb{R}$ are defined by

$$p(x) = \begin{cases} x-1, & x \leq 2, \\ 1, & x \geq 2, \end{cases}$$

and

$$q(x) = \begin{cases} x+1, & x \leq 1, \\ \sqrt{x}+1, & x \geq 1. \end{cases}$$

Clearly, p and q are sub-topical functions. Consider the function $q-p$:

$$q(x)-p(x) = \begin{cases} 2, & x \leq 1, \\ \sqrt{x}-x+2, & 1 \leq x \leq 2, \\ \sqrt{x}, & x \geq 2. \end{cases}$$

Clearly, $\Xi_p = (-\infty,-1]$ and $\varepsilon_y = 1$, for all $y \in \Xi_p$. Also, $\Xi_q = (-\infty, 1]$ and $\eta_z = \frac{3+\sqrt{5-4z}}{2}$, for all $z \in \Xi_q$. Since $m \geq 0$, for every $y \in \Xi_p$, there is $z \in \Xi_q$ such that $\varepsilon_y + m \leq \eta_z$.
According to Theorem 4.7 and Remark 4.8,

$$p(-y+1) = p(-y+\varepsilon_y) = \varepsilon_y = 1 \quad \Rightarrow \quad y = -1,$$

and then,

$$q(2) = q(-y+\varepsilon_y) = \eta_z = m + \varepsilon_y = 1 + m \quad \Rightarrow \quad m = \sqrt{2}.$$

Moreover, $x = -y + \varepsilon_y = 2$ is a global minimizer.

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