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# GENERALIZED DERIVATIONS ON MODULES

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Abstract. Let $A$ be a Banach algebra and $M$ be a Banach right $A$-module. A linear map $\delta : M \to M$ is called a generalized derivation if there exists a derivation $d : A \to A$ such that

$$\delta(xa) = \delta(x)a + xd(a) \quad (a \in A, x \in M).$$

In this paper we associate a triangular Banach algebra $\mathcal{T}$ to a Banach $A$-module $M$ and investigate the relation between generalized derivations on $M$ and derivations on $\mathcal{T}$. In particular, we prove that the so-called generalized first cohomology group of $M$ is isomorphic to the first cohomology group of $\mathcal{T}$.

## 1. Introduction

Recently, a number of analysts [1, 6, 17] have studied various generalized notions of derivations in the context of Banach algebras. There are some applications in the other fields of study [11]. Such maps have been extensively studied in pure algebra; cf. [2, 5, 12].

Let, throughout the paper, $A$ denote a Banach algebra (not necessarily unital) and let $M$ be a Banach right $A$-module. A linear mapping $d : A \to A$ is called a derivation if $d(ab) = d(a)b + ad(b) \quad (a, b \in A)$. If $a \in A$ and we define $d_a$ by $d_a(x) = ax - xa \quad (x \in A)$, then $d_a$ is a derivation and such derivation is called inner. A linear mapping $\delta : M \to M$

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is called a generalized derivation if there exists a derivation $d : A \to A$ such that $\delta(xa) = \delta(x)a + xd(a) \quad (x \in M, a \in A)$. For convenience, we say that such a generalized derivation $\delta$ is a $d$-derivation. In general, the derivation $d : A \to A$ is not unique and it may happen that $\delta$ (resp. $d$) is bounded but $d$ (resp. $\delta$) is not bounded. For instance, assume that the action of $A$ on $M$ is trivial, i.e $MA = \{0\}$. Then every linear mapping $\delta : M \to M$ is a $d$-derivation for each derivation $d$ on $A$.

Our notion is a generalization of both concepts of a generalized derivation (cf. [5, 12]) and of a multiplier (cf. [7]) on an algebra (see also [19]). To see this, regard the algebra as a module over itself. The authors in [1] investigated the generalized derivations on Hilbert $C^*$-modules and showed that these maps may appear as the infinitesimal generators of dynamical systems.

**Example 1.1.** Let $M$ be a right Hilbert $C^*$-module over a $C^*$-algebra $A$ of compact operators acting on a Hilbert space (see [15] for more details on Hilbert $C^*$-modules). By Theorem 4 of [3], $M$ has an orthonormal basis so that each element $x$ of $M$ can be expressed as $x = \sum\limits_{\lambda} v_\lambda < v_\lambda, x >$. If $d$ is a derivation on $A$, then the mapping, $\delta : M \to M$ defined by $\delta(x) = \sum\limits_{\lambda} v_\lambda d(< v_\lambda, x >)$ is a $d$-derivation, since

$$\begin{aligned} \delta(xa) &= \delta\left(\sum_{\lambda} v_\lambda < v_\lambda, xa >\right) \\ &= \sum_{\lambda} v_\lambda d(< v_\lambda, x > a) \\ &= \sum_{\lambda} v_\lambda d(< v_\lambda, x >)a + \sum_{\lambda} v_\lambda < v_\lambda, x > d(a) \\ &= \delta(x)a + xd(a). \end{aligned}$$

The set $\mathcal{B}(M)$ of all bounded module maps on $M$ is a Banach algebra and $M$ is a Banach $\mathcal{B}(M) - A$-bimodule equipped with $T.x = T(x) \quad (x \in M, T \in \mathcal{B}(M))$, since we have $T.(xa) = T(xa) = T(x)a = (T.x)a$ and $\|T.xa\| \leq \|T\| \, \|x\| \, \|a\|$, for all $a \in A, x \in M, T \in \mathcal{B}(M)$.

We call $\delta : M \to M$ a generalized inner derivation if there exist $a \in A$ and $T \in \mathcal{B}(M)$ such that $\delta(x) = T.x - xa = T(x) - xa$. Mathieu in [16] called a map $\delta : A \to A$ a generalized inner derivation if $\delta(x) = bx - xa$ for some $a, b \in A$. If we consider $A$ as a right $A$-module in a natural way, and take $T(x) = bx$, then our definition covers the notion of Mathieu.

In this paper we deal with the derivations on the triangular Banach algebras of the form $\mathcal{T} = \begin{pmatrix} \mathcal{B}(M) & M \\ 0 & A \end{pmatrix}$. Such algebras were introduced by Forrest and Marcoux [8] that in turn are motivated by work of Gilfeather and Smith in [10] (these algebras have been also investigated by Y. Zhang who called them module extension Banach algebras [22]). Among some facts on generalized derivations, we investigate the relation between generalized derivations on $M$ and derivations on $\mathcal{T}$. In particular, we show that the generalized first cohomology group of $M$ is isomorphic to the first cohomology group of $\mathcal{T}$.

## 2. Main results

If we consider $A$ as an $A$-module in a natural way then we have the following lemma about generalized derivations on $A$.

**Lemma 2.1.** *A linear mapping $\delta : A \to A$ is a generalized derivation if and only if there exist a derivation $d : A \to A$ and a module map $\varphi : A \to A$ such that $\delta = d + \varphi$.*

**Proof.** Suppose $\delta$ is a generalized derivation on $A$. Then there exists a derivation $d$ on $A$ such that $\delta$ is a $d$-derivation. On putting $\varphi = \delta - d$, we have

$$\varphi(xa) = \delta(xa) - d(xa) = \delta(x)a + xd(a) - (d(x)a + xd(a))$$

$$= (\delta(x) - d(x))a = \varphi(x)a,$$

for all $a, x \in A$. Thus $\varphi$ is a module map and $\delta = d + \varphi$.

Conversely, let $d$ be a derivation on $A$, $\varphi$ be a module map on $A$ and put $\delta = d + \varphi$. Then clearly $\delta$ is a linear map and

$$\delta(xa) = d(xa) + \varphi(xa) = d(x)a + xd(a) + \varphi(x)a$$

$$= (d(x) + \varphi(x))a + xd(a) = \delta(x)a + xd(a)$$

for all $a, x \in A$. Therefore $\delta$ is a $d$-derivation. $\square$

The next two results concern the boundedness of a generalized derivation.

**Theorem 2.2.** *Let $A$ have a bounded left approximate identity $\{e_\alpha\}_{\alpha\in I}$ and let $\delta$ be a $d$-derivation on $A$. Then $\delta$ is bounded if and only if $d$ is bounded.*

**Proof.** First we show that every module map on $A$ is bounded. Suppose that $\varphi$ is a module map on $A$ and let $\{a_n\}$ be a sequence in $A$ converging to zero in the norm topology. By a consequence of Cohen Factorization Theorem (see Corollary 11.12 of [4]) there exist a sequence $\{b_n\}$ and an element $c$ in $A$ such that $b_n \to 0$ and $a_n = cb_n, \quad (n \in \mathbb{N})$. Then $\varphi(a_n) = \varphi(cb_n) = \varphi(c)b_n \to 0$. Thus by the closed graph theorem, $\varphi$ is bounded. Now let $\delta$ be a $d$-derivation. By Lemma 2.1, $\delta = d + \varphi$ for some module map $\varphi$ on $A$. Therefore $\delta$ is bounded if and only if $d$ is bounded. $\square$

**Corollary 2.3.** *Every generalized derivation on a $C^*$-algebra is bounded.*

**Proof.** Every derivation on a $C^*$-algebra is automatically continuous; cf. [13]. $\square$

Let $\varphi : A \to A$ be a homomorphism (algebra morphism). A linear mapping $T : M \to M$ is called a $\varphi$-morphism if $T(xa) = T(x)\varphi(a) \quad (a \in A, x \in M)$. If $\varphi$ is a isomorphism and T is a bijective mapping then we say T to be a $\varphi$-isomorphism. An $id_A$-morphism is a module map (module morphism). Here $id_A$ denotes the identity operator on $A$.

**Proposition 2.4.** *Suppose $\delta$ is a bounded $d$-derivation on $M$ and $d$ is bounded. Then $T = \exp(\delta)$ is a bi-continuous $\exp(d)$-isomorphism.*

**Proof.** Using induction one can easily show that

$$\delta^{(n)}(xa) = \sum_{r=0}^{n} \binom{n}{r} \delta^{(n-r)}(x) d^{(r)}(a).$$

For each $a \in A, x \in M$ we have

$$\begin{aligned}
T(xa) &= \exp(\delta)(xa) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\delta^{(n)}(xa) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\sum_{r=0}^{n} \tbinom{n}{r}\delta^{(n-r)}(x)d^{(r)}(a) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n} (\frac{1}{(n-r)!}\delta^{(n-r)}(x)(\frac{1}{r!}d^{(r)}(a)) \\
&= (\sum_{n=0}^{\infty} \frac{1}{n!}\delta^{(n)}(x))(\sum_{n=0}^{\infty} \frac{1}{n!}d^{(n)}(a)) \\
&= \exp(\delta)(x)\exp(d)(a).
\end{aligned}$$

The operators $\exp(\delta)$ and $\exp(d)$ are invertible in the Banach algebras of bounded operators on $M$ and $A$, respectively. Hence $T$ is an $\exp(d)$-isomorphism.

**Proposition 2.5.** *Let $\delta$ be a bounded generalized derivation on $M$. Then $\delta$ is a generalized inner derivation if and only if there exists an inner derivation $d_a$ on $A$ such that $\delta$ is $d_a$-derivation.*

**Proof.** Let $\delta$ be a generalized inner derivation. Then there exist $a \in A$ and $T \in \mathcal{B}(M)$ such that $\delta(x) = T(x) - xa \quad (x \in M)$. We have $\delta(x)b+xd_a(b) = (T(x)-xa)b+xab-xba = T(x)b-xba = T(xb)-(xb)a = \delta(xb) \quad (b \in A, x \in M)$. Hence $\delta$ is a $d_a$-derivation.

Conversely, suppose that $\delta$ is a $d_a$-derivation for some $a \in A$. Define $T : M \to M$ by $T(x) = \delta(x) + xa$. Then T is linear, bounded and $T(xb) = \delta(xb)+(xb)a = (\delta(x)b+xd_a(b))+xba = \delta(x)b+xab-xba+xba = (\delta(x) + xa)b = T(x)b$. It follows that $T \in \mathcal{B}(M)$ and $\delta(x) = (\delta(x) + xa) - xa = T(x) - xa$. Therefore $\delta$ is a generalized inner derivation. $\square$

The linear spaces of all bounded generalized derivations and generalized inner derivations on $M$ are denoted by $GZ^1(M,M)$ and $GN^1(M,M)$, respectively. We call the quotient space $GH^1(M,M) = GZ^1(M,M)/GN^1(M,M)$ the generalized first cohomology group of $M$.

**Corollary 2.6.** $GH^1(M,M)=0$ *whenever* $H^1(A,A)=0$.

**Proof.** Let $\delta : M \to M$ be a generalized derivation. Then there exists a derivation $d : A \to A$ such that $\delta$ is a $d$-derivation. Due to $H^1(A,A)=0$, we deduce that $d$ is inner and, by Proposition 2.5, so is $\delta$. Hence $GH^1(M,M)=0$. $\square$

Using some ideas of [8, 18], we give the following notion.

**Definition 2.7.** $\mathcal{T}=\{\begin{pmatrix} T & x \\ 0 & a \end{pmatrix} : T\in\mathcal{B}(M), x\in M, a\in A\}$ equipped with the usual $2\times 2$ matrix addition and formal multiplication and with the norm $\|\begin{pmatrix} T & x \\ 0 & a \end{pmatrix}\| = \|T\|+\|x\|+\|a\|$ is a Banach algebra. We call this algebra the triangular Banach algebra associated to M.

The following two theorems give some interesting relations between generalized derivations on $M$ and derivations on $\mathcal{T}$.

Let $\delta$ be a bounded $d$-derivation on $M$. We define $\Delta_\delta : \mathcal{B}(M)\to\mathcal{B}(M)$ by $\Delta_\delta(T)=\delta T - T\delta$. Then $\Delta_\delta$ is clearly a derivation on $\mathcal{B}(M)$.

**Theorem 2.8.** *Let $\delta$ be a bounded d-derivation on $M$ and let $d$ be bounded. Then the map $D^\delta : \mathcal{T}\to\mathcal{T}$ defined by* $D^\delta\begin{pmatrix} T & x \\ 0 & a \end{pmatrix} = \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix}$ *is a bounded derivation on $\mathcal{T}$. Also $\delta$ is a generalized inner derivation if and only if $D^\delta$ is an inner derivation.*

**Proof.** It is clear that $D^\delta$ is linear. For any $T_1,T_2\in\mathcal{B}(M), x_1,x_2\in M, a_1,a_2\in A$ we have

$$\begin{aligned}
& D^\delta(\begin{pmatrix} T_1 & x_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} T_2 & x_2 \\ 0 & a_2 \end{pmatrix}) = D^\delta\begin{pmatrix} T_1T_2 & T_1.x_2+x_1a_2 \\ 0 & a_1a_2 \end{pmatrix} \\
= & \begin{pmatrix} \Delta_\delta(T_1T_2) & \delta(T_1.x_2+x_1a_2) \\ 0 & d(a_1a_2) \end{pmatrix} \\
= & \begin{pmatrix} \Delta_\delta(T_1T_2) & \delta(T_1(x_2))+\delta(x_1)a_2+x_1d(a_2) \\ 0 & a_1d(a_2)+d(a_1)a_2 \end{pmatrix} \\
= & \begin{pmatrix} T_1\Delta_\delta(T_2)+\Delta_\delta(T_1)T_2 & T_1.\delta(x_2)+x_1d(a_2)+(\delta T_1-T_1\delta)(x_2)+\delta(x_1)a_2 \\ 0 & a_1d(a_2)+d(a_1)a_2 \end{pmatrix} \\
= & \begin{pmatrix} T_1 & x_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} \Delta_\delta(T_2) & \delta(x_2) \\ 0 & d(a_2) \end{pmatrix} + \begin{pmatrix} \Delta_\delta(T_1) & \delta(x_1) \\ 0 & d(a_1) \end{pmatrix}\begin{pmatrix} T_2 & x_2 \\ 0 & a_2 \end{pmatrix} \\
= & \begin{pmatrix} T_1 & x_1 \\ 0 & a_1 \end{pmatrix} D^\delta\begin{pmatrix} T_2 & x_2 \\ 0 & a_2 \end{pmatrix} + D^\delta(\begin{pmatrix} T_1 & x_1 \\ 0 & a_1 \end{pmatrix})\begin{pmatrix} T_2 & x_2 \\ 0 & a_2 \end{pmatrix}.
\end{aligned}$$

Thus $D^\delta$ is a derivation on $\mathcal{T}$. Due to $\| \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix} \| = \|\Delta_\delta(T)\| + \|\delta(x)\| + \|d(a)\| \leq \max\{\|\Delta_\delta\|, \|\delta\|, \|d\|\} \| \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} \|$, we infer that $D^\delta$ is bounded. Now suppose that $\delta$ is a generalized inner derivation. Then there exist $a \in A$ and $T \in \mathcal{B}(M)$ such that $\delta(x) = T(x) - xa \quad (x \in M)$. For all $S \in \mathcal{B}(M), b \in A$ and $y \in M$ we have

$$\begin{aligned} D_{\begin{pmatrix} T & 0 \\ 0 & a \end{pmatrix}} \begin{pmatrix} S & y \\ 0 & b \end{pmatrix} &:= \begin{pmatrix} T & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} S & y \\ 0 & b \end{pmatrix} - \begin{pmatrix} S & y \\ 0 & b \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} TS - ST & T.y - ya \\ 0 & ab - ba \end{pmatrix} \\ &= \begin{pmatrix} \Delta_\delta(S) & \delta(y) \\ 0 & d_a(b) \end{pmatrix} \\ &= D^\delta \begin{pmatrix} S & y \\ 0 & b \end{pmatrix}. \end{aligned}$$

Hence $D^\delta = D_{\begin{pmatrix} T & 0 \\ 0 & a \end{pmatrix}}$ and so $D^\delta$ is an inner derivation.

Conversely, let $\delta$ be a bounded $d$-derivation such that the associated derivation $D^\delta$ is an inner derivation, say $D^\delta = D_{\begin{pmatrix} T_0 & x_0 \\ 0 & a_0 \end{pmatrix}}$. Then for each $T \in \mathcal{B}(M), x \in M, a \in A$ we have

$$\begin{aligned} \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix} &= D^\delta(\begin{pmatrix} T & x \\ 0 & a \end{pmatrix}) \\ &= D_{\begin{pmatrix} T_0 & x_0 \\ 0 & a_0 \end{pmatrix}} \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} T_0T - TT_0 & T_0(x) + x_0a - T(x_0) - xa_0 \\ 0 & a_0a - aa_0 \end{pmatrix} \\ &= \begin{pmatrix} T_0T - TT_0 & T_0(x) + x_0a - T(x_0) - xa_0 \\ 0 & d_{a_0}(a) \end{pmatrix}. \qquad (2.1) \end{aligned}$$

Hence $d = d_{a_0}$ is inner. Putting $a = 0$ and $T = 0$ in (2.1), we conclude that $\delta(x) = T_0(x) - xa_0 \quad (x \in M)$. Hence $\delta$ is a generalized inner derivation. □

The converse of the above theorem is true in the unital case.

**Theorem 2.9.** *Let $A$ be unital and $\mathcal{T}$ be the triangular Banach algebra associated to a unital Banach right $A$-module $M$. Assume that $D : \mathcal{T} \to \mathcal{T}$ is a bounded derivation. Then there exist $m_0 \in M$, a bounded derivation $d : A \to A$ and a bounded $d$-derivation $\delta : M \to M$ such that*

$$D\begin{pmatrix} T & x \\ 0 & a \end{pmatrix} = \begin{pmatrix} \Delta_\delta(T) & \delta(x) + m_0 a - T.m_0 \\ 0 & d(a) \end{pmatrix}.$$

*Moreover, $D$ is inner if and only if $\delta$ is a generalized inner derivation.*

**Proof.** We use some ideas of Proposition 2.1 of [8]. By simple computation one can verify that

(i) $D\begin{pmatrix} 0 & 0 \\ 0 & 1_A \end{pmatrix} = \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix}$ for some $m_0 \in M$;

(ii) $D\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & m_0 a \\ 0 & d(a) \end{pmatrix}$ for some bounded derivation $d$ on $A$;

(iii) $D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta(x) \\ 0 & 0 \end{pmatrix}$ for some linear mapping $\delta$ on $M$;

(iv) $D\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Delta_\delta(T) & -T.m_0 \\ 0 & 0 \end{pmatrix}$;

and finally $D\begin{pmatrix} T & x \\ 0 & a \end{pmatrix} = \begin{pmatrix} \Delta_\delta(T) & \delta(x) + m_0 a - T.m_0 \\ 0 & d(a) \end{pmatrix}$.

We have

$$\begin{aligned}
\begin{pmatrix} 0 & \delta(xa) \\ 0 & 0 \end{pmatrix} &= D(\begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix}) = D(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}) \\
&= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}) + D(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \\
&= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m_0 a \\ 0 & d(a) \end{pmatrix} + \begin{pmatrix} 0 & \delta(x) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \\
&= \begin{pmatrix} 0 & \delta(x)a + xd(a) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus $\delta(xa) = \delta(x)a + xd(a)$ and so $\delta$ is a $d$-derivation. It is clear that $D$ is inner if and only if $d$ is inner and, using Proposition 2.5, the latter holds if and only if $\delta$ is a generalized inner derivation. $\square$

**Theorem 2.10.** *Let $A$ be a unital Banach algebra, $M$ be a unital Banach right $A$-module and* $\mathcal{T} = \begin{pmatrix} \mathcal{B}(M) & M \\ 0 & A \end{pmatrix}$. *Then* $H^1(\mathcal{T},\mathcal{T}) \cong GH^1(M,M)$

**Proof.** Let $\Psi : GZ^1(M,M) \to H^1(\mathcal{T},\mathcal{T})$ be defined by

$$\Psi(\delta) = [D^\delta],$$

where $[D^\delta]$ represents the equivalence class of $D^\delta$ in $H^1(\mathcal{T},\mathcal{T})$. Clearly $\Psi$ is linear. We shall show that $\Psi$ is surjective. To end this, we assume that $D$ is a bounded derivation on $\mathcal{T}$. Let $\delta$, $d$, $\Delta_\delta$ and $m_0 \in M$ be as in the Theorem 2.9. Then

$$\begin{aligned}
(D - D^\delta)\begin{pmatrix} T & x \\ 0 & a \end{pmatrix} &= \begin{pmatrix} \Delta_\delta(T) & \delta(x) + m_0 a - T.m_0 \\ 0 & d(a) \end{pmatrix} \\
&\quad - \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix} \\
&= \begin{pmatrix} 0 & m_0 a - T.m_0 \\ 0 & 0 \end{pmatrix} \\
&= D_{\begin{pmatrix} 0 & -m_0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} T & x \\ 0 & a \end{pmatrix}.
\end{aligned}$$

So $[D] = [D^\delta] = \Psi(\delta)$ and thus $\Psi$ is surjective. Therefore $H^1(\mathcal{T},\mathcal{T}) \cong GZ^1(M,M)/Ker(\Psi)$. Note that $\delta \in Ker(\Psi)$ if and only if $D^\delta$ is inner derivation on $\mathcal{T}$. Hence $Ker(\Psi) = GN^1(M,M)$, by Theorem 2.8. Thus $H^1(\mathcal{T},\mathcal{T}) \cong GH^1(M,M)$. □

**Example 2.11.** Suppose that $A$ is unital and $M = A$. Then $\mathcal{B}(A) = A$ and so $GH^1(A,A) \cong H^1(\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}, \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}) = H^1(A,A)$, by Proposition 4.4 of [9]. In particular, every generalized derivation on a unital commutative semisimple Banach algebra [21], a unital simple $C^*$-algebra [20], or a von Neumann algebra [14] is generalized inner.

We have investigated the interrelation between generalized derivations on a Banach algebra and its ordinary derivations. We also studied generalized derivations on a Banach module in virtue of derivations on its associated triangular Banach algebra. Thus, we established a link between two interesting research areas: Banach algebras and triangular algebras.

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