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MULTI-VALUED OPERATORS WITH RESPECT wt-DISTANCE ON METRIC TYPE SPACES

M. DEMMA, R. SAADATI* AND P. VETRO

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Abstract. Recently, Hussain et al., discussed the concept of wt-distance on a metric type space. In this paper, we prove some fixed point theorems for classes of contractive type multi-valued operators, by using wt-distances in the setting of a complete metric type space. These results generalize a result of Feng and Liu on multi-valued operators.
Keywords: Multivaled operator, Pompeiu-Hausdorff generalized metric type, fixed point theorem.
MSC(2010): Primary: 47H10; Secondary: 54E40, 54E35, 54H25.

1. Introduction and preliminaries

The source of metric fixed point theory for self-mappings is the contraction mapping principle, presenteded in Banach's Ph.D. dissertation, and later published in 1922 [4]. For multi-valued operators the fundamental result is due to Nadler [27], which extended the contraction mapping principle from a single-valued mapping to a multi-valued operator. This fundamental result was largely applied in dealing with various theoretical and practical problems, arising in a number of branches of mathematics. This potentiality attracted many researchers and hence the literature has reached fixed point results, see for example [1, 8, 10, 12, 22, 28–31].

In this exciting context, Bakhtin [3] and Czerwik [11] developed the concept of b-metric spaces and proved some fixed point theorems for single-valued and multi-valued operators in b-metric spaces. Since then, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric and cone b-metric spaces (see [2, 5, 6, 13, 15–17, 23, 26, 32, 33] and references therein).

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Successively, this notion has been reintroduced by Khamsi [24], Khamsi and Hussain [25], with the name of metric-type spaces. In the literature, there are a lot of consequences of this study in metric and cone metric type spaces, see for example [10, 18, 20, 21, 24, 25].

Very recently, Feng and Liu [12] discussed the existence of fixed points for multi-valued operators in the classical setting of metric spaces. Precisely, they proved fixed point theorems, which generalize known results in the literature, by using a suitable semi-continuous function. Successively, Chifu and Petruşel [7] gave a local version of the main result in [12]. Moreover, some very recent results for Feng and Liu type multi-valued operators appeared in [35], with respect to partial metric spaces, and in [9], with respect to metric type spaces.

In view of the above considerations, we investigate the possibility to extend the results in [7, 9, 12, 35] to the setting of metric type spaces endowed with a wt-distance. Also, our results generalize and complement well known results in the literature. An example is given to demonstrate the usefulness of our results over the existing results in metric spaces.

Now, we recall some definitions and some results needed in the sequel.

Definition 1.1. Let X be a nonempty set. A metric type on X is a function $D : X \times X \to [0, +\infty)$ which satisfies the following conditions:

(1) $D(x, y) = 0$ if and only if $x = y$;
(2) $D(x, y) = D(y, x)$, for all $x, y \in X$;
(3) $D(x, y) \leq K(D(x, z) + D(z, y))$, for all $x, y, z \in X$, for some constant $K \geq 1$.

A triplet (X, D, K) is called a metric type space.

Definition 1.2. Let (X, D, K) be a metric type space.

(1) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to+\infty} D(x_n, x) = 0$.
(2) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to+\infty} D(x_n, x_m) = 0$.
(3) (X, D, K) is complete if and only if any Cauchy sequence in X is convergent.

The following are examples of metric type spaces.

Example 1.3. Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 dx < \infty.$$

As usual, we identify two functions if they coincide almost everywhere. Define $D : X \times X \to [0, +\infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx.$$

Then D satisfies the following properties:

(1) $D(f,g) = 0$ if and only if $f = g$;
(2) $D(f,g) = D(g,f)$, for all $f, g \in X$;
(3) $D(f,g) \leq 2(D(f,h) + D(h,g))$, for all functions $f, g, h \in X$.

Thus $(X, D, 2)$ is a metric type space.

Example 1.4. Let $D : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be defined by:

$$D(x,y) = |x-y|^2 \quad \text{for all } x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, D, 2)$ is a metric type spaces.

Definition 1.5. Let (X, D, K) be a metric type space. A subset $A \subset X$ is said to be open if and only if for any $a \in A$, there exists $\varepsilon > 0$ such that the open ball $B(a,\varepsilon) = \{b \in X : D(a,b) < \varepsilon\}$ is contained in A. The family of all open subsets of X will be denoted by τ.

Theorem 1.6. ([25]) *τ defines a topology on (X, D, K).*

Theorem 1.7. ([25]) *Let (X, D, K) be a metric type space and τ be the topology defined above. Then for any nonempty subset $A \subset X$ we have*

(1) *A is closed if and only if for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$;*
(2) *if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\varepsilon > 0$, we have*

$$B(a,\varepsilon) \cap A \neq \emptyset.$$

Corollary 1.8. *Every closed subset of a complete metric type space is complete.*

2. *wt*-distance

Hussain, Saadati and Agarwal [17] introduced in 2014, the concept of *wt*-distance on a metric type space and proved some fixed point theorems. In this section, we recall the definition and some examples of *wt*-distance and we state a lemma which we will use in the main section of this work.

Definition 2.1. Let (X, D, K) be a metric type space. Then a function $P : X \times X \to [0, +\infty)$ is called a *wt*–distance on X if the following are satisfied:

(a) $P(x,z) \leq K(P(x,y) + P(y,z))$ for all $x, y, z \in X$;
(b) for any $x \in X$, $P(x, \cdot) : X \to [0, +\infty)$ is K-lower semi-continuous;
(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z,x) \leq \delta$ and $P(z,y) \leq \delta$ imply $D(x,y) \leq \varepsilon$.

Let us recall that a real-valued function f defined on a metric type space X is said to be K-lower semi-continuous at a point x_0 in X if either $\liminf_{n\to+\infty} f(x_n) = +\infty$ or $f(x_0) \leq \liminf_{n\to+\infty} K f(x_n)$, whenever $\{x_n\} \subset X$ and $x_n \to x_0$ (see [19]).

Let us give some examples of *wt*–distance.

Example 2.2. ([17]) Let (X, D, K) be a metric type space. Then the metric D is a wt–distance on X.

Proof. Conditions (a) and (b) are obvious. We show that (c) holds. Then, for any $\varepsilon > 0$, we put $\delta = \frac{\varepsilon}{2K}$, and hence we have that $D(z, x) \leq \delta$ and $D(z, y) \leq \delta$ imply $D(x, y) \leq \varepsilon$. □

Example 2.3. ([17]) Consider the metric type space $(\mathbb{R}, D, 2)$, where $D(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Then the function $P : X \times X \to [0, +\infty)$ defined by $P(x, y) = |x|^2 + |y|^2$ for every $x, y \in X$ is a wt–distance on X.

Proof. Conditions (a) and (b) are obvious. We show that (c) holds. Then, for any $\varepsilon > 0$, we put $\delta = \frac{\varepsilon}{4}$ so that we have

$$D(x, y) = (x - y)^2 \leq 2|x|^2 + 2|y|^2 \leq 2P(z, x) + 2P(z, y) \leq 2\delta + 2\delta = \varepsilon.$$

□

Example 2.4. ([17]) Consider the metric type space $(\mathbb{R}, D, 2)$, where $D(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Then the function $P : X \times X \to [0, +\infty)$ defined by $P(x, y) = |y|^2$ for every $x, y \in X$ is a wt–distance on X.

Proof. Conditions (a) and (b) are obvious. We show that (c) holds. Then, for any $\varepsilon > 0$, we put $\delta = \frac{\varepsilon}{4}$ and hence we have

$$D(x, y) = (x - y)^2 \leq 2|x|^2 + 2|y|^2 = 2P(z, x) + 2P(z, y) \leq 2\delta + 2\delta = \varepsilon.$$

□

Lemma 2.5. ([17]) *Let (X, D, K) be a metric type space and $P : X \times X \to [0, +\infty)$ be a wt–distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to zero, and let $x, y, z \in X$. Then the following hold:*

(1) *If $P(x_n, y) \leq \alpha_n$ and $P(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $P(x, y) = 0$ and $P(x, z) = 0$, then $y = z$;*
(2) *if $P(x_n, y_n) \leq \alpha_n$ and $P(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $D(y_n, z) \to 0$;*
(3) *if $P(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *if $P(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

3. Main results

Let (X, D, K) be a metric type space. We will use the following notation:

(i) $N(X)$ denotes the set of all nonempty subsets of X;
(ii) $C(X)$ denotes the set of all nonempty closed subsets of X;
(iii) $CB(X)$ denotes the set of all nonempty bounded and closed subsets of X.

For $A, B \in C(X)$, define

$$H(A,B) = \begin{cases} \max\{\delta(A,B), \delta(B,A)\} & \text{if there exists,} \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\delta(A,B) = \sup\{D(a,B) : a \in A\}, \qquad \delta(B,A) = \sup\{D(b,A) : b \in B\},$$

with

$$D(a,C) = \inf\{D(a,x) : x \in C\}.$$

Note that H is called the Pompeiu-Hausdorff generalized metric type induced by the metric type D.

Let $A : X \to C(X)$ be a multi-valued operator. The graph of A is the subset $\{(x,y) : x \in X, y \in A(x)\}$ of $X \times X$; we denote the graph of A by $G(A)$. Then A is a closed multi-valued operator if the graph $G(A)$ is a closed subset of $X \times X$.

Definition 3.1. Let (X, D, K) be a metric type space. Assume that $A : X \to N(X)$ is a multi-valued operator and $P : X \times X \to [0, +\infty)$ is a wt-distance on (X, D, K). Define the function $\Phi : X \times X \to [0, +\infty)$ by

$$\Phi(x, A(x)) = \inf\{P(x,y) : y \in A(x)\}.$$

For a positive constant $b \in (0,1)$ define the set $I_b^x \subset X$ as follows:

$$I_b^x = \{y \in A(x) : \; bP(x,y) \leq \Phi(x, A(x))\}.$$

Now, inspired by [9, 12, 35], we will present a fixed point theorem for multi-valued operators on a complete metric type space endowed with a wt-distance. Our results generalize and extend some recent results presented in [12, 14, 34].

Theorem 3.2. *Let (X, D, K) be a complete metric type space, $A : \; X \to C(X)$ a multi-valued operator, $P : X \times X \to [0, +\infty)$ a wt-distance on X and $b \in (0,1)$. Suppose that there exists $c \in (0,1)$, with $cb^{-1} \in [0, K^{-1})$, such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$cP(x,y) \geq \Phi(y, A(y)). \tag{3.1}$$

If one of the following assertions holds:

(i) *$\Phi(x, A(x)) = 0$ if there exists a sequence $\{x_n\} \subset X$ such that $\Phi(x_n, A(x_n)) \to 0$;*

(ii) *the function Φ is K-lower semi-continuous;*

(iii) *for every $y \in X$ with $y \notin A(y)$, we have*

$$\inf_{x \in X}\{P(x,y) + \Phi(x, A(x))\} > 0;$$

(iv) *A is a closed operator,*

then A has a fixed point in X.

Proof. Since $A(x) \in C(X)$ for any $x \in X$, I_b^x is nonempty for any constant $b \in (0,1)$. Thus for any initial point $x_0 \in X$, there is $x_1 \in I_b^{x_0}$ such that

$$cP(x_0, x_1) \geq \Phi(x_1, A(x_1)).$$

If $x_1 = x_0$ or $x_1 \in A(x_1)$, then x_1 is a fixed point for A and the existence of a fixed point is proved. Now, we assume that $x_1 \neq x_0$ and $x_1 \notin A(x_1)$, then there is $x_2 \in I_b^{x_1}$ such that

$$cP(x_1, x_2) \geq \Phi(x_2, A(x_2)).$$

If $x_2 = x_1$ or $x_2 \in A(x_2)$, then x_2 is a fixed point for A and the existence of a fixed point is proved. Next, we assume that $x_2 \neq x_1$ and $x_2 \notin A(x_2)$. Proceeding in this way, we obtain an iterative sequence $\{x_n\}$ where $x_{n+1} \in I_b^{x_n}$, $x_n \neq x_{n+1}$ and $x_n \notin A(x_n)$ such that

$$cP(x_n, x_{n+1}) \geq \Phi(x_{n+1}, A(x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Since $x_{n+1} \in I_b^{x_n}$, we have

$$bP(x_n, x_{n+1}) \leq \Phi(x_n, A(x_n)) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Form (3.2) and (3.3), we have

$$\frac{c}{b}\Phi(x_n, A(x_n)) \geq \Phi(x_{n+1}, A(x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Then

$$\left(\frac{c}{b}\right)^n \Phi(x_0, A(x_0)) \geq \Phi(x_n, A(x_n)) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

From (3.5), since $c < b$, we deduce that the sequence $\{\Phi(x_n, A(x_n))\}$ converges to 0. On the other hand, by (3.2) and (3.3), we obtain

$$\begin{aligned} \frac{c}{b} bP(x_n, x_{n+1}) &\leq \frac{c}{b}\Phi(x_n, A(x_n)) \\ &\leq \frac{c^2}{b} P(x_{n-1}, x_n) \end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N} \cup \{0\}$, that is,

$$P(x_n, x_{n+1}) \leq \frac{c}{b} P(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

Then, for each $n \in \mathbb{N}$, we have

$$P(x_n, x_{n+1}) \leq \left(\frac{c}{b}\right)^n P(x_0, x_1). \tag{3.8}$$

Now, let $s = cb^{-1}$, for $m, n \in \mathbb{N}$ with $m > n$ we successively have

$$\begin{aligned} P(x_n, x_m) &\leq KP(x_n, x_{n+1}) + K^2 P(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + K^{m-n-1}[P(x_{m-2}, x_{m-1}) + P(x_{m-1}, x_m)] \\ &\leq s^n KP(x_0, x_1) + \cdots + s^{m-1} K^{m-n-1} P(x_0, x_1) \\ &\leq Ks^n(1 + Ks + (Ks)^2 + \cdots) P(x_0, x_1). \end{aligned}$$

Since $Ks < 1$, from the previous inequality, for $m, n \in \mathbb{N}$ with $m > n$ we obtain

$$P(x_n, x_m) \leq \frac{Ks^n}{1-Ks} P(x_0, x_1). \tag{3.9}$$

From (3.9) and Lemma 2.5 (3), since $\frac{Ks^n}{1-Ks} \to 0$ as $n \to +\infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, D, K). Since X is a complete metric type space, there exists $z \in X$ such that the sequence $\{x_n\}$ converges to z. We claim that z is a fixed point of A.

Case 1. The assertion (i) holds. Since the sequence $\{\Phi(x_n, A(x_n))\}$ converges to 0, we have

$$\Phi(z, A(z)) = 0. \tag{3.10}$$

From the definition of function Φ and (3.10), we deduce that for all $n \in \mathbb{N}$ there exists $y_n \in A(z)$ such that

$$P(z, y_n) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

Again, from (3.9) and the K-lower semi-continuity of P, we get

$$P(x_n, z) \leq \frac{K^2 s^n}{1-Ks} P(x_0, x_1) \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

Now, (3.11) and (3.12) imply

$$P(x_n, y_n) \leq K(P(x_n, z) + P(z, y_n)) \leq K\left(\frac{K^2 s^n}{1-Ks} P(x_0, x_1) + \frac{1}{n}\right) \tag{3.13}$$

for all $n \in \mathbb{N}$. By Lemma 2.5 (2), (3.12) and (3.13) we deduce that

$$D(y_n, z) \to 0. \tag{3.14}$$

From $A(x) \in C(X)$ and (3.14) we have that $z \in A(z)$. Hence, A has a fixed point in X.

Case 2. If (ii) holds, then (i) holds and so A has a fixed point.

Case 3. The assertion (iii) holds. Suppose to the contrary that $z \notin A(z)$. Now, by condition (iii), we have

$$\begin{aligned}
0 &< \inf_{x \in X} \{P(x, z) + \Phi(x, A(x))\} \\
&\leq \inf_{n \in \mathbb{N}} \{P(x_n, z) + \Phi(x_n, A(x_n))\} \\
&\leq \inf_{n \in \mathbb{N}} \{P(x_n, z) + P(x_{n-1}, x_n)\} \\
&\leq \inf_{n \in \mathbb{N}} \left\{\frac{K^2 s^n}{1-Ks} P(x_0, x_1) + s^{n-1} P(x_0, x_1)\right\} \\
&= 0
\end{aligned}$$

which is a contradiction and hence $z \in A(z)$, that is, z is a fixed point of A.

Case 4. The assertion (iv) holds. From the fact that $x_{n+1} \in A(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $(x_n, x_{n+1}) \to (z, z)$, we get $z \in A(z)$, that is, z is a fixed point of A.

□

Now, we show that Theorem 3.2 is a generalization of the following version of Nadler's fixed point theorem in metric type spaces.

Theorem 3.3. *Let (X, D, K) be a complete metric type space and let $A : X \to C(X)$ be a multi-valued operator such that for all $x, y \in X$ we have $H(Ax, Ay) \leq cD(x, y)$, where $c \in (0, K^{-1})$, then A has a fixed point.*

Proof. We have to show that the contractive condition (3.1) and condition (i) of Theorem 3.2 are satisfied with respect to *wt*-distance D. Firstly, we prove that A satisfies condition (3.1) of Theorem 3.2. Indeed, for all $x \in X$ and $y \in A(x)$, we write

$$\Phi(y, A(y)) = D(y, A(y)) \leq H(A(x), A(y)) \leq cD(x, y)$$

and hence the assertion holds trivially for each $x \in X$ and $y \in I_b^x$ with $b \in (0, 1)$ such that $c < bK^{-1}$. It would remain to show that Φ satisfies condition (i) of Theorem 3.2. Indeed, let $\{x_n\} \subset X$ be a sequence such that $x_n \to x \in X$ and $\Phi(x_n, A(x_n)) \to 0$. For every $n \in \mathbb{N}$, we choose $y_n \in A(x_n)$ such that

$$D(x_n, y_n) \leq \Phi(x_n, A(x_n)) + \frac{1}{n}.$$

Clearly, we have

$$\begin{aligned}\Phi(x, A(x)) &\leq K^2 D(x, x_n) + K^2 D(x_n, y_n) + KH(A(x_n), A(x)) \\ &\leq K^2 D(x, x_n) + K^2 D(x_n, y_n) + KcD(x_n, x).\end{aligned}$$

Letting $n \to +\infty$, we get that $\Phi(x, A(x)) = 0$. This completes the proof. □

The following theorem is a generalization of Theorem 3.2.

Theorem 3.4. *Let (X, D, K) be a complete metric type space, $A : X \to C(X)$ a multi-valued operator, $P : X \times X \to [0, +\infty)$ a wt-distance on X and $b \in (0, 1)$. Suppose that there exist $a, c \in (0, 1)$, with $b - K(ab + c) > 0$, such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$a\,\Phi(x, A(x)) + cP(x, y) \geq \Phi(y, A(y)).$$

If one of the following assertions holds:

(i) $\Phi(x, A(x)) = 0$ *if there exists a sequence $\{x_n\} \subset X$ such that* $\Phi(x_n, A(x_n)) \to 0$;

(ii) *the function Φ is K-lower semi-continuous;*

(iii) *for every $y \in X$ with $y \notin A(y)$, we have*

$$\inf_{x \in X} \{P(x, y) + \Phi(x, A(x))\} > 0;$$

(iv) *A is a closed operator,*

then A has a fixed point in X.

Proof. Since $A(x) \in C(X)$ for any $x \in X$, I_b^x is nonempty for any constant $b \in (0,1)$. Thus for any initial point $x_0 \in X$, there is $x_1 \in I_b^{x_0}$ such that

$$a\,\Phi(x_0, A(x_0)) + cP(x_0, x_1) \geq \Phi(x_1, A(x_1)).$$

If $x_1 = x_0$ or $x_1 \in A(x_1)$, then x_1 is a fixed point for A and the existence of a fixed point is proved. Now, we assume that $x_1 \neq x_0$ and $x_1 \notin A(x_1)$, then there is $x_2 \in I_b^{x_1}$ such that

$$a\,\Phi(x_1, A(x_1)) + cP(x_1, x_2) \geq \Phi(x_2, A(x_2)).$$

If $x_2 = x_1$ or $x_2 \in A(x_2)$, then x_2 is a fixed point for A and the existence of a fixed point is proved. Next, we assume that $x_2 \neq x_1$ and $x_2 \notin A(x_2)$. Proceeding in this way, we obtain an iterative sequence $\{x_n\}$ where $x_{n+1} \in I_b^{x_n}$, $x_n \neq x_{n+1}$ and $x_n \notin A(x_n)$ such that

$$a\,\Phi(x_n, A(x_n)) + cP(x_n, x_{n+1}) \geq \Phi(x_{n+1}, A(x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.15}$$

Now, we show that the sequence $\{x_n\}$ is Cauchy. Since $x_{n+1} \in I_b^{x_n}$, we have

$$bP(x_n, x_{n+1}) \leq \Phi(x_n, A(x_n)) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.16}$$

Form (3.15) and (3.16), we have

$$\frac{ab + c}{b}\Phi(x_n, A(x_n)) \geq \Phi(x_{n+1}, A(x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.17}$$

Now, let $s = (ab + c)b^{-1}$. From (3.17), we obtain

$$s^n\Phi(x_0, A(x_0)) \geq \Phi(x_n, A(x_n)) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.18}$$

From (3.18), since $s < 1$, we deduce that the sequence $\{\Phi(x_n, A(x_n))\}$ converges to 0.

On the other hand, by (3.16) and (3.17), we have

$$P(x_n, x_{n+1}) \leq \frac{1}{b}\Phi(x_n, A(x_n)) \leq \frac{s^n}{b}\Phi(x_0, A(x_0)) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.19}$$

Now, since $Ks < 1$, for $m, n \in \mathbb{N}$ with $m > n$ we successively have

$$\begin{aligned} P(x_n, x_m) &\leq KP(x_n, x_{n+1}) + K^2P(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + K^{m-n-1}[P(x_{m-2}, x_{m-1}) + P(x_{m-1}, x_m)] \\ &\leq s^nKP(x_0, x_1) + \cdots + s^{m-1}K^{m-n-1}\Phi(x_0, A(x_0)) \\ &\leq \frac{Ks^n}{1 - Ks}\Phi(x_0, A(x_0)). \end{aligned}$$

From $\dfrac{Ks^n}{1 - sK} \to 0$ as $n \to +\infty$, by Lemma 2.5 (3), we conclude that $\{x_n\}$ is a Cauchy sequence in (X, D, K). Since X is a complete metric type space, there exists $z \in X$ such that the sequence $\{x_n\}$ converges to z. Finally, one can proceed as in the proof of Theorem 3.2 to prove that z is a fixed point of A. □

Example 3.5. Let $a, b, c, h \in [0, 1)$ and $K \geq 1$ such that $Kh \leq a+c < b$. Now, consider the complete metric type space $(X, D, 2)$ where $X = \{0, 1\} \cup \{h^n : n \in \mathbb{N}\}$ and $D(x, y) = (x - y)^2$ for all $x, y \in X$. Also, we consider on X a wt-distance defined by $P(x, y) = y^2$ for every $x, y \in X$. Let $A : X \to C(X)$ be a multi-valued operator defined by

$$A(x) = \begin{cases} \{0, h\} & \text{if } x = 0, \\ \{h^n, 1\} & \text{if } x = h^{n-1} \text{ for all } n \in \mathbb{N}. \end{cases}$$

If we choose $y \in I_b^x$ as follows: $y = 0$ if $x = 0$ and $y = h^n$ if $x = h^{n-1}$, then we deduce that

$$bP(x, y) \leq \Phi(x, A(x)) \quad \text{for all } x \in X$$

and

$$\Phi(y, A(y)) \leq a\,\Phi(x, A(x)) + cP(x, y) \quad \text{for all } x \in X.$$

At last, for every $y \in X \setminus \{0, 1\}$, that is, for every $y \in X$ such that $y \notin A(y)$, we have

$$\inf_{x\in X}\{P(x, y) + \Phi(x, A(x))\} \geq \inf_{x\in X}\{P(x, y)\} = y^2 > 0.$$

Hence all conditions of Theorem 3.4 hold and the multi-valued operator A has a fixed point. In this example $x = 0$ and $x = 1$ are fixed points. Note that the multi-valued operator A does not satisfy the hypothesis of Nadler's theorem in the setting of metric type space. In fact, for $x = h$ and $y = h^2$, we have

$$H(A(h), A(h^2)) = h^4(1 - h)^2 = d(h, h^2).$$

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