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σ-DERIVATIONS IN BANACH ALGEBRAS

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ABSTRACT. We introduce the notions of (inner) σ-derivation, (inner) σ-endomorphism and one-parameter group of σ-endomorphisms (σ-dynamics) on a Banach algebra. We associate a σ-derivation to any σ-dynamics as its "σ-infinitesimal generator". We show that the σ-infinitesimal generator of a σ-dynamics of inner σ-endomorphisms is an inner σ-derivation and we study the reverse statement. We also establish a generalized Leibniz formula and generalize Kleinenckr–Sirokov theorem for σ-derivations under certain conditions.

1. **Introduction**

Let $\mathcal{A}$ be a Banach algebra. Recall that a derivation d defined on a (dense) subalgebra $\mathcal{D}$ of $\mathcal{A}$ is a linear mapping satisfying $d(ab) = d(a)b + ad(b)$, where $a, b \in \mathcal{D}$. A derivation d is said to be inner if there exists an element $u \in \mathcal{A}$ such that $d(a) = ua - au$ for all $a \in \mathcal{A}$. There are nonzero derivations defined on a commutative algebra among which we may consider the ordinary derivative $d/dt : C^1([0,1]) \to C([0,1])$, where $C^1([0,1])$ is the algebra of all continuously differentiable functions on $[0,1]$. This example gives an idea to define a derivation on a dense subalgebra of a given algebra $\mathcal{A}$.

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Derivations play essential role in some important branches of mathematics and physics such as dynamical systems. The general theory of dynamical systems is the paradigm for modeling and studying phenomena that undergo spatial and temporal evolution. The application of dynamical systems has nowadays spread to a wide spectrum of disciplines including physics, chemistry, biochemistry, biology, economy and even sociology. In particular, the theory of dynamical systems concerns the theory of derivations in Banach algebras and is motivated by questions in quantum physics and statistical mechanics, cf. [13].

It is known that the relation $TS - ST = I$ is impossible for bounded operators T and S on Banach spaces; cf. [13] and references therein. In fact the study of this relation as a special case of $T\sigma(S) - \sigma(S)T = R$, where σ is a linear mapping, leads the theory of derivations to be extensively developed.

The above considerations motivate us to generalize the notion of derivation as follow. Let $\mathcal{D}$ be a subalgebra of a Banach algebra $\mathcal{A}$ and let $\sigma, d : \mathcal{D} \to \mathcal{A}$ be linear mappings. If $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{D}$ then we say d is a σ-derivation (see [1, 2, 3, 8, 9, 11, 12] and the references therein). There are some interesting questions in this area of research, e.g. one may ask 'What are the σ-derivations of the compact operators acting on a separable Hilbert space?' The paper [4] can be a starting point for answering this question. Note that if σ is the identity map then d (and every so-called inner σ-derivation) is indeed a derivation (inner derivation, respectively) in the usual sense.

In this paper we introduce and study (inner) σ-derivations, (inner) σ-endomorphisms and one-parameter group of σ-endomorphisms (σ-dynamics). The importance of our approach is that σ is a linear mapping, not necessarily an algebra endomorphism. It is shown that the σ-infinitesimal generator of a σ-dynamics of inner σ-endomorphisms is an inner σ-derivation and the converse is true under some conditions. We give a formula for computation $d^n(ab)$, where d is a σ-derivation that is interesting in its own right. We also generalize two known theorems in the context of Banach algebras, namely the Wielandt–Wintner theorem and the Kleinecke–Shirokov theorem.

This paper is self-contained. The reader, however, is referred to [5] for details on Banach algebras and to [6, 7, 13] for more information on dynamical systems.

2. σ-dynamics

Throughout the paper $\mathcal{A}$ denotes a Banach algebra, ι is the identity operator on $\mathcal{A}$, $\mathcal{D}$ denotes a subalgebra of $\mathcal{A}$, and $\sigma, d : \mathcal{D} \to \mathcal{A}$ are linear mappings.

Definition 2.1. d is called a *σ-derivation* if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{D}$.

Example 2.2. Let σ be an arbitrary linear mapping on $\mathcal{D}$ and suppose that u is an element of $\mathcal{A}$ satisfying $u(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))u$ for all $a, b \in \mathcal{D}$. Then the mapping $d : \mathcal{D} \to \mathcal{A}$ defined by $d(a) = u\sigma(a) - \sigma(a)u$ is a σ-derivation.

The above σ-derivation is called inner. Note that if σ is an endomorphism then u can be an arbitrary element of $\mathcal{A}$.

Example 2.3. Let d be an endomorphism on $\mathcal{A}$. Then d is a $\frac{d}{2}$-derivation.

Example 2.4. Let $d, \sigma : C([0,1]) \to C([0,1])$ be defined by $\sigma(f) = \frac{f}{2}$ and $d(f) = fh_0$, respectively. Here h_0 is an arbitrary fixed element in $C([0,1])$. Then easy observations show that $\sigma(1) \neq 1$, $d(1) \neq 0$, the linear mapping σ is not endomorphism, and d is a σ-derivation.

Example 2.5. Suppose that $d, \sigma : C([0,1]) \to C([0,1])$ are defined by

$$\sigma(f)(t) = \begin{cases} \frac{1}{2}f(2t) & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2}f(1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$d(f)(t) = \begin{cases} f(2t)h_0(t) & 0 \leq t \leq \frac{1}{2} \\ f(1)h_0(\frac{1}{2}) & \frac{1}{2} \leq t \leq 1 \end{cases},$$

respectively, where h_0 is an arbitrary fixed element of $C([0,1])$. Then a straightforward verification shows that d is a σ-derivation and that no scalar multiple of σ is an endomorphism.

Definition 2.6. A linear mapping $\alpha : \mathcal{A} \to \mathcal{A}$ is called *σ-endomorphism* if $(\alpha + \sigma - \iota)(ab) - (\alpha + \sigma - \iota)(a)(\alpha + \sigma - \iota)(b) = \sigma(ab) - \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.

Note that if $\sigma = \iota$ then a σ-endomorphism is nothing more than an endomorphism on $\mathcal{A}$ in the usual sense.

Lemma 2.7. *Let α be a linear mapping on $\mathcal{A}$. Then α is a σ-endomorphism if and only if*

$$\alpha(ab) - \alpha(a)\alpha(b) = (\alpha(a) - a)(\sigma(b) - b) + (\sigma(a) - a)(\alpha(b) - b).$$

Proof. Straightforward. □

Definition 2.8. A mapping $t \in \mathbb{R} \mapsto \alpha_t \in B(\mathcal{A})$ denoted by $\{\alpha_t\}_{t\in\mathbb{R}}$ is a *one-parameter group of bounded operators on* $\mathcal{A}$ if it satisfies the following conditions:
(i) $\alpha_t\alpha_s = \alpha_{t+s}$, for all $t, s \in \mathbb{R}$,
(ii) $\alpha_0 = \iota$.

In the case that α_t's are bounded σ-endomorphisms, $\{\alpha_t\}_{t\in\mathbb{R}}$ is called a *one-parameter group of σ-endomorphisms on* $\mathcal{A}$. It is said to be *uniformly continuous* if the map $t \mapsto \alpha_t$ is continuous in the uniform topology, i.e. $\|\alpha_t - \iota\| \to 0$ as $t \to 0$. In this case $\{\mathcal{A}, \alpha\}$ is called a *σ-dynamics.*

Let $\{\mathcal{A}, \alpha\}$ be a σ-dynamics. Then for each $a \in \mathcal{A}$, if the limit of $t^{-1}(\alpha_t(a) - \iota(a))$ as t tends to 0 exists, we can define $d(a)$ to be this limit. This provides a mapping $d : \mathcal{D} \to \mathcal{A}$, where $\mathcal{D}$ is the set of all elements a in $\mathcal{A}$ for which the limit exists. The mapping d is called the *σ-infinitesimal generator of the σ-dynamics* $\{\alpha_t\}_{t\in\mathbb{R}}$.

Proposition 2.9. *Let $\{\mathcal{A}, \alpha\}$ be a σ-dynamics. Then $d = \lim\limits_{t\to 0} t^{-1}(\alpha_t(a) - a)$ is an everywhere defined σ-derivation.*

Proof. We have

$$\begin{aligned} d(ab) &= \lim_{t\to 0} t^{-1}(\alpha_t(ab) - \iota(ab)) \\ &= \lim_{t\to 0} t^{-1}((\alpha_t + \sigma - \iota)(ab) - \sigma(ab)) \\ &= \lim_{t\to 0} t^{-1}((\alpha_t + \sigma - \iota)(a)(\alpha_t + \sigma - \iota)(b) - \sigma(a)\sigma(b)) \\ &= lim_{t\to 0}(t^{-1}((\alpha_t + \sigma - \iota)(a) - \sigma(a))\sigma(b) \\ &\quad +(\alpha_t + \sigma - \iota)(a)t^{-1}((\alpha_t + \sigma - \iota)(b) - \sigma(b))) \\ &= d(a)\sigma(b) + \sigma(a)d(b). \end{aligned}$$

It follows from Proposition 3.1.1 of [6] that d is everywhere defined. $\square$

Definition 2.10. A linear mapping α is called an *inner σ-endomorphism* if there is an element $u \in \mathcal{A}$ such that $(\alpha+\sigma-\iota)(a) = e^u\sigma(a)e^{-u}, a \in \mathcal{A}$ and $u(\sigma(ab) - \sigma(a)\sigma(b)) = (\sigma(ab) - \sigma(a)\sigma(b))u, a, b \in \mathcal{A}$.

Lemma 2.11. *Each inner σ-endomorphism is indeed a σ-endomorphism.*

Proof. We have

$$\begin{aligned}
&(\alpha+\sigma-\iota)(ab) - (\alpha+\sigma-\iota)(a)(\alpha+\sigma-\iota)(b)\\
&= e^u(\sigma(ab) - \sigma(a)\sigma(b))e^{-u}\\
&= e^ue^{-u}(\sigma(ab) - \sigma(a)\sigma(b))\\
&= \sigma(ab) - \sigma(a)\sigma(b).
\end{aligned}$$

$\square$

Theorem 2.12. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter group of inner σ-endomorphisms. Then the σ-infinitesimal generator d of σ-dynamics $(\mathcal{A}, \alpha)$ is an inner σ-derivation.*

Proof. We have

$$\begin{aligned}
\lim_{t\to 0} t^{-1}(\alpha_t(a) - a) &= \lim_{t\to 0} t^{-1}((\alpha_t+\sigma-\iota)(a) - \sigma(a))\\
&= \lim_{t\to 0} t^{-1}(e^{tu}\sigma(a)e^{-tu} - \sigma(a))\\
&= \lim_{t\to 0}(ue^{tu}\sigma(a)e^{-tu} - e^{tu}\sigma(a)ue^{-tu})\\
&= u\sigma(a) + \sigma(a)u.
\end{aligned}$$

Note that we use L'Hospital's rule to get the third equality. $\square$

Lemma 2.13. *Let $d : \mathcal{A} \to \mathcal{A}$ be the inner σ-derivation $d(a) = u\sigma(a) - \sigma(a)u$. If $\sigma^2 = \sigma$ and $\sigma(au) = \sigma(a)u, \sigma(ua) = u\sigma(a)$ for all $a \in \mathcal{A}$. Then*

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}u^k\sigma(a)u^{r-k} = (-1)^r d^r(a) \tag{2.1}$$

for all $a \in \mathcal{A}, 0 \le k \le r$, where $r \ge 1$.

Proof. We use induction on r. For $r = 1$ there is nothing to do. Assume that (2.1) holds for r. We have

$$\begin{aligned}
(-1)^{r+1}d^{r+1}(a) &= (-1)^{r+1}d(d^r(a)) \\
&= -u\sigma((-1)^r d^r(a)) + \sigma((-1)^r d^r(a))u \\
&= -\sum_{k=0}^{r}(-1)^k\binom{r}{k}u^{k+1}\sigma(a)u^{r-k} \\
&\quad +\sum_{k=0}^{r}(-1)^k\binom{r}{k}u^{k}\sigma(a)u^{r-k+1} \\
&= -\sum_{k=0}^{r}(-1)^k\binom{r}{k}u^{k+1}\sigma(a)u^{r-k} \\
&\quad -\sum_{k=-1}^{r-1}(-1)^{k+1}\binom{r}{k+1}u^{k+1}\sigma(a)u^{r-k} \\
&= (-1)^{r+1}u^{r+1}\sigma(a) + \sigma(a)u^{r+1} \\
&\quad -\sum_{k=0}^{r-1}(-1)^k(\binom{r}{k}+\binom{r}{k+1})u^{k+1}\sigma(a)u^{r-k} \\
&= (-1)^{r+1}u^{r+1}\sigma(a) + \sigma(a)u^{r+1} \\
&\quad -\sum_{k=0}^{r-1}(-1)^k\binom{r+1}{k+1}u^{k+1}\sigma(a)u^{r-k} \\
&= (-1)^{r+1}u^{r+1}\sigma(a) + \sigma(a)u^{r+1} + \\
&\quad \sum_{k=1}^{r}(-1)^k\binom{r+1}{k}u^{k}\sigma(a)u^{r+1-k} \\
&= \sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}u^{k}\sigma(a)u^{r+1-k}.
\end{aligned}$$

□

Theorem 2.14. *Let $d : \mathcal{A} \to \mathcal{A}$ be the inner σ-derivation $d(a) = u\sigma(a) - \sigma(a)u$. If $\sigma^2 = \sigma$ and $\sigma(au) = \sigma(a)u, \sigma(ua) = u\sigma(a)$ for all $a \in \mathcal{A}$, then there exists a one-parameter group of operators $\{\alpha_t\}_{t\in\mathbb{R}}$ such that d is its σ-infinitesimal generator and $\alpha_t - \sigma + \iota$ is an inner σ-homomorphism for all $t \in \mathbb{R}$.*

Proof. Put $\alpha_t(a) = \sum_{n=0}^{\infty} \frac{t^n d^n(a)}{n!}$. Using Lemma 2.13 we have

$$\begin{aligned}
e^{tu}\sigma(a)e^{-tu} &= \left(\sum_{n=0}^{\infty} \frac{t^n u^n}{n!}\right)\sigma(a)\left(\sum_{m=0}^{\infty} \frac{(-t)^m u^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n(-t)^m}{n!m!} u^n\sigma(a)u^m \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{t^k(-t)^{r-k}}{k!(r-k)!} u^k\sigma(a)u^{r-k} \\
&= \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} u^k\sigma(a)u^{r-k} \\
&= \sum_{r=0}^{\infty} \frac{t^r}{r!} d^r(a) \\
&= \alpha_t(a).
\end{aligned}$$

Since $((\alpha_t - \sigma + \iota) + \sigma - \iota)(a) = \alpha_t(a) = e^{tu}\sigma(a)e^{-tu}$, we deduce that $\alpha_t - \sigma + \iota$ is an inner σ-endomorphism. Obviously $\alpha_t\alpha_s = \alpha_{t+s}$ and $\alpha_0 = \iota$. In addition, $\lim_{t\to 0}\frac{\alpha_t(a) - a}{t} = \lim_{t\to 0}\sum_{n=1}^{\infty}\frac{t^n d^n(a)}{n!} = d(a)$. $\square$

Definition 2.15. Let d be a σ-derivation. We say d *multiplizes* σ if $\sigma(ab) - \sigma(a)\sigma(b) \subseteq \ker(d)$. In this case d is called a *multiplizing* σ-derivation.

Example 2.16. Each inner σ-derivation d is multiplizing. Let $d(a) = u\sigma(a) - \sigma(a)u$ for some $u \in \mathcal{A}$. Then we have $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, and so $u\sigma(ab) - \sigma(ab)u = (u\sigma(a) - \sigma(a)u)\sigma(b) + \sigma(a)(u\sigma(b) - \sigma(b)u)$, which implies that $u(\sigma(ab) - \sigma(a)\sigma(b)) - (\sigma(ab) - \sigma(a)\sigma(b))u = 0$ or $d(\sigma(ab) - \sigma(a)\sigma(b)) = 0$. Thus d is a multiplizing σ-derivation.

Proposition 2.17. *Let $\mathcal{A}$ be an algebra with no zero divisor. Then d is a multiplizing σ-derivation if and only if $\sigma(b\sigma(ab)) = \sigma(b)\sigma^2(ab)$ for all $a, b \in \mathcal{A}$.*

Proof. For each $a, b, c \in \mathcal{D}$ we have

$$\begin{aligned} d(abc) &= d(ab)\sigma(c) + \sigma(ab)d(c) \\ &= (d(a)\sigma(b) + \sigma(a)d(b))\sigma(c) + \sigma(ab)d(c) \\ &= d(a)\sigma(b)\sigma(c) + \sigma(a)d(b)\sigma(c) + \sigma(ab)d(c). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(abc) &= d(a)\sigma(bc) + \sigma(a)d(bc) \\ &= d(a)\sigma(bc) + \sigma(a)(d(b)\sigma(c) + \sigma(b)d(c)) \\ &= d(a)\sigma(bc) + \sigma(a)d(b)\sigma(c) + \sigma(a)\sigma(b)d(c). \end{aligned}$$

Therefore,

$$d(a)(\sigma(bc) - \sigma(b)\sigma(c)) = (\sigma(ab) - \sigma(a)\sigma(b))d(c),$$

for each $a, b, c \in \mathcal{D}$. Putting $c = \sigma(ab) - \sigma(a)\sigma(b)$, we have $\sigma(ab) - \sigma(a)\sigma(b) \in \ker(d)$ if and only if $\sigma(b(\sigma(ab) - \sigma(a)\sigma(b))) - \sigma(b)\sigma(\sigma(ab) - \sigma(a)\sigma(b)) = 0$, which implies the result. □

3. Generalized Leibniz rule

In the rest of the paper we need a family of mappings $\{\varphi_{n,k}\}_{n\in N, 0\leq k\leq 2^n-1}$ to simplify the notations. We introduce these mappings by representing the natural numbers in base 2.

Let n be a natural number and $0 \leq k \leq 2^n - 1$. Note that $2^n - 1 = (\underbrace{1\dots1}_{n\ times})_2$ and each $0 \leq k \leq 2^n - 1$ has at most n digits in base 2. Now assume that $\varphi_{n,k}$ is the mapping derived from writing k in base 2 with exactly n digits and put d for 1's and σ for 0's.

To illustrate the mappings $\varphi_{n,k}$'s, let us give an example. Let $n = 5$ and $k = 11$. Then we can write $k = (01011)_2$ and so $\varphi_{5,11} = \sigma d\sigma dd = \sigma d\sigma d^2$.

Lemma 3.1 *Let n be a natural number and $0 \leq k \leq 2^n - 1$. Then*

(i) $d\varphi_{n,k} = \varphi_{n+1,2^n+k}$,
(ii) $d\varphi_{n,2^n-1-k} = \varphi_{n+1,2^{n+1}-1-k}$,
(iii) $\sigma\varphi_{n,k} = \varphi_{n+1,k}$,
(iv) $\sigma\varphi_{n,2^n-1-k} = \varphi_{n+1,2^{n+1}-1-(2^n+k)}$.

Proof. Assume that $k = (c_n \dots c_1)_2$.

(i) $d\varphi_{n,k} = \varphi_{n+1,(1c_n\dots c_1)_2} = \varphi_{n+1,2^n+k}$.

(ii) $2^n - 1 - k = (\bar{c_n}\dots\bar{c_1})_2$, where $\bar{c_i} + c_i = 1$, since $(2^n - 1 - k) + k = 2^n = (1\dots 1)_2$. Thus we infer that $d\varphi_{n,2^n-1-k} = d\varphi_{n,(\bar{c_n}\dots\bar{c_1})_2} = \varphi_{n+1,(1\bar{c_n}\dots\bar{c_1})_2} = \varphi_{n+1,2^n+(2^n-1-k)} = \varphi_{n+1,2^{n+1}-1-k}$.

(iii) $\sigma\varphi_{n,k} = \varphi_{n+1,(0c_n\dots c_1)_2} = \varphi_{n+1,k}$.

(iv) $\sigma\varphi_{n,2^n-1-k} = \varphi_{n+1,(0\bar{c_n}\dots\bar{c_1})_2} = \varphi_{n+1,2^n-1-k} = \varphi_{n+1,2^{n+1}-1-(2^n+k)}$. $\square$

Theorem 3.2. *For each $a, b \in \mathcal{D}$,*

$$d^n(ab) = \sum_{k=0}^{2^n-1} \varphi_{n,k}(a)\varphi_{n,2^n-1-k}(b). \tag{3.1}$$

Proof. We prove the assertion by induction on n. For $n = 1$ we have

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b) = \varphi_{1,1}(a)\varphi_{1,0}(b) + \varphi_{1,0}(a)\varphi_{1,1}(b).$$

Now suppose (3.1) is true for n. By Lemma 3.1 we obtain

$$\begin{aligned} d^{n+1}(ab) &= d(d^n(ab)) = d(\sum_{k=0}^{2^n-1} \varphi_{n,k}(a)\varphi_{n,2^n-1-k}(b)) \\ &= \sum_{k=0}^{2^n-1} d(\varphi_{n,k}(a)\varphi_{n,2^n-1-k}(b)) \\ &= \sum_{k=0}^{2^n-1} d(\varphi_{n,k}(a))\sigma(\varphi_{n,2^n-1-k}(b)) + \sigma(\varphi_{n,k}(a))d(\varphi_{n,2^n-1-k}(b)) \\ &= \sum_{k=0}^{2^n-1} \varphi_{n+1,2^n+k}(a)\varphi_{n+1,2^{n+1}-1-(2^n+k)}(b) \\ &\quad + \sum_{k=0}^{2^n-1} \varphi_{n+1,k}(a)\varphi_{n+1,2^{n+1}-1-k}(b) \\ &= \sum_{l=2^n}^{2^{n+1}-1} \varphi_{n+1,l}(a)\varphi_{n+1,2^{n+1}-1-l}(b) \\ &\quad + \sum_{l=0}^{2^n-1} \varphi_{n+1,l}(a)\varphi_{n+1,2^{n+1}-1-l}(b) \\ &= \sum_{l=0}^{2^{n+1}-1} \varphi_{n+1,l}(a)\varphi_{n+1,2^{n+1}-1-l}(b). \end{aligned}$$

$\square$

Example 3.3. As an illustration, consider the case $n = 3$. We have

$$\begin{aligned} d^3(ab) &= \varphi_{3,0}(a)\varphi_{3,7}(b) + \varphi_{3,1}(a)\varphi_{3,6}(b) \\ &\quad +\varphi_{3,2}(a)\varphi_{3,5}(b) + \varphi_{3,3}(a)\varphi_{3,4}(b) \\ &\quad +\varphi_{3,4}(a)\varphi_{3,3}(b) + \varphi_{3,5}(a)\varphi_{3,2}(b) \\ &\quad +\varphi_{3,6}(a)\varphi_{3,1}(b) + \varphi_{3,7}(a)\varphi_{3,0}(b) \\ &= \sigma^3(a)d^3(b) + \sigma^2 d(a)d^2\sigma(b) \\ &\quad +\sigma d\sigma(a)d\sigma d(b) + \sigma d^2(a)d\sigma^2(b) \\ &\quad +d\sigma^2(a)\sigma d^2(b) + d\sigma d(a)\sigma d\sigma(b) \\ &\quad +d^2\sigma(a)\sigma^2 d(b) + d^3(a)\sigma^3(b). \end{aligned}$$

Corollary 3.4. *If $d\sigma = \sigma d = d$, then*

$$d^n(ab) = \sum_{r=0}^{n} \binom{n}{r} d^r(a)d^{n-r}(b),$$

for each $a, b \in \mathcal{D}$.

Proof. If the representation of k to base 2 has r 1's, then $\varphi_{n,k} = d^r$. But we have $\binom{n}{r}$ terms in the summand with exactly r 1's in the representation of k. $\square$

Note that by putting $\sigma = \iota$, we get the known results concerning ordinary derivations.

Our next result generalizes Theorem 3.2. As before, let k be represented as $(c_n \ldots c_1)_2$ to base 2. If the number of 1's in this representation is r_k, we can construct 2^{r_k} numbers t with the property that 1 occurs in t only if the corresponding position at the representation of k is 1. More precisely, we can write

$$T_k = \{t = (d_n \ldots d_1)_2 \ : \quad d_i = 1 \text{ implies } c_i = 1 \text{ for each } 1 \leq i \leq n\}.$$

To illustrate T_k's, let $k = 19 = (10011)_2$. Then

$$\begin{aligned} T_{19} &= \{(00000)_2, (00001)_2, (00010)_2, (00011)_2, \\ &\quad (10000)_2, (10001)_2, (10010)_2, (10011)_2\} \\ &= \{0, 1, 2, 3, 16, 17, 18, 19\}. \end{aligned}$$

Here T_k has $2^3 = 8$ elements.

Lemma 3.5 *Suppose n, k are two natural numbers. Then*

(i) $T_0 = \{0\}, T_{2^n} = \{0, 2^n\}$ *and* $T_{2^n-1} = \{0, 1, 2, \dots, 2^n - 1\}$,

(ii) $T_k = T_{k-2^n} \cup (2^n + T_{k-2^n}) = T_{k-2^n} \cup \{2^n + t \ : \ t \in T_{k-2^n}\}$, *provided that* $2^n \leq k \leq 2^{n+1} - 1$.

Proof. (i) This is clear.

(ii) Let $k = (c_n \dots c_1)_2$. We have $c_n = 1$, since $k \geq 2^n$. Let $(d_n \dots d_1)_2 \in T_k$. If $d_n = 0$ then $(d_n \dots d_1)_2 = (d_{n-1} \dots d_1)_2 \in T_{k-2^n}$, and if $d_n = 1$ then $(d_n \dots d_1)_2 = 2^n + (d_{n-1} \dots d_1)_2$, where $(d_{n-1} \dots d_1)_2 \in T_{k-2^n}$. □

Definition 3.6. A linear mapping σ is called *a semi-endomorphism* if

$$\sigma(a\sigma(b)) = \sigma(a)\sigma^2(b), \qquad \sigma(ad(b)) = \sigma(a)\sigma(d(b))$$

for all $a, b \in \mathcal{D}$. Obviously any endomorphism is semi-endomorphism.

Theorem 3.7. *Let σ be an endomorphism. Then for each $n, k \in \mathbb{N}$ with $0 \leq k \leq 2^n - 1$ and $a, b \in \mathcal{D}$ we have*

$$\varphi_{n,k}(ab) = \sum_{\ell \in T_k} \varphi_{n,\ell}(a)\varphi_{n,k-\ell}(b). \tag{3}$$

Proof. We use induction on n. For $n = 1$, if $k = 0$ then (3) is clear and if $k = 1$ then $T_1 = \{0, 1\}$ and

$$\varphi_{1,1}(ab) = d(ab) = d(a)\sigma(b) + \sigma(a)d(b) = \varphi_{1,1}(a)\varphi_{1,0}(b) + \varphi_{1,0}(a)\varphi_{1,1}(b).$$

Now suppose that (3) is true for n. For $0 \leq k = (c_{n+1}c_n \dots c_1)_2 \leq 2^{n+1}$, two cases occur.

Case1. $1 \leq k < 2^n$.

In this case, $c_{n+1} = 0$ and $\varphi_{n+1,k} = \sigma\varphi_{n,k}$. Hence

$$\begin{aligned}
\varphi_{n+1,k}(ab) &= \sigma\varphi_{n,k}(ab) \\
&= \sigma\Big(\sum_{\ell \in T_k} \varphi_{n,\ell}(a)\varphi_{n,k-\ell}(b)\Big) \\
&= \sum_{\ell \in T_k} \sigma\varphi_{n,\ell}(a)\sigma\varphi_{n,k-\ell}(b) \\
&= \sum_{\ell \in T_k} \varphi_{n+1,\ell}(a)\varphi_{n+1,k-\ell}(b).
\end{aligned}$$

Case2. $2^n \leq k < 2^{n+1} - 1$.

In this case, $c_{n+1} = 1$ and so $\varphi_{n+1,k} = d\varphi_{n,k-2^n}$. Thus

$$\begin{aligned}
\varphi_{n+1,k}(ab) &= d\varphi_{n,k-2^n}(ab) \\
&= d(\sum_{\ell\in T_{k-2^n}} \varphi_{n,\ell}(a)\varphi_{n,k-2^n-\ell}(b)) \\
&= \sum_{\ell\in T_{k-2^n}} [d\varphi_{n,\ell}(a)\sigma\varphi_{n,k-2^n-\ell}(b) \\
&\quad +\sigma\varphi_{n,\ell}(a)d\varphi_{n,k-2^n-\ell}(b)] \\
&= \sum_{\ell\in T_{k-2^n}} [\varphi_{n+1,2^n+\ell}(a)\varphi_{n+1,k-2^n-\ell}(b) \\
&\quad +\varphi_{n+1,\ell}(a)\varphi_{n+1,k-\ell}(b)] \\
&= \sum_{2^n\leq m\in T_k} \varphi_{n+1,m}(a)\varphi_{n+1,k-m}(b) \\
&\quad + \sum_{2^n> m\in T_k} \varphi_{n+1,m}(a)\varphi_{n+1,k-m}(b) \\
&= \sum_{m\in T_k} \varphi_{n+1,m}(a)\varphi_{n+1,k-m}(b).
\end{aligned}$$

□

Remark 3.8. Putting $k = 2^n - 1$ in the above theorem we get Theorem 3.2.

The following theorem with $\sigma = \iota$ is a generalization of Wielandt–Wintner theorem (cf. Theorem 2.2.1 of [13]).

Theorem 3.9. *Let σ be a bounded endomorphism on a Banach algebra $\mathcal{A}$, d be a bounded σ-derivation such that $d\sigma = \sigma d = d$ and $d^2(a) = 0$. Then $d(a)$ is a quasinilpotent, i.e. $r(d(a)) = 0$.*

Proof. Using induction on n we can establish that $d^n(a^n) = n!d(a)^n$ holds for all positive integer n. Indeed if $d^{n-1}(a^{n-1}) = (n-1)!d(a)^{n-1}$, then we infer from Corollary 3.4 that

$$\begin{aligned}
d^n(a^n) = d^n(a^{n-1}a) &= \sum_{r=0}^{n} \binom{n}{r} d^r(a^{n-1})d^{n-r}(a) \\
&= nd^{n-1}(a^{n-1})d(a) + d^n(a^{n-1}) \\
&= n(n-1)!d(a)^{n-1}d(a) + d(d^{n-1}(a^{n-1})) \\
&= n!d(a)^n + d((n-1)!d(a)^{n-1}) = n!d(a)^n + 0.
\end{aligned}$$

Hence $r(d(a)) = \lim_{n\to\infty} \|d(a)^n\|^{1/n} = \lim_{n\to\infty} \|d^n(a^n)\|/n!)^{\frac{1}{n}} \le \frac{\|d\|\|a\|}{(n!)^{1/n}} = 0$. ☐

We are ready to extend the Wielandt–Wintner theorem which states that there are no two elements a and b in a Banach algebra such that $ab - ba = 1$ (see Corollary 2.2.2 of [13]).

Theorem 3.10. *Suppose that σ is a bounded endomorphism on a Banach algebra $\mathcal{A}$. Then there are no three elements $a, b, c \in \mathcal{A}$ satisfying $a\sigma(b) - \sigma(b)a = c$ provided that*

(i) $\sigma(a)\sigma^2(b) - \sigma^2(b)\sigma(a) = a\sigma(b) - \sigma(b)a$,
(ii) $(\sigma^2(b) - \sigma(b))a = a(\sigma^2(b) - \sigma(b))$,
(iii) $a\sigma(c) - \sigma(c)a = 0$,
(iv) *c is not quasinilpotent.*

Proof. Use the previous theorem with the inner σ-derivation $d_a(u) = a\sigma(u) - \sigma(u)a, u \in \mathcal{A}$. In fact, the conditions implies that $d_a^2(b) = 0$ and so $d_a(b) = c$ would be quasinilpotent which is a contradiction. ☐

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