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EXTRINSIC SPHERE AND UMBILICAL SUBMANIFOLDS IN FINSLER SPACES

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Abstract. Based on a definition for circle in Finsler space, recently proposed by one of the present authors and Z. Shen, a natural definition of extrinsic sphere in Finsler geometry is given and it is shown that a connected submanifold of a Finsler manifold is totally umbilical and has non-zero parallel mean curvature vector field, if and only if its circles coincide with circles of the ambient manifold.
Finally, some examples of extrinsic sphere in Finsler geometry, particularly in Randers spaces are given.
Keywords: Finsler space, development, mean curvature, umbilical, extrinsic sphere.
MSC(2010): Primary: 53C60; Secondary: 58B20.

1. Introduction

The concept of a Euclidean sphere has two axiomatic useful extensions in Riemannian geometry, namely, intrinsic sphere and extrinsic sphere. An *intrinsic sphere* is locally isometric to an ordinary sphere in a Euclidean space. An $n(\geq 2)$-dimensional submanifold of an arbitrary Riemannian manifold is said to be an *extrinsic sphere* if it is totally umbilical and has non-zero parallel mean curvature vector. The notions of intrinsic and extrinsic spheres coincide in a Euclidean space. However, in general, an extrinsic sphere is not always an intrinsic sphere, namely, an extrinsic sphere is not always isometric with a sphere.

Recently the present authors have defined an axiom of sphere as follows.
Axiom of r-spheres. *Let (M,F) be a Finsler manifold of dimension $n \geq 3$. For each point x in M and any r-dimensional subspace E_r of T_xM, there exists an r-dimensional umbilical submanifold S with parallel mean curvature vector*

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field such that $x \in S$ *and* $T_xS = E_r$, cf., [13, 14]. Next it is proved;

Theorem A. ([13]). *If a Finsler manifold of dimension* $n \geq 3$ *satisfies the axiom of* r*-spheres for some* r, $2 \leq r < n$, *then* M *has constant flag curvature.*

Here, following a definition for circles in Finsler geometry given by one of the present authors in a joint work with Z. Shen, cf., [6], and the notion of an umbilical submanifold in Theorem A, a natural definition of extrinsic sphere in Finsler geometry is given and the following theorem is proved.

Theorem 1.1. *Let* (M, F) *be a Finsler manifold. A connected submanifold* S *of* M *is totally umbilical and has non-zero parallel mean curvature if and only if every circle of* S *is a circle of* M.

Next, some explicit examples of Finslerian spheres, particularly in Randers spaces are furnished.

2. Notations and preliminaries

Let M be an n-dimensional manifold and F a *Finsler structure* and g the corresponding Finslerian metric. Let $i : S \longrightarrow M$ be an immersion and S a submanifold of dimension k of M. We identify any point $x \in S$ by its image $i(x)$ and any tangent vector $X \in T_xS$ by its image $i_*(X)$, where i_* is the linear tangent mapping. Thus T_xS becomes a sub-space of T_xM. Let TS_0 be the fiber bundle of non-zero tangent vectors on S. TS_0 is a sub-vector bundle of TM_0 and the restriction of $p : TM_0 \longrightarrow M$ to TS_0 is denoted by $q : TS_0 \longrightarrow S$. We denote by $\bar{T}(S) := i^*TM$, the pull back induced vector bundle of TM by i. The Finslerian metric of M induces a Finslerian metric on S where we denote it again by g. At a point $x = qz \in S$, where $z \in TS_0$, the orthogonal complement of $T_{qz}S$ in $\bar{T}_{qz}S$ is denoted by $N_{qz}S$, namely, $\bar{T}_x(S) = T_x(S) \oplus N_{qz}S$, where $T_x(S) \cap N_{qz}S = 0$. We have the following decomposition:

$$q^*\bar{T}S = q^*TS \oplus N, \tag{2.1}$$

where N is called the normal fiber bundle. If TTS_0 is the tangent vector bundle to TS_0, we denote by ϱ, the canonical linear mapping $\varrho : TTS_0 \longrightarrow q^*TS$. Let $\hat{X}$ and $\hat{Y}$ be the two vector fields on TS_0. For $z \in TS_0$, $(\nabla_{\hat{X}}Y)_z$ belongs to $\bar{T}_{qz}S$. Attending to (2.1) we have

$$\nabla_{\hat{X}}Y = \bar{\nabla}_{\hat{X}}Y + \alpha(\hat{X}, Y), \quad Y = \varrho(\hat{Y}), \quad X = \varrho(\hat{X}), \tag{2.2}$$

where ∇ is the covariant derivative of Cartan connection and $\alpha(\hat{X}, Y)$ is the second fundamental form of the submanifold S. It belongs to N and is bilinear in $\hat{X}$ and Y. It results from (2.2) that the induced connection $\bar{\nabla}$ is a metric compatible covariant derivative with respect to the induced metric g in the vector bundle $q^*TS \longrightarrow TS_0$, cf., [2].

2.1. **Shape operator or Weingarten formula in Finsler spaces.** Let (M,F) be a Finsler manifold and S an immersed submanifold of (M,F). For any $\hat{X}\in\chi(TS_0)$ and $W\in\Gamma(N)$ we set

$$\nabla_{\hat{X}}W=-A_W\hat{X}+\bar{\nabla}^{\perp}_{\hat{X}}W, \tag{2.3}$$

where $A_W\hat{X}\in\Gamma(q^*TS)$ and $\bar{\nabla}^{\perp}_{\hat{X}}W\in\Gamma(N)$ and we have partially used notations of [4]. It follows that $\bar{\nabla}^{\perp}$ is a linear connection on the normal bundle N. We also consider the bilinear map

$$\begin{aligned}A:&\Gamma(N)\otimes\Gamma(TTS_0)\longrightarrow\Gamma(q^*TS),\\ &A(W,\hat{X})=A_W\hat{X}.\end{aligned}$$

For any $W\in\Gamma(N)$, the operator $A_W:\Gamma(TTS_0)\longrightarrow\Gamma(q^*TS)$ is called the *shape operator* or the *Weingarten map* with respect to W. Finally, (2.3) is said to be the *Weingarten formula* for the immersion of S in M.

2.2. **Umbilical submanifolds in Finsler spaces.** The *mean curvature* vector field η of the isometric immersion $i:S\longrightarrow M$ is defined by

$$\eta=\frac{1}{n}tr_g\alpha({}^h\hat{X},Y), \tag{2.4}$$

where $X,Y\in\Gamma(q^*TS)$ and ${}^h\hat{X}$ is the horizontal lift of X, cf., [1]. We say that the mean curvature vector field η is *parallel* in all directions if $\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta=0$ for all $X\in\Gamma(q^*TS)$.

Definition 2.1 ([1]). A submanifold of a Finsler manifold is said to be *totally umbilical*, or simply *umbilical*, if it is equally curved in all tangent directions.

More precisely, let $i:S\longrightarrow M$ be an isometric immersion. Then i is totally umbilical if there exists a normal vector field $\xi\in N$ along i such that its second fundamental form α with values in the normal bundle satisfies

$$\alpha({}^h\hat{X},Y)=g(X,Y)\xi, \tag{2.5}$$

for all $X,Y\in\Gamma(q^*TS)$, where ${}^h\hat{X}$ is the horizontal lift of X. For more details and examples on totally umbilical Finsler submanifolds one can refer to [8] and [10]. Clearly by definition we have the following remark.

Remark 2.2. Let $i:S\longrightarrow M$ be an isometric immersion. If S is totally umbilical, then the normal vector field ξ coincides with the mean curvature vector field η.

3. Circle and its development in Finsler space

3.1. **Circle on Finsler spaces.** Let (M, F) be a Finsler manifold of class C^∞, $c : I \subset \mathbb{R} \longrightarrow M$ a curve parameterized by the arc length s and $X := \dot{c} = \frac{dc}{ds}$ the unitary tangent vector field at each point $c(s)$. Let us denote by $\hat{c}$ the horizontal lift of c on TM_0 and ${}^h\hat{X}$ its tangent vector field. In fact ${}^h\hat{X}$ is the horizontal lift of X on TM_0.

Definition 3.1 ([5, 6])**.** A curve c in a Finsler space (M, F) is said to be a *circle* if there exists a unitary vector field Y along c and a positive constant κ such that

$$\nabla_{{}^h\hat{X}} X = \kappa Y,$$
$$\nabla_{{}^h\hat{X}} Y = -\kappa X,$$

where $\nabla_{{}^h\hat{X}}$ denotes the Cartan covariant derivative along c. The number $\frac{1}{\kappa}$ is called the radius of circle.

Lemma 3.2 ([6])**.** *Let $c = c(s)$ be a unit speed curve on an n-dimensional Finsler manifold (M, F). If c is a circle, then it satisfies the following ODE*

$$\nabla_{{}^h\hat{X}} \nabla_{{}^h\hat{X}} X + g(\nabla_{{}^h\hat{X}} X, \nabla_{{}^h\hat{X}} X)X = 0, \tag{3.1}$$

where $g(.,.)$ denotes the scalar product determined by Finsler structure F. Conversely, if c satisfies (3.3), then it is either a geodesic or a circle.

3.2. **Parallel transport and absolute derivative.** In order to define notion of parallelism on a manifold one should identify all tangent spaces at any two points of a curve joining these two points. This identification has to preserve the linear structure of tangent spaces. Let $c(s)$ be a piecewise differentiable curve in M parameterized by the arc length s, joining two points $p, q \in M$. A *parallel transport along* c from p to q is defined to be a linear isomorphism $\tau_p^q : T_pM \longrightarrow T_qM$ such that $\tau_p^r \circ \tau_r^q = \tau_p^q$, where r is a point on c and $(\tau_p^q)^{-1} = \tau_q^p$. A vector field X along a curve c is said to be *parallel vector field along* c if $X_{c(s)} = \tau_{p_0}^{c(s)} X_{p_0}$, for all s, where X_{p_0} is a vector field at a point $p_o = c(s_0)$. One can use the notion of parallel transport to define a connection or an absolute derivative and vice versa in the sense that if a vector field is parallel along a curve, then its absolute derivative vanishes. Let $X_s := X_{c(s)}$ be a vector field along c. For $h > 0$, we denote by $\tau_{s+h}^s X_{s+h}$ the parallel transport of X_{s+h} along c from $c(s+h)$ to $c(s)$. Hence the *absolute derivative* of X along c at $c(s)$ is defined by

$$\frac{DX}{ds} = \lim_{h \to 0} \frac{\tau_{s+h}^s X_{s+h} - X_s}{h} = \frac{d}{dh}\{\tau_{s+h}^s X_{s+h}\}\,|_{h=0}\,. \tag{3.2}$$

A vector field X is parallel along a curve c with parameter s if and only if $\frac{DX}{ds} = 0$, cf., [3, 7].

3.3. **Development of a curve into the tangent space.** We recall the classical definition of development into the tangent bundle of a smooth manifold with an affine connection. Let $\tau : x_s$ be a smooth curve on M and $\tau_s^0 : T_{x_s}M \longrightarrow T_{x_0}M$ the parallel transport along τ from x_s to x_0. *Development* of τ into the Euclidean tangent space $T_{x_0}M$ is the unique curve $\tau_s^0(x_s)$ in $T_{x_0}M$. Equivalently, if $X_s^* := \tau_s^0(X_s)$, where X_s is tangent to τ at x_s, then the *development* $\overset{*}{\tau}$ *of* τ is the unique curve $\overset{*}{x}_s$ in $T_{x_0}M$ starting at the origin x_0 of $T_{x_0}M$ such that its tangent vector $\frac{d\overset{*}{x}_s}{ds}$ is parallel to X_s in the Euclidean sense, cf., [9].

For our purpose we prefer to work with a Riemannian connection on $T_{x_0}M$, associated to the Riemannian metric $\tilde{g} := g_{ij}(x_0, y)dy^i dy^j$, defined by the vertical part of Sasakian metric on TM. At the point $z = (x_0, y) \in T_{x_0}M$ the coefficients of Riemannian connection associated to the vertical metric $\tilde{g}$ are given by $C^i_{jk}(x_0, y)$, where C is the Cartan torsion tensor, cf., [2]. We denote the corresponding Riemannian covariant derivative by $\dot{D}_{\dot{\partial}_k}$, where

$$\dot{D}_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}(x_0, y)\dot{\partial}_i. \tag{3.3}$$

In the following definition, we extend the definition of development of a curve in M into the tangent space $T_{x_0}M$ with respect to the Riemannian metric $\tilde{g}$ on $T_{x_0}M$ arises from the Finsler metric.

Definition 3.3. Let $\tau : x_s$ be an arc length parameterized smooth curve on Finsler space (M, g), and X_s the unit tangent vector field at each point x_s. Denote by $\tau_s^0 : T_{x_s}M \longrightarrow T_{x_0}M$ the linear parallel transport along τ from x_s to x_0. Let $X_s^* = \tau_s^0(X_s)$ be the parallel transport of X_s along τ on M. *Development* $\overset{*}{\tau}$ of τ is an arc length parameterized curve $(\overset{*}{x}{}_s^i)$ on $(T_{x_0}M, \tilde{g})$ starting from x_0, such that its tangent vector $\frac{d\overset{*}{x}_s}{ds}$ is parallel to X_s^* along $\overset{*}{\tau}$, that is, $X_s^* = \overset{*}{\tau}{}_s^0(\frac{d\overset{*}{x}_s}{ds})$. Here, $\overset{*}{\tau}{}_s^0$ is parallel transport along $\overset{*}{\tau}$ from $\overset{*}{x}_s$ to $\overset{*}{x}_0$.

Proposition 3.4. *A curve on a Finsler manifold* (M, F) *is a circle if and only if its development into the Riemannian tangent space* $(T_{x_0}M, \tilde{g})$ *is a circle.*

Proof. Let $\tau : x_s$ be a circle on (M, F) parameterized by arc length and X_s, Y_s unitary tangent and normal vector fields at x_s, respectively. By definition of a circle we have

$$\begin{aligned} \nabla_s X_s &= \kappa Y_s, \\ \nabla_s Y_s &= -\kappa X_s, \end{aligned} \tag{3.4}$$

where $\nabla_s := \nabla_{{}^h\hat{X}_s}$ is the Cartan covariant derivative along τ in M. Let $\tau_s^0 : T_{x_s}M \longrightarrow T_{x_0}M$ be the parallel transport along τ from x_s to x_0. We denote by $X_s^* := \tau_s^0(X_s)$ and $Y_s^* := \tau_s^0(Y_s)$ the parallel transport of X_s and Y_s along τ from x_s to x_0, respectively. Let $\overset{*}{\tau} : \overset{*}{x}_s$ be the development of τ on $(T_{x_0}M, \tilde{g})$ and $\overset{*}{\tau}{}_0^s : V_{\overset{*}{x}_0}T_{x_0}M \longrightarrow V_{\overset{*}{x}_s}T_{x_0}M$, the parallel transport along the curve $\overset{*}{\tau}$ in

$T_{x_0}M$ from $\overset{*}{x}_0$ to $\overset{*}{x}_s$. Putting $\bar{X}^*_s := \overset{*}{\tau}{}^s_0(X^*_s)$ and $\bar{Y}^*_s := \overset{*}{\tau}{}^s_0(Y^*_s)$, by means of linearity of the mapping $\overset{*}{\tau}{}^s_0$, the Riemannian covariant derivative $\dot{D}_s\bar{X}^*_s$ on $(T_{x_0}M, \tilde{g})$ along the curve $\overset{*}{\tau}$ is written

$$\begin{aligned}\dot{D}_s\bar{X}^*_s &= \lim_{h\to 0}\frac{\overset{*}{\tau}{}^s_{s+h}(\bar{X}^*_{s+h}) - \bar{X}^*_s}{h} = \lim_{h\to 0}\frac{\overset{*}{\tau}{}^s_{s+h}(\overset{*}{\tau}{}^{s+h}_0 X^*_{s+h}) - \overset{*}{\tau}{}^s_0 X^*_s}{h}\\ &= \lim_{h\to 0}\frac{\overset{*}{\tau}{}^s_0 X^*_{s+h} - \overset{*}{\tau}{}^s_0 X^*_s}{h} = \overset{*}{\tau}{}^s_0(\lim_{h\to 0}\frac{X^*_{s+h} - X^*_s}{h}).\end{aligned}\tag{3.5}$$

Next, by means of (3.2) for Cartan covariant derivative ∇ along the curve τ and linearity of the mapping τ^0_s, we have

$$\begin{aligned}\lim_{h\to 0}\frac{X^*_{s+h} - X^*_s}{h} = \lim_{h\to 0}\frac{\tau^0_s(\tau^s_{s+h}X_{s+h} - X_s)}{h} &= \tau^0_s(\lim_{h\to 0}\frac{\tau^s_{s+h}X_{s+h} - X_s}{h})\\ &= \tau^0_s(\nabla_s X_s).\end{aligned}\tag{3.6}$$

Therefore, from (3.5) and (3.6), we obtain

$$\dot{D}_s\bar{X}^*_s = \overset{*}{\tau}{}^s_0\tau^0_s(\nabla_s X_s).\tag{3.7}$$

Similarly, we have

$$\dot{D}_s\bar{Y}^*_s = \overset{*}{\tau}{}^s_0\tau^0_s(\nabla_s Y_s).\tag{3.8}$$

By virtue of (3.4), the relation (3.7) and (3.8) imply

$$\begin{aligned}\dot{D}_s\bar{X}^*_s &= \kappa\bar{Y}^*_s,\\ \dot{D}_s\bar{Y}^*_s &= -\kappa\bar{X}^*_s.\end{aligned}\tag{3.9}$$

Above equations show that $\overset{*}{\tau}_s : \overset{*}{x}_s$ is a circle in $T_{x_0}M$. Conversely, assume that the development $\overset{*}{\tau}$ of τ in $T_{x_0}M$ is a circle. By definition there is a family of unit vectors $\bar{Y}^*_s$ and a constant $\kappa > 0$ together with the tangent vectors $\bar{X}^*_s$ of $\overset{*}{\tau}_s : \overset{*}{x}_s$ satisfying (3.9) . Using the fact that τ^0_s and $\overset{*}{\tau}{}^s_0$ are isomorphisms by means of (3.7) and (3.8) we have

$$\begin{aligned}\nabla_s X_s &= (\tau^0_s)^{-1}(\overset{*}{\tau}{}^s_0)^{-1}(\dot{D}_s\bar{X}^*_s) = \tau^s_0\overset{*}{\tau}{}^0_s(\dot{D}_s\bar{X}^*_s),\\ \nabla_s Y_s &= (\tau^0_s)^{-1}(\overset{*}{\tau}{}^s_0)^{-1}(\dot{D}_s\bar{Y}^*_s) = \tau^s_0\overset{*}{\tau}{}^0_s(\dot{D}_s\bar{Y}^*_s),\end{aligned}$$

where $\nabla_s := \nabla_{{}^h\hat{X}_s}$ is the Cartan covariant derivative along τ in M. By means of (3.9), we obtain

$$\begin{aligned}\nabla_s X_s &= \kappa Y_s,\\ \nabla_s Y_s &= -\kappa X_s.\end{aligned}$$

Hence $\tau : x_s$ is a circle in M. This completes the proof. □

Proposition 3.4 has the following consequence.

Remark 3.5. Let x be an arbitrary point of M. For any orthonormal pair of vectors X and Y in T_xM and for any given constant $\kappa > 0$, there is a circle x_s of radius $\frac{1}{\kappa}$, defined for $\mid s \mid < \epsilon$ for some $\epsilon > 0$, passing through the point $x_0 = x$ and tangent to the vector field X for which its covariant derivative along c is proportional to Y, that is,

$$(\nabla_{{}^h\hat{X}_s} X_s)_{s=0} = \kappa Y,$$

where X_s is the tangent vector of x_s and ${}^h\hat{X}_s$ is the horizontal lift of X_s.

4. Extrinsic sphere in Finsler spaces

Extrinsic spheres in Riemannian geometry have been geometrically characterized by K. Nomizu and K. Yano, cf., [11, 12]. Now, we are in a position to define an sphere in Finsler spaces.

Definition 4.1. A submanifold S of an arbitrary Finsler manifold (M, F) is said to be an *extrinsic sphere* if it is umbilical and has non-zero parallel mean curvature in all directions.

That is, for all $X \in \Gamma(q^*TS)$, we have $\bar{\nabla}^{\perp}_{{}^h\hat{X}} \eta = 0$, where ${}^h\hat{X}$ is the horizontal lift of X.
Here, as well as in Riemannian geometry we use the expression "extrinsic" for a sphere because of its dependence to the extrinsic properties of the submanifold. To prove Theorem 1.1 we need the following lemma.

Lemma 4.2. *Let S be a submanifold of a Finsler manifold M. If $\alpha({}^h\hat{X}, Y) = 0$ for any orthonormal pair of vectors $X, Y \in \Gamma(q^*TS)$ at a point $x = qz \in S$, where ${}^h\hat{X}$ is the horizontal lift of X, then the following statements hold;*
(1) *For any orthonormal $X, Y \in \Gamma(q^*TS)$ at a point $x = qz \in S$ we have*

$$\alpha({}^h\hat{X}, X) = \alpha({}^h\hat{Y}, Y), \tag{4.1}$$

where ${}^h\hat{Y}$ is the horizontal lift of Y.
(2) *At an arbitrary point $x = qz \in S$ we have*

$$\eta = \alpha({}^h\hat{X}, X), \tag{4.2}$$

where $X \in \Gamma(q^*TS)$ is an arbitrary unit vector field in T_xS.
(3) *S is umbilical at $x = qz$.*

Proof. Case (1). Let X and Y be two orthonormal sections of q^*TS with respect to the induced Finsler metric g on S, one can easily check that $\frac{1}{\sqrt{2}}(X + Y)$ and $\frac{1}{\sqrt{2}}(X - Y)$ are orthonormal too. Using the fact that the horizontal lift of a vector field is a bundle isomorphism and the assumption $\alpha({}^h\hat{X}, Y) = 0$, we have

$$\alpha(\frac{1}{\sqrt{2}}({}^h\hat{X} + {}^h\hat{Y}), \frac{1}{\sqrt{2}}(X - Y)) = 0.$$

By bilinearity of α, we have

$$\alpha({}^h\hat{X}, X) + \alpha({}^h\hat{Y}, X) - \alpha({}^h\hat{X}, Y) - \alpha({}^h\hat{Y}, Y) = 0.$$

Dropping zero terms we obtain

$$\alpha({}^h\hat{X}, X) = \alpha({}^h\hat{Y}, Y).$$

Case (2). Consider an orthonormal basis $\{X_1, X_2, ..., X_n\}$ on T_xS as a fiber of q^*TS. By means of (4.1), we have $\alpha({}^h\hat{X}_1, X_1) = \alpha({}^h\hat{X}_2, X_2) = ... = \alpha({}^h\hat{X}_n, X_n)$. Thus by definition, the mean curvature vector η at a point $x = qz \in S$ is given by

$$\eta = \frac{1}{n}\sum_{i=1}^{n} \alpha({}^h\hat{X}_i, X_i) = \alpha({}^h\hat{X}_1, X_1).$$

Case (3). Again let $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis for T_xS as a fiber of q^*TS and $X, Y \in \Gamma(q^*TS)$, hence we have $X = \sum_{i=1}^n a_i X_i, Y = \sum_{i=1}^n b_i X_i$. Since the horizontal lift of a vector field is a bundle isomorphism and by definition, restriction of a bundle isomorphism to each fiber is a linear map and we have ${}^h\hat{X} = \sum_{i=1}^n a_i({}^h\hat{X}_i)$. By means of bilinearity of α and (4.2), we obtain

$$\begin{aligned}\alpha({}^h\hat{X}, Y) &= \alpha(\sum_{i=1}^{n} a_i({}^h\hat{X}_i), \sum_{j=1}^{n} b_j X_j) = \sum_{i,j=1}^{n} a_i b_j \alpha({}^h\hat{X}_i, X_j)\\ &= \sum_{i=1}^{n} a_i b_i \alpha({}^h\hat{X}_i, X_i) = (\sum_{i=1}^{n} a_i b_i)\alpha({}^h\hat{X}_1, X_1)\\ &= g(X, Y)\eta.\end{aligned}$$

This completes the proof. □

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let x be an arbitrary point of S and X and Y orthonormal vectors in T_xS. By means of Remark 3.5, there is a circle x_s, $| s |< \epsilon$, of radius $\frac{1}{\kappa}$ passing through the point $x_0 = x$ and tangent to the vector field X such that

$$(\bar{\nabla}_{{}^h\hat{X}_s} X_s)_{s=0} = \kappa Y, \tag{4.3}$$

where $\bar{\nabla}$ is the induced connection for S, X_s the tangent vector of x_s and ${}^h\hat{X}_s$ the horizontal lift of X_s. By means of Lemma 3.2, we have the differential equation

$$\bar{\nabla}_{{}^h\hat{X}_s}\bar{\nabla}_{{}^h\hat{X}_s} X_s + g(\bar{\nabla}_{{}^h\hat{X}_s} X_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) X_s = 0. \tag{4.4}$$

By assumption, x_s is a circle in M too and satisfies the differential equation

$$\nabla_{{}^h\hat{X}_s}\nabla_{{}^h\hat{X}_s} X_s + g(\nabla_{{}^h\hat{X}_s} X_s, \nabla_{{}^h\hat{X}_s} X_s) X_s = 0, \tag{4.5}$$

where ∇ denotes the Cartan covariant derivative on (M, F). Let α be the second fundamental form of S in M. Then we have

$$\nabla_{{}^h\hat{X}_s} X_s = \bar{\nabla}_{{}^h\hat{X}_s} X_s + \alpha({}^h\hat{X}_s, X_s). \tag{4.6}$$

By operating $\nabla_{{}^h\hat{X}_s}$ on (4.6) and using the Wiengarten formula (2.3), we have

$$\begin{aligned}\nabla_{{}^h\hat{X}_s}\nabla_{{}^h\hat{X}_s} X_s &= \nabla_{{}^h\hat{X}_s}(\bar{\nabla}_{{}^h\hat{X}_s} X_s) + \nabla_{{}^h\hat{X}_s}(\alpha({}^h\hat{X}_s, X_s)) \\ &= \bar{\nabla}_{{}^h\hat{X}_s}\bar{\nabla}_{{}^h\hat{X}_s} X_s + \alpha({}^h\hat{X}_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) - A_{\alpha({}^h\hat{X}_s, X_s)}{}^h\hat{X}_s \\ &\quad + \bar{\nabla}^{\perp}_{{}^h\hat{X}_s}\alpha({}^h\hat{X}_s, X_s).\end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) and taking into account (4.4), we obtain

$$\begin{aligned}\alpha({}^h\hat{X}_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) - A_{\alpha({}^h\hat{X}_s, X_s)}{}^h\hat{X}_s + \bar{\nabla}^{\perp}_{{}^h\hat{X}_s}\alpha({}^h\hat{X}_s, X_s) \\ + g(\alpha({}^h\hat{X}_s, X_s), \alpha({}^h\hat{X}_s, X_s))X_s = 0.\end{aligned}$$

For the tangent components of S on the above equation, by dropping normal components we have

$$A_{\alpha({}^h\hat{X}_s, X_s)}{}^h\hat{X}_s = g(\alpha({}^h\hat{X}_s, X_s), \alpha({}^h\hat{X}_s, X_s))X_s. \tag{4.8}$$

For the normal components of S, we have

$$\alpha({}^h\hat{X}_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) + \bar{\nabla}^{\perp}_{{}^h\hat{X}_s}\alpha({}^h\hat{X}_s, X_s) = 0. \tag{4.9}$$

Denoting $X_s\,|_{s=0} = X$ and ${}^h\hat{X}_s\,|_{s=0} = {}^h\hat{X}$ at $s = 0$, by means of (4.3), we may rewrite (4.9) in the following form:

$$\alpha({}^h\hat{X}, Y) = -\frac{1}{\kappa}\bar{\nabla}^{\perp}_{{}^h\hat{X}}\alpha({}^h\hat{X}, X). \tag{4.10}$$

The last equation shows that, for a given unit vector $X \in T_xS$, $\alpha({}^h\hat{X}, Y)$ is independent of the unit vector $Y \in T_xS$, provided Y is orthogonal to X. In particular, changing Y into $-Y$ we see that $\alpha({}^h\hat{X}, Y) = 0$, provided X and Y are orthonormal. By virtue of the third part of Lemma 4.2, we know that S is umbilical in $x \in S$. Since x is arbitrary, S is umbilical in M. By means of metric compatibility, i.e. $\bar{\nabla} g = 0$, we have $g(X_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) = 0$ and since S is totally umbilical, we have $\alpha({}^h\hat{X}_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) = 0$. Thus (4.9) gives

$$\bar{\nabla}^{\perp}_{{}^h\hat{X}_s}\alpha({}^h\hat{X}_s, X_s) = 0. \tag{4.11}$$

By the second part of Lemma 4.2, $\alpha({}^h\hat{X}_s, X_s)$ is equal to the mean curvature vector η along the curve x_s. At $s = 0$, (4.11) leads to $\bar{\nabla}^{\perp}_{{}^h\hat{X}}\eta = 0$. Since x and $X \in T_xS$ are arbitrary, the mean curvature vector η of S is parallel and S is by definition an extrinsic sphere.

Conversely, assume that S is an extrinsic sphere in M and x_s is a circle in S. Hence the equation (4.4) holds. Since S is umbilical, we have

$$\alpha({}^h\hat{X}_s, X_s) = g(X_s, X_s)\eta_{x_s} = \eta_{x_s},$$

and (4.6) shows that

$$\begin{aligned} g(\nabla_{{}^h\hat{X}_s} X_s, \nabla_{{}^h\hat{X}_s} X_s) &= g(\bar{\nabla}_{{}^h\hat{X}_s} X_s + \eta_{x_s}, \bar{\nabla}_{{}^h\hat{X}_s} X_s + \eta_{x_s}) \\ &= g(\bar{\nabla}_{{}^h\hat{X}_s} X_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) + g(\eta_{x_s}, \bar{\nabla}_{{}^h\hat{X}_s} X_s) \\ &\quad + g(\bar{\nabla}_{{}^h\hat{X}_s} X_s, \eta_{x_s}) + g(\eta_{x_s}, \eta_{x_s}) \\ &= g(\bar{\nabla}_{{}^h\hat{X}_s} X_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) + g(\eta_{x_s}, \eta_{x_s}). \end{aligned} \tag{4.12}$$

By the fact that S is totally umbilical and η is parallel, we have

$$\begin{aligned} &\alpha({}^h\hat{X}_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s) = g(X_s, \bar{\nabla}_{{}^h\hat{X}_s} X_s)\eta_{x_s} = 0, \\ &\bar{\nabla}^{\perp}_{{}^h\hat{X}_s} \alpha({}^h\hat{X}_s, X_s) = \bar{\nabla}^{\perp}_{{}^h\hat{X}_s} \eta_{x_s} = 0. \end{aligned}$$

By means of (4.8), we have

$$A_{\alpha({}^h\hat{X}_s, X_s)} {}^h\hat{X}_s = A^h_{\eta_{x_s}} \hat{X}_s = g(\eta_{x_s}, \eta_{x_s}) X_s.$$

Thus (4.7) reduces to

$$\nabla_{{}^h\hat{X}_s} \nabla_{{}^h\hat{X}_s} X_s = \bar{\nabla}_{{}^h\hat{X}_s} \bar{\nabla}_{{}^h\hat{X}_s} X_s - g(\eta_{x_s}, \eta_{x_s}) X_s. \tag{4.13}$$

The equation (4.5) is satisfies as a consequence of (4.4), (4.12) and (4.13). Thus x_s is a circle in M and the proof is complete. □

Example 4.3. As a 1-dimensional example we show, in the following proposition, every circle in (M, F) is an extrinsic sphere.

Proposition 4.4. *Let (M, F) be an n-dimensional Finsler manifold. Every circle in (M, F) as a 1-dimensional immersed submanifold is an extrinsic sphere.*

Proof. By definition of circle in Finsler spaces, there exists a unitary vector field Y along c and a positive constant κ such that

$$\nabla_{{}^h\hat{X}} X = \kappa Y, \tag{4.14}$$

$$\nabla_{{}^h\hat{X}} Y = -\kappa X, \tag{4.15}$$

where $X := \dot{c} = \frac{dc}{ds}$ is the unitary tangent vector field at each point $c(s)$ and ${}^h\hat{X}$ the horizontal lift of X. Then (2.2) and (4.14) show that

$$\alpha({}^h\hat{X}_s, X_s) = \kappa Y_s = g(X_s, X_s)\kappa Y_s, \tag{4.16}$$

where α is the second fundamental form of the 1-dimensional submanifold c of M. By means of (4.16) and definition of totally umbilical submanifold, we conclude that c is totally umbilical. On the other hand by Remark 2.2, the mean curvature vector η is κY_s. Then (4.15) and Weingarten formula (2.3) show that η is parallel relative to the normal connection $\bar{\nabla}^{\perp}$. Therefore c is an extrinsic sphere in M. □

Example 4.5. To give another example of an extrinsic sphere in Finsler space, we use a theorem on totally umbilical submanifolds given in [8]. There is shown that if $(M^{n+1}, \alpha+\beta)$ is a Randers space, where α is a Euclidean metric and β is a closed 1-form, then any complete and connected n-dimensional totally umbilical submanifold of $(M^{n+1}, \alpha+\beta)$ must be either a plane or an Euclidean sphere. Excluding the trivial case of plane, since the Euclidean sphere is of non-zero constant sectional curvature by means of Definition 4.1, it is an extrinsic sphere in Randers space $(M^{n+1}, \alpha+\beta)$.

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