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WEAK CONVERGENCE THEOREMS FOR SYMMETRIC GENERALIZED HYBRID MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we prove some theorems related to properties of generalized symmetric hybrid mappings in Banach spaces. Using Banach limits, we prove a fixed point theorem for symmetric generalized hybrid mappings in Banach spaces. Moreover, we prove some weak convergence theorems for such mappings by using Ishikawa iteration method in a uniformly convex Banach space.
Keywords: Fixed point, hybrid method, Opial's condition, uniformly convex Banach space, weak convergence.
MSC(2010): Primary: 47J25; Secondary: 47H10, 47H09, 47J05.

1. Introduction

Let C be a nonempty, closed convex subset of a real Banach space E. The self mapping T of C is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T.

There exist some iteration processes which are often used to approximate a fixed point of a nonexpansive mapping: Picard iteration, Krasnoselskii iteration, Halpern iteration, Mann iteration and Ishikawa iteration. During the recent years, Mann and Ishikawa iterative schemes [6, 8] have been studied by a number of authors.

Let E be a nonempty closed convex subset of a Banach space. In 1953, for a self mapping T of E, Mann [8] defined the following iteration procedure:

$$(1.1) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \end{cases}$$

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where $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Let K be a closed convex subset of a Hilbert space H. In 1974, for a Lipschitzian pseudocontractive self mapping T of K, Ishikawa [6] defined the following iteration procedure:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \end{cases} \tag{1.2}$$

where $0 \leq \beta_n \leq \alpha_n \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and he proved strong convergence of the sequence $\{x_n\}$ generated by the above iterative scheme if $\lim_{n\to\infty} \beta_n = 1$ and $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = \infty$. Taking $\beta_n = 1$ for all $n \geq 0$ in (1.2), Ishikawa iteration process reduces to Mann iteration process.

In general, to gain the convergence in Mann and Ishikawa iteration processes, we must assume that underlying the space E has elegant properties. For example, Reich [10] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the Mann iteration scheme converges weakly to a fixed point of T. However, we know that the Mann iteration process is weakly convergent even in a Hilbert space [4]. Also, Tan and Xu [16] proved that if E is a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable and if $\{\alpha_n\}$ and $\{\beta_n\}$ are such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)$ diverges, $\sum_{n=1}^{\infty} \alpha_n(1-\beta_n)$ converges and $\limsup \beta_n < 1$, then Ishikawa iteration process converges weakly to a fixed point of T.

It easy to see that process (1.2) is more general than the process (1.1). Also, for a Lipschitz pseudocontractive mapping in a Hilbert space, process (1.1) is not known to converge to a fixed point while the process (1.2) is convergent. In spite of these facts, researchers are interested to study the convergence theorems by process (1.1), because the formulation of process (1.1) is simpler than that of (1.2). If $\{\beta_n\}$ satisfies suitable conditions, we can gain a convergence theorem for process (1.2) on a convergence theorem for process (1.1).

In recent years, many authors have proved weak or strong convergence theorems for some nonlinear mappings by using various iteration processes in the framework of Hilbert spaces and Banach spaces, see, [1, 9, 11, 14].

Let C be a nonempty, closed convex subset of a real Banach space E. A mapping S from C into E is called *symmetric generalized hybrid* [15] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx - Sy\|^2 + \beta(\|x - Sy\|^2 + \|Sx - y\|^2) + \gamma\|x - y\|^2 \\ + \delta(\|x - Tx\|^2 + \|y - Sy\|^2) \leq 0, \end{aligned} \tag{1.3}$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-*symmetric generalized hybrid mapping.*

In this paper, motivated by Takahashi and Yao [14], we prove some theorems related to properties of generalized symmetric hybrid mappings in Banach spaces. Moreover, we prove a fixed point theorem for symmetric generalized hybrid mappings in a Banach space . Also, we prove some weak convergence theorems for symmetric generalized hybrid mappings in a uniformly convex Banach space.

2. **Preliminaries**

Let E be a real Banach space with $\|.\|$ and dual space E^*. We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

for all $x \in E$, where $\langle ., . \rangle$ denotes the generalized duality pairing between E and E^*. E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if for every $\epsilon \in (0, 2]$, there exists a $\delta > 0$, such that $\|\frac{x+y}{2}\| < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. Furthermore, E is called *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $x, y \in B_E = \{x \in E : \|x\| = 1\}$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in E$. For more details see [12].

Denote by l^∞ the set of all bounded sequences equipped with supremum norm. A continuous linear functional μ on l^∞ is called a *Banach limit* if

(i) $\mu(e) = \|e\| = 1$, where $e = (1, 1, 1, \ldots)$;
(ii) $\mu_n(x_n) = \mu_n(x_{n+1})$ for all $x = (x_1, x_2, \ldots) \in l^\infty$, where $\mu_n(x_{n+m}) = \mu(x_{m+1}, x_{m+2}, x_{m+3}, \ldots, x_{m+n}, \ldots)$.

As usual, we denote by $\mu_n(x_n)$ the value of μ at $x = (x_1, x_2, \ldots)$. It is well known that there exists a Banach limit on l^∞. Let μ be a Banach limit, then

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n, \quad x = (x_1, x_2, \ldots) \in l^\infty.$$

Moreover, if $x_n \to a$, then $\mu_n(x_n) = a$. For more details we refer readers to [12].

We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

A Banach space E satisfies the *Opial's condition* if for every sequence $\{x_n\}$ in E such that $x_n \rightharpoonup x \in E$, then

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

for all $y \in E$, $y \neq x$.

A self mapping T of $C \subseteq E$ is called: (i) *firmly nonexpansive* [2], if $\|Tx - Ty\|^2 \leq \langle x - y, j \rangle$ for all $x, y \in C$, where $j \in J(Tx - Ty)$; (ii) *nonspreading*

[7], if $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in E$; (iii) *hybrid* [13], if $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for all $x, y \in E$. Also, a self mapping T of C with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$.

It easy to see that:

- a $(1, 0, -1, 0)$-symmetric generalized hybrid mapping is nonexpansive;
- a $(2, -1, 0, 0)$-symmetric generalized hybrid mapping is nonspreading;
- a $(3, -1, -1, 0)$-symmetric generalized hybrid mapping is hybrid.

The following result is given in [12].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping of E. Then*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

for all $x, y \in E$ where $j \in Jy$.

Theorem 2.2 ([5])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and T be a self mapping of C. Let $\{x_n\}$ be a bounded sequence of E and μ be a mean on l^∞. If*

$$\mu_n\|x_n - Tu\|^2 \leq \mu_n\|x_n - u\|^2,$$

for all $u \in C$, then T has a fixed point in C.

Theorem 2.3 ([17])**.** *Let E be a uniformly convex Banach space and let r be a positive real number. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$$

for all $x, y \in B_r$ and t with $0 \leq t \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

3. Main results

Theorem 3.1. *Let E be a real Banach space, C be a nonempty closed convex subset of E and T be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid self mapping of C such that $F(T) \neq \emptyset$ and the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta > 0$ and (3) $\delta \geq 0$ hold. Then T is quasi-nonexpansive.*

Proof. Since T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid self mapping of C, we have

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|Tx - y\|^2) + \gamma\|x - y\|^2 + \delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0, \tag{3.1}$$

for all $x, y \in E$. Since $F(T) \neq \emptyset$, there exists $x \in E$ such that $x = Tx$. So,

$$\alpha\|x - Ty\|^2 + \beta(\|x - Ty\|^2 + \|x - y\|^2) + \gamma\|x - y\|^2 + \delta\|y - Ty\|^2 \leq 0,$$

for all $y \in E$. Therefore we can conclude that

$$(\alpha+\beta)\|x-Ty\|^2+(\beta+\gamma)\|x-y\|^2 \leq 0, \tag{3.2}$$

for all $y \in E$. It follows from condition (2) and (3.2) that $-(\beta+\gamma) \geq 0$. So conditions (1) and (2) imply that

$$0 \leq \frac{-(\beta+\gamma)}{\alpha+\beta} \leq 1. \tag{3.3}$$

Then, form (3.2) and (3.3), we derive that $\|x-Ty\| \leq \|x-y\|$, i.e., T is quasi-nonexpansive. □

Theorem 3.2. *Let E be a real Banach space, C be a nonempty subset of E and ζ, η be nonnegetive real numbers. Then a firmly nonexpansive self mapping of C is a $(2\zeta+\eta, -\zeta, -\eta, 0)$-symmetric generalized hybrid mapping.*

Proof. Assume that T is a firmly nonexpasive self mapping of C. Then we have

$$\|Tx-Ty\|^2 \leq \langle x-y, j\rangle,$$

for all $x, y \in C$ and $j \in J(Tx-Ty)$. By using Theorem 2.1 we get

$$\begin{aligned}
\|Tx-Ty\|^2 \leq \langle x-y, j\rangle &\Longleftrightarrow 0 \leq 2\langle x-Tx-(y-Ty), j\rangle \\
&\Longrightarrow 0 \leq \|x-y\|^2-\|Tx-Ty\|^2 \\
&\Longleftrightarrow \|Tx-Ty\|^2 \leq \|x-y\|^2 \\
&\Longleftrightarrow \|Tx-Ty\| \leq \|x-y\|.
\end{aligned}$$

Hence for $\zeta \geq 0$, we have

$$\zeta\|Tx-Ty\| \leq \zeta\|x-y\|. \tag{3.4}$$

On the other hand, for all $x, y \in C$ and $j \in J(Tx-Ty)$ we get

$$\begin{aligned}
&\|Tx-Ty\|^2 \leq \langle x-y, j\rangle \\
&\qquad\Longleftrightarrow 0 \leq 2\langle x-Tx-(y-Ty), j\rangle \\
&\qquad\Longleftrightarrow 0 \leq 2\langle x-Tx, j\rangle+2\langle Ty-y, j\rangle \\
&\qquad\Longrightarrow 0 \leq \|x-Ty\|^2-\|Tx-Ty\|^2+\|Tx-y\|^2-\|Tx-Ty\|^2 \\
&\qquad\Longleftrightarrow 0 \leq \|x-Ty\|^2-\|y-Tx\|^2-2\|Tx-Ty\|^2 \\
&\qquad\Longleftrightarrow 2\|Tx-Ty\|^2 \leq \|x-Ty\|^2+\|y-Tx\|^2.
\end{aligned}$$

So, for $\eta \geq 0$ we get

$$2\eta\|Tx-Ty\|^2 \leq \eta\|x-Ty\|^2+\eta\|y-Tx\|^2. \tag{3.5}$$

Hence, summing both sides of (3.4) and (3.5) we obtain

$$(\zeta+2\eta)\|Tx-Ty\|^2 \leq \eta\|x-Ty\|^2+\eta\|y-Tx\|^2+\zeta\|x-y\|^2,$$

and therefore

$$(\zeta+2\eta)\|Tx-Ty\|^2-\eta(\|x-Ty\|^2+\|y-Tx\|^2)-\zeta\|x-y\|^2 \leq 0.$$

This yields that T is a $(\zeta+2\eta,-\eta,-\zeta,0)$-symmetric generalized hybrid mapping. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Banach space E and T be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid self mapping of C and the conditions (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\beta>0$ and (3) $\delta\geq 0$ hold. Then the following are equivalent:*

(i) $F(T)\neq\phi$;
(ii) $\{T^n x\}$ *is bounded for some* $x\in C$.

Proof. $(i)\Longrightarrow(ii)$: It is obvious.
$(ii)\Longrightarrow(i)$: Since T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid self mapping of C, the inequality (3.1) is satisfied. Let $u\in C$ such that $\{T^n u\}$ is bounded. Replacing x by $T^n u$ in (3.1), we have

$$\begin{aligned}&\alpha\|T^{n+1}u-Ty\|^2+\beta\|T^n u-Ty\|^2\\&\quad\leq-\beta\|T^{n+1}u-y\|^2-\gamma\|T^n u-y\|^2-\delta(\|T^n u-T^{n+1}u\|^2+\|y-Ty\|^2)\\&\quad\leq-\beta\|T^{n+1}u-y\|^2-\gamma\|T^n u-y\|^2,\end{aligned}$$

for all $y\in C$ and all $n\in\mathbb{N}$. Since $\{T^n u\}$ is bounded, by taking a Banach limit μ on both sides of the last inequlity, we get

$$\begin{aligned}&\mu_n(\alpha\|T^{n+1}u-Ty\|^2+\beta\|T^n u-Ty\|^2)\\&\qquad\qquad\leq\mu_n(-\beta\|T^{n+1}u-y\|^2-\gamma\|T^n u-y\|^2).\end{aligned}$$

So, by using the properties of Banach limit, we have

$$\begin{aligned}&\alpha\mu_n\|T^n u-Ty\|^2+\beta\mu_n\|T^n u-Ty\|^2\\&\qquad\qquad\leq-\beta\mu_n\|T^n u-y\|^2-\gamma\mu_n\|T^n u-y\|^2.\end{aligned}$$

From the last inequality, we can conclude that

$$(\alpha+\beta)\mu_n\|T^n u-Ty\|^2\leq-(\beta+\gamma)\mu_n\|T^n u-y\|^2.$$

Similar to the proof of Theorem 3.1, we derive that

$$\mu_n\|T^n u-Ty\|^2\leq\mu_n\|T^n u-y\|^2,$$

for all $y\in C$. So Theorem 2.2 implies that T has a fixed point.

□

Theorem 3.4. *Let C be a nonempty closed convex subset of a Banach space E satisfying Opial's condition. Assume that T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid self mapping of C such that the conditions* (1) $\alpha+2\beta+\gamma\geq 0$, (2) $\alpha+\beta>0$, (3) $\beta\leq 0$ *and* (4) $\delta\geq 0$ *hold. Then $I-T$ is demiclosed (at 0), i.e., $x_n\rightharpoonup u$ and $x_n-Tx_n\to 0$ imply $u\in F(T)$.*

Proof. Since T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid self mapping of C, the inequality (3.1) is satisfied. Assume that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since $x_n \rightharpoonup u$, we can conclude that $\{x_n\}$ is bounded and by $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we obtain that $\{Tx_n\}$ is bounded. Substituting x and y by x_n and u in (3.1), respectively, we have

$$\alpha\|Tx_n - Tu\|^2 + \beta(\|x_n - Tu\|^2 + \|Tx_n - u\|^2) + \gamma\|x_n - u\|^2 + \delta(\|x_n - Tx_n\|^2 + \|u - Tu\|^2) \le 0.$$

Therefore

$$\alpha\|Tx_n - Tu\|^2 \le -\beta\|x_n - Tu\|^2 - \beta\|Tx_n - u\|^2 - \gamma\|x_n - u\|^2,$$

and hence

$$\alpha\|Tx_n - Tu\|^2 \le -\beta(\|Tx_n - x_n\| + \|x_n - u\|)^2 - \gamma\|x_n - u\|^2 - \beta(\|x_n - Tx_n\| + \|Tx_n - Tu\|)^2.$$

So, we can conclude that

$$\begin{aligned} \|Tx_n - Tu\|^2 &\le \frac{-(\beta+\gamma)}{\alpha+\beta}\|x_n - u\|^2 - \frac{2\beta}{\alpha+\beta}\|x_n - Tx_n\|^2 \\ &\le \|x_n - u\|^2 - \frac{2\beta}{\alpha+\beta}\|x_n - Tx_n\|^2. \end{aligned} \tag{3.6}$$

Assume that $Tu \neq u$. So, using boundedness of $\{x_n\}$ and $\{Tx_n\}$, Opial's condition and (3.6), we have

$$\begin{aligned} \liminf_{n\to\infty} \|x_n - u\|^2 &< \liminf_{n\to\infty} \|x_n - Tu\|^2 \\ &= \liminf_{n\to\infty} \|Tx_n - Tu\|^2 \\ &\le \liminf_{n\to\infty} (\|x_n - u\|^2 - \frac{2\beta}{\alpha+\beta}\|x_n - Tx_n\|^2 \\ &\le \liminf_{n\to\infty} \|x_n - u\|^2, \end{aligned} \tag{3.7}$$

which is a contradiction. Hence we get $Tu = u$ and therefore $I - T$ is demiclosed. □

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition. Assume that T is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid self mapping of C such that $F(T) \neq \emptyset$ and the conditions (1) $\alpha + 2\beta + \gamma \ge 0$, (2) $\alpha + \beta > 0$, (3) $\beta \le 0$ and (4) $\delta \ge 0$ hold. Assume that $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\lambda_n)x_n + \lambda_n Tx_n, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n Ty_n, \end{cases}$$

where $0 \leq \lambda_n \leq 1$, $0 < a \leq \gamma_n \leq 1$ *for all* $n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$. *Then* $x_n \rightharpoonup x_0 \in F(T)$.

Proof. Since T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping such that $F(T) \neq \emptyset$, so by Theorem 3.1, T is quasi-nonexpansive. Then, for all $q \in F(T)$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|y_n - q\| &= \|(1-\lambda_n)x_n + \lambda_n T x_n - q\| \\ &= \|(1-\lambda_n)(x_n - q) + \lambda_n \|T x_n - q\| \\ &\leq (1-\lambda_n)\|x_n - q\| - \lambda_n \|x_n - q\| \\ &= \|x_n - q\|, \end{aligned} \tag{3.8}$$

and hence using (3.8), we get

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\gamma_n)x_n + \gamma_n T y_n - q\| \\ &\leq (1-\gamma_n)\|x_n - q\| + \gamma_n \|T y_n - q\| \\ &\leq (1-\gamma_n)\|x_n - q\| + \gamma_n \|y_n - q\| \\ &\leq \|x_n - q\|. \end{aligned} \tag{3.9}$$

Then, we can conclude that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ and $\{y_n\}$ are bounded. Since T is quasi-nonexpansive, $\{T x_n\}$ and $\{T y_n\}$ are also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}} \|x_n - q\|,\ \sup_{n\in\mathbb{N}} \|T x_n - q\|,\ \sup_{n\in\mathbb{N}} \|y_n - q\|,\ \sup_{n\in\mathbb{N}} \|T y_n - q\|\}.$$

Hence, by Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_r$ and t with $0 \leq t \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. Then, for all $q \in F(T)$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} \|y_n - q\|^2 &= \|(1-\lambda_n)x_n + \lambda_n T x_n - q\|^2 \\ &= \|(1-\lambda_n)(x_n - q) + \lambda_n (T x_n - q)\|^2 \\ &\leq (1-\lambda_n)\|x_n - q\|^2 + \lambda_n \|x_n - q\|^2 \\ &\qquad - \lambda_n(1-\lambda_n)g(\|x_n - T x_n\|) \\ &= \|x_n - q\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T x_n\|), \end{aligned} \tag{3.10}$$

and hence

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\gamma_n)x_n+\gamma_n Ty_n-q\|^2\\
&= \|(1-\gamma_n)(x_n-q)+\gamma_n(Ty_n-q)\|^2\\
&\le (1-\gamma_n)\|x_n-q\|^2+\gamma_n\|y_n-q\|^2\\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n-Ty_n\|)\\
&\le (1-\gamma_n)\|x_n-q\|^2+\gamma_n\|x_n-q\|^2\\
&\quad -\gamma_n\lambda_n(1-\lambda_n)g(\|x_n-Tx_n\|)\\
&\quad -\gamma_n(1-\gamma_n)g(\|x_n-Ty_n\|)\\
&\le \|x_n-q\|^2-\gamma_n\lambda_n(1-\lambda_n)g(\|x_n-Tx_n\|).
\end{aligned}
\tag{3.11}
$$

Since $0<a\le\gamma_n\le 1$, it is easy to see that

$$\|x_{n+1}-q\|^2\le\|x_n-q\|^2-a\lambda_n(1-\lambda_n)g(\|x_n-Tx_n\|).$$

So,

$$0\le a\lambda_n(1-\lambda_n)g(\|x_n-Tx_n\|)\le\|x_n-q\|^2-\|x_{n+1}-q\|^2\to 0,$$

as $n\to\infty$, since $\liminf_{n\to\infty}\lambda_n(1-\lambda_n)>0$. Therefore

$$\lim_{n\to\infty}g(\|x_n-Tx_n\|)=0.$$

From the properties of g, we get

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.12}$$

Now, we conclude from boundedness of $\{x_n\}$ and reflexivity of E that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q\in C$. So Theorem 3.4 and (3.12) imply that $Tq=q$. We will prove that the sequence $\{x_n\}$ converges weakly to some point of $F(T)$. Suppose that there exist two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q$ and $x_{n_i}\rightharpoonup p$. Assume that $q\neq p$. We know that $\lim_{n\to\infty}\|x_n-q\|$ and $\lim_{n\to\infty}\|x_n-p\|$ exist, since $q,p\in F(T)$. So, Opial's condition on E implies that

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n-q\|=\lim_{i\to\infty}\|x_{n_i}-q\|<\lim_{i\to\infty}\|x_{n_i}-p\| &= \lim_{n\to\infty}\|x_n-p\|\\
&= \lim_{j\to\infty}\|x_{n_j}-p\|\\
&< \lim_{j\to\infty}\|x_{n_j}-q\|\\
&= \lim_{n\to\infty}\|x_n-q\|.
\end{aligned}
$$

This is a contradiction. Therefore, we obtain $q=p$. This yields that $\{x_n\}$ converges weakly to a point of $F(T)$. □

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition. Suppose that T is a hybrid self mapping of C with $F(T) \neq \emptyset$. Assume that $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n T y_n, \end{cases}$$

where $0 \leq \lambda_n \leq 1$, $0 < a \leq \gamma_n \leq 1$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty} \lambda_n(1-\lambda_n) > 0$. Then $x_n \rightharpoonup x_0 \in F(T)$.

Proof. Since T is a hybrid self mapping of C, so T is a $(3,-1,-1,0)$-symmetric generalized hybrid mapping. Therefore by Theorem 3.5, we get the desired result. □

Remark 3.7. Since nonexpansive mappings are $(1,0,-1,0)$-symmetric generalized hybrid mappings and nonspreading mappings are $(2,-1,0,0)$-symmetric generalized hybrid mappings, then Theorem 3.5 holds for these mappings.

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