ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 771–780

Title:

A characterization of curves in Galilean 4-space $\mathbb{G}_4$

Author(s):

G. Öztürk, S. Büyükkütük and İ. Kişi

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 771–780
Online ISSN: 1735-8515

A CHARACTERIZATION OF CURVES IN GALILEAN 4-SPACE $\mathbb{G}_4$

G. ÖZTÜRK*, S. BÜYÜKKÜTÜK AND İ. KIŞI

(Communicated by Jost-Hinrich Eschenburg)

Abstract. In the present study, we consider a regular curve in Galilean 4-space $\mathbb{G}_4$ whose position vector is written as a linear combination of its Frenet vectors. We characterize such curves in terms of their curvature functions. Further, we obtain some results of rectifying, constant ratio, T-constant and N-constant curves in $\mathbb{G}_4$.
Keywords: Frenet frame, constant-ratio curves, rectifying curves.
MSC(2010): Primary: 53A04; Secondary: 53A35.

1. Introduction

Rectifying curves in Euclidean 3-space $\mathbb{E}^3$ were introduced by B.Y. Chen in [5] as space curves whose position vectors (denoted also by α) lie in their rectifying planes spanned by the tangent and binormal vector fields $t(s)$ and $b(s)$ of the curve. In the same paper, B.Y. Chen gave a simple characterization of rectifying curves. In particular, it is shown in [6] that there exists a simple relation between rectifying curves and centrodes which play an important role in mechanics, kinematics as well as in differential geometry in defining the curves of constant procession.

In recent years, researchers have begun to investigate the curves and surfaces in Galilean space and thereafter psedou-Galilean space. The theory of the curves in Galilean space is extensively studied in Röschel [17]. Furthermore, many works related to Galilean geometry have been done by several authors. In [11], the authors studied helices in Galilean space $\mathbb{G}_3$, and in [15] authors studied some curves in Galilean space. Similar studies about Galilean geometry are found in [1,2] and [12]. In [18], the authors constructed Frenet-Serret frame of a curve in the Galilean 4-space. Also in [13], the author studied Bertrand curves in Galilean 4-space.

Article electronically published on 30 June, 2017.
Received: 19 December 2015, Accepted: 19 February 2016.
*Corresponding author.

For a unit speed regular curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$, the hyperplanes at each point of $\alpha(s)$ spanned by $\{t, n, b\}$, $\{t, n, e\}$ are known as *osculating hyperplanes.* If the position vector α lies on its *osculating hyperplane,* then $\alpha(s)$ is called as an *osculating curve.* In [19], D. W. Yoon, J. W. Lee and C.W. Lee considered the osculating curves in the Galilean 4-space $\mathbb{G}_4$.

For a regular curve $\alpha(s)$, the position vector α can be decomposed into its tangential and normal components at each point:

$$\alpha = \alpha^T + \alpha^N. \tag{1.1}$$

A curve $\alpha(s)$ with $\kappa(s) > 0$ is said to be of *constant ratio* if the ratio $\|\alpha^T\| : \|\alpha^N\|$ is constant on $\alpha(I)$, where $\|\alpha^T\|$ and $\|\alpha^N\|$ denote the length of α^T and α^N, respectively [3].

A curve in $\mathbb{E}^n$ is called *T-constant* (resp. *N-constant*) if the tangential component α^T (resp. the normal component α^N) of its position vector α is of constant length [4]. Regarding this study, in [8], the authors gave necessary and sufficient conditions for twisted curves in Euclidean 3-space $\mathbb{E}^3$ to become T-constant and N-constant.

In the present study, we consider a curve in the Galilean 4-space $\mathbb{G}_4$ as a curve whose position vector satisfies the parametric equation

$$\alpha(s) = m_0(s)t(s) + m_1(s)n(s) + m_2(s)b(s) + m_3(s)e(s), \tag{1.2}$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 3$, where $\{t, n, b, e\}$ is the curve's Frenet frame. We characterize such curves in terms of their curvature functions $m_i(s)$ and give necessary and sufficient conditions for such curves to become rectifying, constant ratio, T-constant and N-constant curves in $\mathbb{G}_4$.

2. **Basic notations**

In this section, some fundamental properties of curves in four dimensional Galilean space are given for the purpose of the requirements [18].

In the affine coordinates, the Galilean scalar product between two points

$$P_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}), \quad i = 1, 2 \tag{2.1}$$

is defined as

$$g(P_1, P_2) = \left\{ \begin{array}{ll} |p_{21} - p_{11}|, & \text{if } p_{21} \neq p_{11} \\ \left((p_{22} - p_{12})^2 + (p_{23} - p_{13})^2 + (p_{24} - p_{14})^2\right)^{\frac{1}{2}}, & \text{if } p_{21} = p_{11} \end{array} \right\}. \tag{2.2}$$

The Galilean cross product in $\mathbb{G}_4$ for the vectors $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4)$ and $\vec{w} = (w_1, w_2, w_3, w_4)$ is defined as

$$\vec{u} \times \vec{v} \times \vec{w} = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}, \tag{2.3}$$

where e_i, $1 \leq i \leq 4$, are the standard basis vectors.

The scalar product of two vectors $\vec{u} = (u_1, u_2, u_3, u_4)$ and $\vec{v} = (v_1, v_2, v_3, v_4)$ in $\mathbb{G}_4$ is defined as

$$\left\langle \vec{u}, \vec{v} \right\rangle_{\mathbb{G}_4} = \left\{ \begin{array}{ll} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ u_2 v_2 + u_3 v_3 + u_4 v_4 & \text{if } u_1 = 0 \ , \ \ v_1 = 0. \end{array} \right\}. \tag{2.4}$$

The norm of vector $\vec{u} = (u_1, u_2, u_3, u_4)$ is defined as

$$\left\| \vec{u} \right\|_{\mathbb{G}_4} = \sqrt{\left| \left\langle \vec{u}, \vec{u} \right\rangle_{\mathbb{G}_4} \right|}. \tag{2.5}$$

Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4, \alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arclength parameter s in $\mathbb{G}_4$. The first vector of the Frenet-Serret frame, that is, the tangent vector of α, is defined as

$$t = \alpha'(s) = (1, y'(s), z'(s), w'(s)) \tag{2.6}$$

Since t is a unit vector, we can express

$$\langle t, t \rangle_{\mathbb{G}_4} = 1. \tag{2.7}$$

Differentiating (2.7) with respect to s, we have

$$\langle t', t \rangle_{\mathbb{G}_4} = 0. \tag{2.8}$$

The vector function t' gives us the rotation measurement of the curve α. The real valued function

$$\kappa(s) = \|t'(s)\| = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2} \tag{2.9}$$

is called the first curvature of the curve α. For all $s \in I$, we assume that $\kappa(s) \neq 0$. Similar to space $\mathbb{G}_3$, the principal normal vector is defined as

$$n(s) = \frac{t'(s)}{\kappa(s)}; \tag{2.10}$$

in other words,

$$n(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s), w''(s)). \tag{2.11}$$

By the aid of the differentiation of the principal normal vector given in (2.11), the second curvature function is defined as

$$\tau(s) = \|n'(s)\|_{\mathbb{G}_4}. \tag{2.12}$$

This real valued function is called torsion of the curve α. The third vector field, namely, binormal vector field of the curve α, is defined as

$$b(s) = \frac{1}{\tau(s)} \left(0, \left(\frac{y''(s)}{\kappa(s)} \right)', \left(\frac{z''(s)}{\kappa(s)} \right)', \left(\frac{w''(s)}{\kappa(s)} \right)' \right). \tag{2.13}$$

Thus, the vector $b(s)$ is perpendicular to both t and n. The fourth unit vector is defined as

$$e(s) = \mu t(s) \times n(s) \times b(s). \tag{2.14}$$

Here, the coefficient μ is taken ± 1 to make the determinant of the $[t, n, b, e]$ matrix $+1$.

The third curvature of the curve α is defined as

$$\sigma = \langle b', e \rangle_{\mathbb{G}_4}. \tag{2.15}$$

Here, as well known, the set $\{t, n, b, e, \kappa, \tau, \sigma\}$ is called the Frenet-Serret apparatus of the curve α. We know that the vectors $\{t, n, b, e\}$ are mutually orthogonal vectors satisfying

$$\begin{aligned} \langle t, t \rangle_{\mathbb{G}_4} &= \langle n, n \rangle_{\mathbb{G}_4} = \langle b, b \rangle_{\mathbb{G}_4} = \langle e, e \rangle_{\mathbb{G}_4} = 1, \\ \langle t, n \rangle_{\mathbb{G}_4} &= \langle t, b \rangle_{\mathbb{G}_4} = \langle t, e \rangle_{\mathbb{G}_4} = 0, \\ \langle n, b \rangle_{\mathbb{G}_4} &= \langle n, e \rangle_{\mathbb{G}_4} = \langle b, e \rangle_{\mathbb{G}_4} = 0. \end{aligned} \tag{2.16}$$

For the curve α in $\mathbb{G}_4$, we have the following Frenet-Serret equations:

$$\begin{aligned} t' &= \kappa(s) n(s), \\ n' &= \tau(s) b(s), \\ b' &= -\tau(s) n(s) + \sigma(s) e(s), \\ e' &= -\sigma(s) b(s). \end{aligned} \tag{2.17}$$

If the Serret-Frenet curvatures $\kappa(s), \tau(s)$ and $\sigma(s)$ of α are constant functions, then α is called a screw line or a helix [7]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations, F. Klein and S. Lie called them *W-curves* [9]. If the tangent vector t of the curve α makes a constant angle with the unit vector u of $\mathbb{G}_4$, then this curve is called a general helix (or inclined curve) in $\mathbb{G}_4$ [16]. It is known that a regular curve in $\mathbb{G}_4$ is said to have constant curvature ratios if the ratios of the consecutive curvatures are constant [10]. The Frenet curves with constant curvature ratios are called ccr-curves [16]. We remark that a regular curve in $\mathbb{G}_4$ is a ccr-curve if $H_1(s) = \frac{\kappa}{\tau}(s)$ and $H_2(s) = \frac{\sigma}{\tau}(s)$ are constant functions.

3. Characterization of curves in $\mathbb{G}_4$

In the present section, we characterize a curve given with the arclength parameter s in $\mathbb{G}_4$ in terms of their curvatures. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed regular curve with curvatures $\kappa(s) \geq 0$, $\tau(s)$ and $\sigma(s)$. For some differentiable functions $m_i(s)$, $0 \leq i \leq 3$, the position vector of the curve satisfies the vectorial equation (1.2). Differentiating (1.2) with respect to arclength parameter s and using the Serret-Frenet equations (2.17), we obtain

$$
\begin{aligned}
\alpha'(s) &= m_0'(s)t(s) \\
&\quad + \left(m_1'(s) + \kappa(s)m_0(s) - \tau(s)m_2(s)\right) n(s) \\
&\quad + \left(m_2'(s) + \tau(s)m_1(s) - \sigma(s)m_3(s)\right) b(s) \\
&\quad + \left(m_3'(s) + \sigma(s)m_2(s)\right) e(s).
\end{aligned}
\tag{3.1}
$$

Hence,

$$
\begin{aligned}
m_0'(s) &= 1, \\
m_1'(s) + \kappa(s)m_0(s) - \tau(s)m_2(s) &= 0, \\
m_2'(s) + \tau(s)m_1(s) - \sigma(s)m_3(s) &= 0, \\
m_3'(s) + \sigma(s)m_2(s) &= 0.
\end{aligned}
\tag{3.2}
$$

Theorem 3.1 ([18])**.** *Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arclength parameter s in $\mathbb{G}_4$ with the Frenet-Serret equations (2.17). Tangent vector of α in $\mathbb{G}_4$ satisfies a vector differential equation of fourth order as follows:*

$$
\left\{\frac{1}{\sigma}\left[\frac{1}{\tau}\left(\frac{t'}{\kappa}\right)'\right]'\right\}' + \left[\frac{\tau}{\kappa\sigma}\right]' + \frac{\sigma}{\tau}\left(\frac{t'}{\kappa}\right)' = 0. \tag{3.3}
$$

Theorem 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a curve with $\kappa > 0$, and let s be its arclength function. If α is a W-curve, then the curvature functions are given as*

$$
\begin{aligned}
m_0(s) &= s + c, \\
m_1(s) &= \frac{\tau c_1}{a}\sin as - \frac{\tau c_2}{a}\cos as - \frac{\kappa\sigma^2}{a^2}\left(\frac{s^2}{2} + cs + c_3\right), \\
m_2(s) &= c_1\cos as + c_2\sin as + \frac{\kappa\tau}{a^2}\left(s + c\right), \\
m_3(s) &= -\frac{\sigma c_1}{a}\sin as + \frac{\sigma c_2}{a}\cos as - \frac{\kappa\tau\sigma}{a^2}\left(\frac{s^2}{2} + cs + c_4\right),
\end{aligned}
\tag{3.4}
$$

where c_i, $0 \le i \le 4$, are integral constants and $a = \sqrt{\tau^2 + \sigma^2}$ is a real constant.

Proof. Let α be a W-curve in $\mathbb{G}_4$, then by the use of the equations (3.2), we get

$$
m_2''(s) + (\tau^2 + \sigma^2)m_2(s) = \kappa\tau\left(s + c\right). \tag{3.5}
$$

One can show that equation (3.5) has a non-trivial solution:

$$
m_2(s) = c_1\cos\sqrt{\tau^2 + \sigma^2}s + c_2\sin\sqrt{\tau^2 + \sigma^2}s + \frac{\kappa\tau}{\tau^2 + \sigma^2}\left(s + c\right).
$$

Further, substituting this equation into (3.2), we get the result. $\square$

Definition 3.3 ([1])**.** Let α be a regular curve in Galilean space $\mathbb{G}_4$ given with the Frenet frame $\{t, n, b, e\}$ and κ be its curvature. If $\kappa = 0$, then α is called a straight line with respect to the Frenet frame.

3.1. Osculating curves in $\mathbb{G}_4$.

Definition 3.4 ([19])**.** Let α be a unit speed curve in $\mathbb{G}_4$. If its position vector always lies in the orthogonal complement $b^{\perp}$ or $e^{\perp}$ of b or e, then α is called an osculating curve in $\mathbb{G}_4$. Consequently, for some smooth functions, $m_i(s)$, $0 \leq i \leq 3$, an osculating curve can be expressed as

$$\alpha(s) = m_o(s)t(s) + m_1(s)n(s) + m_3(s)e(s) \tag{3.6}$$

or

$$\alpha(s) = m_o(s)t(s) + m_1(s)n(s) + m_2(s)b(s). \tag{3.7}$$

Theorem 3.5 ([19])**.** *Let α be a unit speed curve in $\mathbb{G}_4$ with non-zero curvatures κ, τ and σ. Then α is an osculating curve if and only if*

$$-\frac{d}{\kappa}\left(H_2'\right) = s + c, \tag{3.8}$$

where $H_2(s) = \frac{\sigma(s)}{\tau(s)}$ and c,d are non-zero constants.

Corollary 3.6 ([19])**.** *None of a unit speed W-curve with non-zero curvatures κ, τ and σ in $\mathbb{G}_4$ is an osculating curve.*

Corollary 3.7. *None of a unit speed ccr-curve with non-zero curvatures κ, τ and σ in $\mathbb{G}_4$ is an osculating curve.*

Proof. Let α be a unit speed *ccr*-curve in $\mathbb{G}_4$, then we know, H_2 is constant. If this curve is an osculating curve, H_2 cannot be a constant in equation (3.8). That is a contradiction. So, none of a unit speed *ccr*-curve in $\mathbb{G}_4$ is an osculating curve. □

3.2. Rectifying curves in $\mathbb{G}_4$.

Similar to [14], we give the following definition:

Definition 3.8. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a regular curve in $\mathbb{G}_4$ given with the arclength parameter s. If the position vector α lies in the hyperplane spanned by $\{t, b, e\}$, then α is called a rectifying curve in $\mathbb{G}_4$.

Theorem 3.9. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. Then α is a rectifying curve if and only if the position vector of α satisfies the equality:*

$$\alpha(s) = (s + c)\, t(s) + H_1\,(s + c)\, b(s) + \frac{1}{\sigma}\left(H_1'\,(s + c) + H_1\right) e(s), \tag{3.9}$$

where $H_1(s) = \frac{\kappa}{\tau}(s)$.

Proof. Assume that $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{G}_4$ is a rectifying curve in $\mathbb{G}_4$ given with the arclength parameter s. By definition, the curvature function m_1 vanishes identically. Therefore, from (3.2), we get

$$\begin{aligned} m_0'(s) &= 1, \\ \kappa(s)m_0(s) - \tau(s)m_2(s) &= 0, \\ m_2'(s) - \sigma(s)m_3(s) &= 0, \\ m_3'(s) + \sigma(s)m_2(s) &= 0, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} m_0(s) &= s + c, \\ m_2(s) &= H_1\left(s + c\right), \\ m_3(s) &= \frac{1}{\sigma}\left(H_1'\left(s + c\right) + H_1\right), \end{aligned} \tag{3.11}$$

where $H_1(s) = \frac{\kappa}{\tau}(s)$. Thus, the position vector of α satisfies the equality (3.9). □

By the use of (3.10) with (3.11), we obtain the following result.

Proposition 3.10. *$\alpha : I \subset \mathbb{R} \rightarrow \mathbb{G}_4$ is congruent to a unit speed rectifying curve in $\mathbb{G}_4$ if and only if*

$$\left(H_1\left(s + c\right)\right)^2 + \frac{1}{\sigma^2}\left(H_1'\left(s + c\right) + H_1\right)^2 = c_1, \tag{3.12}$$

where $H_1(s) = \frac{\kappa}{\tau}(s)$ and $c, c_1 \in \mathbb{R}$.

As a consequence of (3.12), we obtain the following result.

Corollary 3.11. *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{G}_4$ be a unit speed rectifying curve. If α is a ccr-curve, then*

$$\sigma(s) = \frac{H_1}{\sqrt{c_1 - \left(H_1\left(s + c\right)\right)^2}},$$

where $c, c_1 \in \mathbb{R}$.

3.3. **Curves of constant-ratio in $\mathbb{G}_4$.**

Similar to [3], we give the following definition:

Definition 3.12. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{G}_4$ be a unit speed curve given with the arclength parameter s. Then the position vector α can be decomposed into its tangential and normal components at each point, as in (1.1). If the ratio $\left\|\alpha^T\right\| : \left\|\alpha^N\right\|$ is constant on $\alpha(I)$ then α is said to be of *constant-ratio.*

Clearly, for a constant ratio curve in Galilean space $\mathbb{G}_4$, we have

$$\frac{m_0^2(s)}{m_1^2(s) + m_2^2(s) + m_3^2(s)} = c_1, \tag{3.13}$$

where $c_1 \in \mathbb{R}$.

Theorem 3.13. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. Then α is a constant ratio curve if and only if the position vector of the curve α has a parametrization of the form:*

$$\begin{aligned}\alpha(s) &= (s+c)t(s) - \left(\frac{1}{c_1\kappa}\right) n(s) + \left(\frac{\kappa'}{c_1\kappa^2\tau} + \frac{\kappa}{\tau}(s+c)\right) b(s) \\ &\quad + \frac{1}{\sigma}\left(\left[\frac{\kappa'}{c_1\kappa^2\tau} + \frac{\kappa}{\tau}(s+c)\right]' - \frac{\tau}{c_1\kappa}\right) e(s).\end{aligned}$$

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve. Then by (3.13), the curvature functions satisfy

$$m_1(s)m_1'(s) + m_2(s)m_2'(s) + m_3(s)m_3'(s) = \frac{s+c}{c_1}. \tag{3.14}$$

Also, by the use of the equations (3.2) with (3.14), we have

$$m_1(s) = -\frac{1}{c_1\kappa}.$$

Then, we get

$$\begin{aligned} m_2(s) &= \frac{\kappa'}{c_1\kappa^2\tau} + \frac{\kappa}{\tau}(s+c), \\ m_3(s) &= \frac{1}{\sigma}\left(\left[\frac{\kappa'}{c_1\kappa^2\tau} + \frac{\kappa}{\tau}(s+c)\right]' - \frac{\tau}{c_1\kappa}\right), \end{aligned}$$

which completes the proof. □

Corollary 3.14. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. Then α is a constant ratio curve if and only if*

$$\left\{\frac{1}{\sigma}\left(\left[\frac{\kappa'}{c_1\kappa^2\tau} + \frac{\kappa}{\tau}(s+c)\right]' - \frac{\tau}{c_1\kappa}\right)\right\}' + \frac{\sigma\kappa'}{c_1\kappa^2\tau} + \frac{\sigma\kappa}{\tau}(s+c) = 0.$$

3.4. *T*-constant curves in $\mathbb{G}_4$.

Similar to [4], we give the following definition:

Definition 3.15. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. If $\left\|\alpha^T\right\|$ is constant, then α is called a *T-constant curve*. Further, a T-constant curve α is called of the first kind if $\left\|\alpha^T\right\| = 0$; otherwise it is called of the second kind.

As a consequence of (1.2) with (3.2), we get the following result.

Proposition 3.16. *There is no unit speed T-constant curve in the Galilean space $\mathbb{G}_4$.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed T-constant curve in $\mathbb{G}_4$. Then $\left\|\alpha^T\right\| = m_0$ is zero or is a non-zero constant. However, $m_0(s) = s + c$ by equations (3.2). As this is a contradiction, the assumption is wrong. □

3.5. N-constant curves in $\mathbb{G}_4$.

Similar to [4], we give the following definition:

Definition 3.17. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. If $\left\|\alpha^N\right\|$ is constant then α is called a N-constant curve. For a N-constant curve α, either $\left\|\alpha^N\right\| = 0$ or $\left\|\alpha^N\right\| = \mu$ for some non-zero smooth function μ. Further, a N-constant curve α is called of the first kind if $\left\|\alpha^N\right\| = 0$; otherwise it is called of the second kind.

Note that, for a N-constant curve α in $\mathbb{G}_4$, we can write

$$\left\|\alpha^N(s)\right\|^2 = m_1^2(s) + m_2^2(s) + m_3^2(s) = c_1, \tag{3.15}$$

where c_1 is a real constant.

As a consequence of (1.2), (3.2) and (3.15), we get the following result.

Lemma 3.18. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. Then, α is a N-constant curve if and only if*

$$\begin{aligned} m_0'(s) &= 1, \\ m_1'(s) + \kappa(s)m_0(s) - \tau(s)m_2(s) &= 0, \\ m_2'(s) + \tau(s)m_1(s) - \sigma(s)m_3(s) &= 0, \\ m_3' + \sigma(s)m_2(s) &= 0, \\ m_1(s)m_1'(s) + m_2(s)m_2'(s) + m_3(s)m_3'(s) &= 0, \end{aligned} \tag{3.16}$$

hold.

Proposition 3.19. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_4$ be a unit speed curve in $\mathbb{G}_4$. Then, α is a N-constant curve if and only if α is congruent to a straight line or a rectifying curve.*

Proof. By the use of the equations (3.16), we get $\kappa m_0 m_1 = 0$. Thus, $\kappa = 0$ or $m_1 = 0$. If $\kappa = 0$, α is a straight line. If $m_1 = 0$, then α is a rectifying curve. □

Acknowledgement

The authors would like to thank the referee(s) for their careful readings, valuable suggestions and comments, which helped to improve the presentation of the paper.

References

[1] T.A. Ahmad, Position vectors of curves in Galilean space $\mathbb{G}_3$, *Mat. Vesnik* **64** (2012), no. 3, 200–210.

[2] A.Z. Azak, M. Akyiğit and S. Ersoy, Involute-Evolute curves in Galilean space $\mathbb{G}_3$, *Sci. Magna* **4** (2010), no. 6, 75–80.

[3] B.Y. Chen, Constant ratio hypersurfaces, *Soochow J. Math.* **28** (2001), no. 4, 353–362.

[4] B.Y. Chen, Geometry of warped products as Riemannian submanifolds and related problems, *Soochow J. Math.* **28** (2002), no. 2, 125–156.

[5] B.Y. Chen, When does the position vector of a space curve always lie in its rectifying plane?, *Amer. Math. Monthly* **110** (2003), no. 2, 147–152.

[6] B.Y. Chen and F. Dillen, Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Acad. Sin.* **33** (2005), no. 2, 77–90.

[7] A. Gray, Modern Differential Geometry of Curves and Surfaces, CRC Press, 1993.

[8] S. Gürpınar, K. Arslan and G. Öztürk, A characterization of constant-ratio curves in Euclidean 3-space $\mathbb{E}^3$, *Acta Univ. Apulensis Math. Inform.* **44** (2015) 39–51.

[9] F. Klein and S. Lie, Uber diejenigen ebenenen kurven welche durch ein geschlossenes system von einfach unendlich vielen vartauschbaren linearen Transformationen in sich übergehen. acte, *Math. Ann.* **4** (1871) 50–84.

[10] J. Monterde, Curves with constant curvature ratios, *Bol. Soc. Mat. Mex.* **13** (2007), no. 1, 177–186.

[11] A.O. Öğrenmiş, M. Ergüt and M. Bektaş, On the helices in the Galilean space $\mathbb{G}_3$, *Iran. J. Sci. Technol. Trans. A Sci.* **31** (2007), no. 2, 177–181.

[12] A.O. Öğrenmiş, H. Öztekin and M. Ergüt, Bertrand curves in Galilean space and their characterizations, *Kragujevac J. Math* **32** (2009) 139–147.

[13] H. Öztekin, Special Bertrand curves in 4D Galilean space, *Math. Probl. Eng.* **2014** (2014), Article ID 318458, 7 pages.

[14] H. Öztekin and A.O. Öğrenmiş, Normal and rectifying curves in pseudo-Galilean space $\mathbb{G}_3^1$ and their characterizations, *J. Math. Comput. Sci.* **2** (2012), no. 1, 91–100.

[15] H.B. Öztekin and S. Tatlıpınar, On some curves in Galilean plane and 3-dimensional Galilean space, *J. Dyn. Syst. Geom. Theor.* **10** (2012), no. 2, 189–196.

[16] G. Öztürk, K. Arslan and H.H. Hacisalihoğlu, A characterization of ccr-curves in $\mathbb{R}^n$, *Proc. Est. Acad. Sci.* **57** (2008), no. 4, 217–224.

[17] O. Röschel, Die Geometrie des Galileischen Raumes, Forschungszentrum Graz Research Centre, Austria, 1986.

[18] S. Yılmaz, Construction of the Frenet-Serret frame of a curve in 4D Galilean space and some applications, *Int. J. Phys. Sci.* **5** (2010), no. 5, 1284–1289.

[19] D.W. Yoon, J.W. Lee and C.W. Lee, Osculating curves in the Galilean 4-Space, *Int. J. Pure Appl. Math.* **100** (2015), no. 4, 497–506.

(Günay Öztürk) Kocaeli University, Art and Science Faculty, Department of Mathematics, Kocaeli, Turkey.

E-mail address: `ogunay@kocaeli.edu.tr`

(Sezgin Büyükkütük) Kocaeli University, Art and Science Faculty, Department of Mathematics, Kocaeli, Turkey.

E-mail address: `sezgin.buyukkutuk@kocaeli.edu.tr`

(İlim Kişi) Kocaeli University, Art and Science Faculty, Department of Mathematics, Kocaeli, Turkey.

E-mail address: `ilim.ayvaz@kocaeli.edu.tr`