ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 3, pp. 943–949

Title:

Simple axiomatization of reticulations on residuated lattices

Author(s):

M. Kondo

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 3, pp. 943–949
Online ISSN: 1735-8515

SIMPLE AXIOMATIZATION OF RETICULATIONS ON RESIDUATED LATTICES

M. KONDO

(Communicated by Ali Enayat)

Abstract. We give a simple and independent axiomatization of reticulations on residuated lattices, which were axiomatized by five conditions in [C. Mureşan, The reticulation of a residuated lattice, Bull. Math. Soc. Sci. Math. Roumanie 51 (2008), no. 1, 47–65]. Moreover, we show that reticulations can be considered as lattice homomorphisms between residuated lattices and bounded distributive lattices. Consequently, the result proved by Mureşan in 2008, for any two reticulattions $(L_1, \lambda_1), (L_2, \lambda_2)$ of a residuated lattice X there exists an isomorphism $f : L_1 \to L_2$ such that $f \circ \lambda_1 = \lambda_2$, can be considered as a homomorphism theorem.
Keywords: Reticulation, residuated lattice, principal filter.
MSC(2010): Primary: 03G10; Secondary: 06F35.

1. **Introduction**

A notion of reticulation which provides topological properties on algebras has been introduced on commutative rings in 1980 by Simmons in [5]. For a given commutative ring A, a pair (L, λ) of a bounded distributive lattice L and a map $\lambda : A \to L$ satisfying some conditions is called a reticulation of A, and the map λ gives a homeomorphism between the topological space $Spec(A)$ consisting of all prime filters of A and the topological space $Spec(L)$ consisting of all prime filters on L. The concept of reticulation is generalized to non-commutative rings, MV-algebras, BL-algebras, quantale and so on (see [1–3]). Since these algebras are axiomatic extensions of residuated lattices which are algebraic semantics of so-called fuzzy logic, it is natural to consider properties of reticulations on residuated lattices. In 2008, Mureşan has published a paper [4] about reticulations on residuated lattices and she has provided an axiomatic definition of reticulations on residuated lattices, in which five conditions are needed. In this short note, we show that only two independent conditions of reticulation are enough to axiomatize reticulations on residuated lattices

Article electronically published on 30 June, 2017.
Received: 14 October 2015, Accepted: 19 April 2016.

and also prove that reticulations on residuated lattices can be considered as homomorphisms between residuated lattices and bounded distributive lattices.

2. **Residuated lattices and reticulations**

An algebraic system $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a *residuated lattice* if

(1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(A, \odot, 1)$ is a commutative monoid, that is, for all $a, b \in A$,
$a \odot b = b \odot a$ and $a \odot 1 = 1 \odot a = a$;
(3) For all $a, b, c \in A$,

$$a \odot b \leq c \Longleftrightarrow a \leq b \rightarrow c.$$

We have basic results about residuated lattices.

Proposition 2.1. *Let A be a residuated lattice. For all $a, b, c \in A$, we have*

(1) $a \leq b \Longleftrightarrow a \rightarrow b = 1$;
(2) $a \rightarrow (b \rightarrow c) = a \odot b \rightarrow c = b \rightarrow (a \rightarrow c)$;
(3) $a \odot (a \rightarrow b) \leq b$;
(4) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$;
(5) $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$;
(6) $(a \vee b)^{m+n} \leq a^m \vee b^n$, *where $m, n \in \mathbb{N}$ and* $a^m = \overbrace{a \odot a \odot \cdots \odot a}^{m}$.

Proof. We only show the case (6): $(a \vee b)^{m+n} \leq a^m \vee b^n$. Since every term in the expansion of $(a \vee b)^{m+n}$ is a sequence of a and b with length $m+n$ and $a^m \odot b^n = (a^m \odot b^n) \vee (a^m \odot b^n)$, we have

$$\begin{aligned} (a \vee b)^{m+n} &= \overbrace{(a \vee b) \odot \cdots \odot (a \vee b)}^{m+n} \\ &= \overbrace{a^{m+n} \vee (a^{m+n-1} \odot b) \vee \cdots \vee (a^m \odot b^n)}^{n+1} \\ &\vee \overbrace{(a^m \odot b^n) \vee \cdots \vee (a \odot b^{m+n-1}) \vee b^{m+n}}^{m} \\ &\leq \overbrace{a^m \vee a^m \vee \cdots \vee a^m}^{n+1} \vee \overbrace{b^n \vee \cdots \vee b^n \vee b^n}^{m} \\ &= a^m \vee b^n. \end{aligned}$$

□

A non-empty subset $F \subseteq A$ of a residuated lattice A is called a *filter* if

(F1) If $a, b \in F$ then $a \odot b \in F$;
(F2) If $a \in F$ and $a \leq c$ then $c \in F$.

For an element $a \in A$, we set

$$[a) = \{b \in A \mid \exists n \in \mathbb{N} \text{ s.t. } a^n \leq b\}$$

and it is called a *principal filter*. By $\mathcal{F}(A)$ (or $\mathcal{PF}(A)$), we mean the set of all filters (or principal filters, respectively) of A.

Moreover, a filter $P(\neq A)$ is called *prime* if it satisfies a condition that $a \in P$ or $b \in P$ whenever $a \vee b \in P$. We denote the set of all prime filters of A by $Spec(A)$.

For a bounded lattice L, a non-empty subset F of L is called a *lattice filter* if

(LF1) If $x, y \in F$ then $x \wedge y \in F$;
(LF2) If $x \in F$ and $x \leq y$ then $y \in F$.

A lattice filter $F(\neq L)$ is called *prime* if it satisfies the condition that if $x \vee y \in F$ then $x \in F$ or $y \in F$. By $Spec(L)$ we mean the set of all prime lattice filters of L. It is trivial that every filter is also a lattice filter.

In the following, let A be a residuated lattice and L be a bounded distributive lattice. For any subset $S \subseteq A$, we define

$$D(S) = \{P \in Spec(A) \mid S \not\subseteq P\}.$$

It is easy to show that

Proposition 2.2. *$\tau_A = \{D(S) \mid S \subseteq A\}$ is a topology on $Spec(A)$ and $\{D(a)\}_{a \in A}$, where $D(a) = \{P \in Spec(A) \mid a \notin P\}$, forms a base of the topology τ_A.*

Similarly, we also define a topology on $Spec(L)$ for a bounded distributive lattice L as follows. For any subset $S \subseteq L$, we define

$$D(S) = \{P \in Spec(L) \mid S \not\subseteq P\}.$$

Proposition 2.3. *$\sigma_L = \{D(S) \mid S \subseteq L\}$ is a topology on $Spec(L)$ and $\{D(x)\}_{x \in L}$, where $D(x) = \{P \in Spec(L) \mid x \notin P\}$, forms a base for σ_L.*

According to [4], we define a reticulation. A pair (L, λ) of a bounded distributive lattice L and a map $\lambda : A \to L$ is called a *reticulation* on a residuated lattice A if the map satisfies the five conditions

(R1) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$;
(R2) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$;
(R3) $\lambda(0) = 0,\ \lambda(1) = 1$;
(R4) $\lambda : A \to L$ is surjective;
(R5) $\lambda(a) \leq \lambda(b)$ if and only if there exists $n \in \mathbb{N}$ such that $a^n \leq b$.

Proposition 2.4 ([4])**.** *Let (L, λ) be a reticulation of A. Then we have*

(1) *λ is order-preserving, that is, if $a \leq b$ then $\lambda(a) \leq \lambda(b)$;*
(2) *$\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$;*
(3) *For all $n \in \mathbb{N}$, $\lambda(a^n) = \lambda(a)$;*
(4) *$\lambda(a) = \lambda(b) \Longleftrightarrow [a) = [b)$.*

We also have the following results.

Proposition 2.5. *Let (L, λ) be a reticulation of A. Then we have*

(1) $\lambda(a \wedge b) = \lambda(a \odot (a \to b))$;
(2) $\lambda[a) = [\lambda a)$.

The next fundamental result about reticulation is very important.

Theorem 2.6 ([4])**.** *For a reticulation (L, λ) of A,*

(a) *$Spec(A)$ and $Spec(L)$ are topological spaces;*
(b) *$\lambda^* : Spec(L) \to Spec(A)$ is a homeomorphism, where λ^* is defined by $\lambda^*(P) = \lambda^{-1}(P)$ $(P \in Spec(L))$;*
(c) *If (L_1, λ_1) and (L_2, λ_2) are reticulations of a residuated lattice A, then there exists an isomorphism $f : L_1 \to L_2$ such that $f \circ \lambda_1 = \lambda_2$;*
(d) *$(\mathcal{PF}(A), \eta)$ is a reticulation on A, where $\eta : A \to \mathcal{PF}(A)$ is a map defined by $\eta(a) = [a)$.*

3. Simple axiomatization of reticulation

In this section we prove that the conditions (R1)-(R3) of reticulations can be proved from the rest (R4) and (R5), that is, reticulation can be defined by only two conditions (R4) and (R5). We note that the condition (R4) is independent from the conditions (R1)-(R3) and (R5) is also independent from (R1)-(R4) by [4]. It follows from our result that the conditions (R4) and (R5) are independent to each other. Let A be a residuated lattice and L be a bounded distributive lattice. Let $f : A \to L$ be a map satisfying the following conditions

(R4) $f : A \to L$ is surjective;
(R5) $f(a) \leq f(b) \Longleftrightarrow \exists n \in \mathbb{N}$ such that $a^n \leq b$.

Lemma 3.1. *For a map f satisfying* (R4) *and* (R5)*, we have*

(1) $a \leq b \Longrightarrow f(a) \leq f(b)$;
(2) $f(a \wedge b) = f(a \odot b)$;
(3) (R1) $f(a \wedge b) = f(a) \wedge f(b)$;
(4) (R2) $f(a \vee b) = f(a) \vee f(b)$;
(5) (R3) $f(0) = 0$, $f(1) = 1$.

Proof. (1) If $a \leq b$, using $a = a^1$, we have $a = a^1 \leq b$ and consequently using (R5) $f(a) \leq f(b)$.

(2) Since $a \odot b \leq a \wedge b$, we get $f(a \odot b) \leq f(a \wedge b)$ from (1). Moreover, since $(a \wedge b)^2 = (a \wedge b) \odot (a \wedge b) \leq a \odot b$, we also, using (R5), have $f(a \wedge b) \leq f(a \odot b)$. This implies that $f(a \wedge b) = f(a \odot b)$.

(3) It is trivial that $f(a \wedge b) \leq f(a), f(b)$, that is, $f(a \wedge b)$ is a lower bound of the set $\{f(a), f(b)\}$. For any lower bound l of $\{f(a), f(b)\}$, since f is surjective (R4), there is an element $c \in A$ such that $f(c) = l$. This implies that $f(c) \leq f(a), f(b)$; consequently, $c^m \leq a, c^n \leq b$ for some $m, n \in \mathbb{N}$ by (R5). Since

$c^{m+n} = c^m \odot c^n \leq a \odot b$, we get from (R5) that $l = f(c) = f(c^{m+n}) \leq f(a \odot b) = f(a \wedge b)$. Therefore, $f(a \wedge b) = \inf_L\{f(a), f(b)\} = f(a) \wedge f(b)$.

(4) It is obvious that $f(a), f(b) \leq f(a \vee b)$. For any $u \in L$, if $f(a), f(b) \leq u$, since $u = f(d)$ for some $d \in A$ by (R4), then we have $f(a), f(b) \leq f(d)$. It follows from (R5) that there exist $m, n \in \mathbb{N}$ such that $a^m \leq d, b^n \leq d$. Since $(a \vee b)^{m+n} \leq a^m \vee b^n \leq d \vee d = d$, we get that $f(a \vee b) \leq f(d) = u$ and consequently $f(a \vee b) = \sup_L\{f(a), f(b)\} = f(a) \vee f(b)$.

(5) For every $x \in L$, since f is surjective, there is an element $a \in A$ such that $f(a) = x$. It follows from $0 \leq a$ that $f(0) \leq f(a) = x$. If we take $x = 0$ then we have $f(0) = 0$. Similarly, we have $f(1) = 1$. □

The result means that the definition of reticulation is given only two conditions (R4) and (R5).

4. **Reticulation and homomorphism**

Let A be a residuated lattice and (L, λ) its reticulation. As proved above, the map λ satisfies the following conditions:

(h1) $\lambda(0) = 0,\ \lambda(1) = 1$;
(h2) $\lambda(a \wedge b) = \lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$;
(h3) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$.

This means that the map λ is an onto homomorphism from A to L of its reticulation with respect to the lattice operations. Let

$$\ker(\lambda) = \{(a, b) \mid \lambda(a) = \lambda(b), a, b \in A\}.$$

Proposition 4.1. $\ker(\lambda)$ *is a congruence on a residuated lattice A with respect to $\wedge, \vee, \odot$.*

We put $a/\ker(\lambda) = \{b \in A \mid (a, b) \in \ker(\lambda)\}$ and $A/\ker(\lambda) = \{a/\ker(\lambda) \mid a \in A\}$. Since $\ker(\lambda)$ is the filter, we define operators $\sqcap, \sqcup$ for $a/\ker(\lambda), b/\ker(\lambda) \in A/\ker(\lambda)$ and constants $\mathbf{0}, \mathbf{1}$ as follows:

$$\begin{aligned} a/\ker(\lambda) \sqcap b/\ker(\lambda) &= (a \wedge b)/\ker(\lambda) \\ &= (a \odot b)/\ker(\lambda); \\ a/\ker(\lambda) \sqcup b/\ker(\lambda) &= (a \vee b)/\ker(\lambda); \\ \mathbf{0} &= 0/\ker(\lambda); \\ \mathbf{1} &= 1/\ker(\lambda). \end{aligned}$$

Then we have from the result above that

Theorem 4.2 (Homomorphism Theorem)**.** *Let (L, λ) be a reticulation of A. Then $(A/\ker(\lambda), \sqcap, \sqcup, \mathbf{0}, \mathbf{1})$ is a bounded distributive lattice. If we define a map $\nu : A \to A/\ker(\lambda)$ by $\nu(a) = a/\ker(\lambda)$, then the pair $(A/\ker(\lambda), \nu)$ of the quotient structure $A/\ker(\lambda)$ and the map ν is a reticulation of a residuated lattice A and thus*

$$A/\ker(\lambda) \cong L.$$

Proof. Since it is obvious that $A/\ker(\lambda)$ is a bounded lattice, we only show that $A/\ker(\lambda)$ is distributive. For all $a/\ker(\lambda), b/\ker(\lambda), c/\ker(\lambda) \in A/\ker(\lambda)$, we have

$$\begin{aligned}
&a/\ker(\lambda) \sqcap (b/\ker(\lambda) \sqcup c/\ker(\lambda))\\
&= (a \odot (b \vee c))/\ker(\lambda)\\
&= ((a \odot b) \vee (a \odot c))/\ker(\lambda)\\
&= (a \odot b)/\ker(\lambda) \vee (a \odot c)/\ker(\lambda)\\
&= (a/\ker(\lambda) \sqcap b/\ker(\lambda)) \sqcup (a/\ker(\lambda) \sqcap c/\ker(\lambda)).
\end{aligned}$$

This means that $A/\ker(\lambda)$ is the distributive lattice.

□

On the other hand, in [4] a binary relation $\equiv$ on A is defined by

$$a \equiv b \Longleftrightarrow D(a) = D(b),$$

where $D(a) = \{P \in Spec(A) \mid a \notin P\}$. Since the binary relation $\equiv$ is a congruence on A with respect to lattice operations $\wedge$ and $\vee$, we consider its quotient algebra by $\equiv$. We take $[a] = \{b \in A \mid a \equiv b\}, A/_{\equiv} = \{[a] \mid a \in A\}$. For $[a], [b] \in A/_{\equiv}$, if we define

$$\begin{aligned}
[a] \vee [b] &= [a \vee b]\\
[a] \wedge [b] &= [a \wedge b],
\end{aligned}$$

then $(A/_{\equiv}, \wedge, \vee, [0], [1])$ is a bounded distributive lattice and $(A/_{\equiv}, \eta)$ is a reticulation of A ([4]), where η is a canonical map $A \to A/_{\equiv}$ defined by $\eta(a) = [a]$ for $a \in A$.

We have another view point, namely, if we note $\lambda(a) = \lambda(b) \Longleftrightarrow [a) = [b)$, then we have

$$\begin{aligned}
a \equiv b &\Longleftrightarrow D(a) = D(b)\\
&\Longleftrightarrow a \notin P \text{ iff } b \notin P \ (\forall P \in Spec(A))\\
&\Longleftrightarrow a \in P \text{ iff } b \in P \ (\forall P \in Spec(A))\\
&\Longleftrightarrow [a) = [b)\\
&\Longleftrightarrow \lambda(a) = \lambda(b)\\
&\Longleftrightarrow (a, b) \in \ker(\lambda).
\end{aligned}$$

This means that the binary relation $\equiv$ defined in [4] is the same as the kernel $\ker(\lambda)$ of the lattice homomorphism λ.

Moreover, we introduce an partial order $\sqsubseteq$ on the class $\mathcal{PF}(A)$ of all principal filters of A by

$$[a) \sqsubseteq [b) \Longleftrightarrow [b) \subseteq [a).$$

It is easy to show that

$$\inf_{\sqsubseteq}\{[a),[b)\} = [a \vee b);$$
$$\sup_{\sqsubseteq}\{[a),[b)\} = [a \wedge b) = [a \odot b);$$
$$\mathbf{0} = [1) = \{1\};$$
$$\mathbf{1} = [0) = A.$$

Hence $\mathcal{PF}(A)$ is a bounded distributive lattice. Moreover if we define a map $\xi : A \to \mathcal{PF}(A)$ by $\xi(a) = [a)$, then $(\mathcal{PF}(A), \xi)$ is a reticulation of A. Since the reticulation is unique up to isomorphism ([4]), we see that

$$A/\ker(\lambda) \cong \mathcal{PF}(A).$$

5. Conclusion

In this short note, we show that a reticulation map f can be defined only by two independent conditions:

(R4) $f : A \to L$ is surjective
(R5) $f(a) \le f(b) \Longleftrightarrow \exists n \in \mathbb{N}$ s.t. $a^n \le b$,

and the reticulation map is only a lattice homomorphism from a (residuated) lattice A to a bounded distributive lattice L. Moreover, since the implication $\to$ does not play a role in the definition of reticulation, we note that the argument in this short note can be generalized to the algebra $(A, \wedge, \vee, \odot, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and $(A, \odot, 1)$ is a commutative monoid satisfying the axiom $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$ for all $x, y, z \in A$,

Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 15K00024.

References

[1] L.P. Belluce, Semisimple algebras of infinite valued logic and bold fuzzy set theory, *Canad. J. Math.* **38** (1986), no. 6, 1356–1379.
[2] G. Georgescu, The reticulation of a quantale, *Rev. Roumaine Math. Pures Appl.* **40** (1995), no. 7-8, 619–631.
[3] L. Leuştean, The prime and maximal spectra and the reticulation of BL-algebras, *Central Europen J. Math.* **1** (2003), no. 3, 382–397.
[4] C. Mureşan, The reticulation of a residuated lattice, *Bull. Math. Soc. Sci. Math. Roumanie* **51** (2008), no. 1, 47–65.
[5] H. Simmons, Reticulated rings, *J. Algebra* **66** (1980), no. 1, 169–192.

(Michiro Kondo) Department of mathematics, School of System Design and Technolodgy, Tokyo Denki University, Japan.
E-mail address: mkondo@mail.dendai.ac.jp