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**Existence and blow-up of Solutions for the sixth-order damped Boussinesq equation**

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## EXISTENCE AND BLOW-UP OF SOLUTIONS FOR THE SIXTH-ORDER DAMPED BOUSSINESQ EQUATION

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**ABSTRACT.** In this paper, we consider the existence and uniqueness of the global solution for the sixth-order damped Boussinesq equation. Moreover, the finite-time blow-up of the solution for the equation is investigated by the concavity method.

**Keywords:** Boussinesq equation, Cauchy problem, global solution, blow-up.

**MSC(2010):** Primary: 35L60; Secondary: 35K55, 35Q80.

### 1. Introduction

The subject of this paper is to study the Cauchy problem of the following sixth-order damped Boussinesq equation

$$(1.1) \quad u_{tt} - u_{ttxx} - u_{xx} + u_{xxxx} - u_{xxxxx} - ru_{txx} = f(u)_{xx}, x \in R, t > 0,$$

$$(1.2) \quad u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), x \in R.$$

where  $u(x; t)$ ,  $f(s)$  and  $r$  denote the unknown function, the given nonlinear function and a constant, respectively.

The effects of small nonlinearity and dispersion are taken into consideration in the derivation of Boussinesq equations, but in many real situations, damping effects are compared in strength to the nonlinear and dispersive ones. Therefore, the damped Boussinesq equation is considered as well:

$$(1.3) \quad u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(f(u))_{xx},$$

where the second term on the left-hand side is responsible for dissipation. For  $f(u) = u^2$ , Varlamov [12–14] has constructed the classical solution of the problem equation (1.3) and obtained the long-time asymptotics in explicit form. Using the eigenfunction expansion method, he also studied the long-time asymptotics of a damped Boussinesq equation which is similar to equation (1.3).

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In [17], the Cauchy problem for a class of Boussinesq equation

$$(1.4) \quad u_{tt} - ku_{txx} + u_{xxxx} - u_{xx} - u_{xxtt} = (f(u))_{xx},$$

was studied. The well-posedness of the local and global solutions and the blow-up of the solution were established. Polat [8,9] studied the locally and globally existence, blow-up and the asymptotic behavior of the solutions for the Cauchy problem of equation (1.4). In order to investigate the water wave problem with surface tension, Schneider and Eugene [10] considered a class of Boussinesq equation which models the water wave problem with surface tension as follows

$$(1.5) \quad u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + (u^2)_{xx},$$

where  $x, t, \mu \in R$  and  $u(x, t) \in R$ . The model can also be formally derived from the 2D water wave problem. For a degenerate case, they proved that the long wave limit can be described approximately by two decoupled Kawahara-equations. In [18,19], Wang studied the well-posedness of the local and globally solutions, the blow-up of solutions and nonlinear scattering for small amplitude solutions to the Cauchy problem of equation (1.5). In [7], the authors considered the Cauchy problem of the following Boussinesq equation

$$(1.6) \quad u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{xxxxtt} + f(u)_{xx} + ku_{txx},$$

the existence, both locally and globally in time, the global nonexistence and the asymptotic behavior of solutions for the Cauchy problem of equation (1.6) are established in  $n$ -dimensional space.

Recently, Wang [20] proved the global existence and asymptotic behavior of solutions of the Cauchy problem for equation (1.1) provided that the initial value is suitably small. In [15], the authors obtained the global existence and asymptotic decay of solutions to the problem equation (1.1). For the initial boundary value problem of equation (1.1) with  $f(u) = u^2$ , Zhang [21] and Lai [3,4] established the well-posedness of strong solution and constructed the solution in the form of series in the small parameter present in the initial conditions. The long-time asymptotics was also obtained in the explicit form. The main purpose of this paper is to study the well-posedness of the global solution for the Cauchy problem of (1.1)-(1.2), and the results in this paper generalize the results established in [8,17]. Because of the complexity of equation (1.1), we yield the existence of local solution by transforming equation (1.1) in another way and establishing the corresponding estimate which is different from that in [8,17].

Throughout this paper, we use  $L_p$  to denote the space of all  $L^p$ - functions on  $R$  with the norm  $\|f\|_p = \|f\|_{L^p}$  and  $\|f\| = \|f\|_2$ ,  $H^s$  denotes the Sobolev space with norm  $\|f\|_{H^s} = \|(I - \partial_x^2)^{\frac{s}{2}} f\|$ , where  $1 \leq p \leq \infty, s \in R$ .

At first, using the contraction mapping principle, we obtain the following existence of the local solution to problem (1.1) and (1.2).

**Theorem 1.1.** *Assume that  $s > \frac{1}{2}, \phi \in H^s, \psi \in H^{s-2}$  and  $f(s) \in C^{[s]+1}(R)$ , then the problem (1.1)-(1.2) admits a unique local solution  $u(x, t)$  defined on a maximal time interval  $[0, T_0)$  with  $u \in C([0, T_0), H^s) \cap C^1([0, T_0), H^{s-2})$ . Moreover, if*

$$(1.7) \quad \sup_{t \in [0, T_0)} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}}) < \infty.$$

then  $T_0 = \infty$ .

Secondly, under some assumptions, we study the well-posedness and blow-up of the global solution for the the problem (1.1)-(1.2).

**Theorem 1.2.** *Suppose that the assumptions of Theorem 1.1 hold and  $T_0 > 0$  is the maximal existence time of the corresponding solution  $u \in C([0, T_0), H^s) \cap C^1([0, T_0), H^{s-2})$  to (1.1) and (1.2). Then  $T_0 < \infty$  if and only if*

$$(1.8) \quad \lim_{t \rightarrow T_0} \sup \|u(\cdot, t)\|_{L^\infty} = \infty.$$

**Theorem 1.3.** *Assume that  $s \geq 1, \phi \in H^s, \psi \in H^{s-2}, (-\partial_x^2)^{-\frac{1}{2}}\phi \in L^2, F(u) = \int_0^u f(z)dz, F(\phi) \in L^1, s \geq 1$  and  $F(u)$  or  $f'(u)$  is bounded below, i.e there is a constant  $A_0$  such that  $f'(s) \geq A_0$ . Then the problem (1.1)-(1.2) has a unique global solution  $u \in C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-2})$ .*

**Theorem 1.4.** *Assume that  $k \geq 0, f(u) \in C(R), \phi \in H^2, \psi \in L^2, (-\partial_x^2)^{-1/2}\phi, (-\partial_x^2)^{-1/2}\psi \in H^1, F(u) = \int_0^u f(s)ds, F(\psi) \in H^1$ , and there exists a constant  $\alpha > 0$  such that*

$$(1.9) \quad f(u)u \leq (\alpha + r + 2)F(u) + \frac{\alpha}{2}u^2, \forall u \in R.$$

Then the solution  $u(x, t)$  of the problem (1.1)-(1.2) blows up in finite time if one of the following conditions is valid:

- (i)  $E(0) = \|(-\partial_x^2)^{-1/2}\psi\|_2^2 + \|\psi\|_2^2 + \|\phi\|_2^2 + \|\phi_x\|_2^2 + \|\phi_{xx}\|_2^2 + 2 \int_R F(u)dx < 0,$
- (ii)  $E(0) = 0$  and  $((-\partial_x^2)^{-\frac{1}{2}}\phi, (-\partial_x^2)^{-\frac{1}{2}}\psi) + (\phi, \psi) > 0,$
- (iii)  $E(0) > 0$  and

$$((-\partial_x^2)^{-\frac{1}{2}}\phi, (-\partial_x^2)^{-\frac{1}{2}}\psi) + (\phi, \psi) > \sqrt{2 \frac{4 + 2r + 2\alpha}{\alpha + 2} E(0) (\|(-\partial_x^2)^{-\frac{1}{2}}\phi\|_2^2 + \|\phi\|_2^2)}.$$

The remaining of this paper is organized as follows. In Section 2, we prove the existence and the uniqueness of the local solution by the contraction mapping principle. The well-posedness of the global solution is given in Section 3. Finally, Section 4 is devoted to the blow-up of solution to the problem.

### 2. Existence of local solution

In this section, the existence and uniqueness of the local solution to the problem (1.1)-(1.2). are proved by the contraction mapping principle. For this purpose, we write equation (1.1) as

$$(2.1) \quad u_{tt} + u_{xxxx} = \Gamma[f(u) + u] + r\Gamma[u_t],$$

where  $\Gamma = (1 - \partial_x^2)^{-1}\partial_x^2$ . Using the Fourier transform, we can obtain

$$\Gamma f = \partial_x^2(G * f) = G * f - f,$$

where  $G(x) = \frac{1}{2}e^{-|x|}$ ,  $u * v = \int_{-\infty}^{+\infty} u(y)v(x - y)dy$  denotes the convolution of  $u$  and  $v$ .

In order to prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1.** *Let  $s \in R$ ,  $\phi \in H^s, \psi \in H^{s-2}$  and  $q \in L^1([0, T]; H^{s-2})$ . Then for any  $T > 0$ , the Cauchy problem for the linear wave equation*

$$u_{tt} + u_{xxxx} = q(x, t), x \in R, t > 0,$$

*with the initial value condition (1.2) has a unique solution  $u \in C([0, T], H^s) \cap C^1([0, T], H^{s-2})$ . Moreover,  $u$  satisfies*

$$(2.2) \quad \begin{aligned} & \|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}} \\ & \leq C(1 + T)(\|\phi\|_{H^s} + \|\psi\|_{H^{s-2}} + \int_0^t \|q(\tau)\|_{H^{s-2}}d\tau), 0 \leq t \leq T, \end{aligned}$$

where  $C$  only depends on  $s$ .

*Proof.* The argument used to prove the existence and uniqueness of the solution of the Cauchy problem for the linear wave equation is similar to that in [11], we omit it. And the solution of the linear wave equation is given in Fourier space by

$$\hat{u}(\xi, t) = \cos(t\xi^2)\hat{\phi}(\xi) + \frac{\sin(t\xi^2)}{\xi^2}|\hat{\psi}|^2 + \int_0^t \frac{\sin(t - \tau)\xi^2}{\xi^2}\hat{q}(\xi, \tau)d\tau,$$

where  $\hat{\cdot}$  denotes Fourier transform with respect to  $t$ . Since

$$\|(1 + \xi^2)^{\frac{s}{2}} \cos(t\xi^2)\hat{\phi}(\xi)\| \leq \|(1 + \xi^2)^{\frac{s}{2}}\hat{\phi}(\xi)\| = \|\phi\|_{H^s}$$

and

$$\begin{aligned}
& \|(1 + \xi^2)^{\frac{s}{2}} \frac{\sin(t\xi^2)}{\xi^2} \hat{\psi}(\xi)\|^2 \\
&= \int_{|\xi| < 1} (1 + \xi^2)^s \frac{\sin^2(t\xi^2)}{\xi^4} |\hat{\psi}(\xi)|^2 d\xi + \int_{|\xi| \geq 1} (1 + \xi^2)^s \frac{\sin^2(t\xi^2)}{\xi^4} |\hat{\psi}(\xi)|^2 d\xi \\
&\leq t^2 \int_{|\xi| < 1} (1 + \xi^2)^s |\hat{\psi}(\xi)|^2 d\xi + \int_{|\xi| \geq 1} (1 + \xi^2)^s \frac{1}{\xi^4} |\hat{\psi}(\xi)|^2 d\xi \\
&\leq 4t^2 \int_{|\xi| < 1} (1 + \xi^2)^{s-2} |\hat{\psi}(\xi)|^2 d\xi + 4 \int_{|\xi| \geq 1} (1 + \xi^2)^s \frac{1}{\xi^4} |\hat{\psi}(\xi)|^2 d\xi \\
&\leq 4(1 + t^2) \int_R (1 + \xi^2)^{s-2} |\hat{\psi}(\xi)|^2 d\xi \\
&= 4(1 + t^2) \|\psi\|_{H^{s-2}}^2,
\end{aligned}$$

we obtain

$$\|u(t)\|_{H^s} \leq \|\phi\|_{H^s} + 2(1 + t) \|\psi\|_{H^{s-2}} + 2(1 + t) \int_0^t \|q(\tau)\|_{H^{s-2}} d\tau$$

and

$$\|u(t)_t\|_{H^{s-2}} \leq \|\phi\|_{H^s} + \|\psi\|_{H^{s-2}} + \int_0^t \|q(\tau)\|_{H^{s-2}} d\tau.$$

Therefore (2.2) holds. This completes the proof of the lemma.  $\square$

**Lemma 2.2.** *The operator  $\Gamma$  is bounded on  $H^s$  for all  $s \geq 0$  and*

$$\|\Gamma u\|_{H^s} \leq \|u\|_{H^s}, \forall u \in H^s.$$

*Proof.* For  $u \in H^s$ ,  $s \geq 0$ , we get

$$\|\Gamma u\|_{H^s}^2 = \int_{R^n} (1 + \xi^2)^s \frac{\xi^4}{(1 + \xi^2)^2} |u(\hat{\xi})|^2 d\xi \leq \|u\|_{H^s}^2.$$

$\square$

**Lemma 2.3** ([1, 16]). *Suppose that  $g(u) \in C^N(R)$  is a function vanishing at zero, where  $N \geq 0$  is an integer. Then for any  $s$  with  $0 \leq s \leq N$  and any  $u, v \in H^s \cap L^\infty$ , it holds that*

$$\begin{aligned}
& \|g(u)\|_{H^s} \leq G(\|u\|_{L^\infty}) \|u\|_{H^s}, \\
& \|g(u) - g(v)\|_{H^s} \leq \bar{G}(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \|u - v\|_{H^s},
\end{aligned}$$

where  $G : [0, \infty) \rightarrow R$  and  $\bar{G} : [0, \infty) \times [0, \infty) \rightarrow R$  are continuous function.

*Proof of Theorem 1.1.* Now we are going to prove the existence and uniqueness of local solutions for the problem (1.1)-(1.2) by contraction mapping argument. For this purpose, we define the function space

$$X(T) = \{C([0, T]; H^s) \cap C^1([0, T], H^{s-2})\}$$

with  $s > \frac{1}{2}$ , equipped with the norm defined by

$$\|u\|_{X(T)} = \max_{t \in [0, T]} [\|u(\cdot, t)\|_{H^s} + \|u_t(\cdot, t)\|_{H^{s-2}}].$$

Since  $H^s \hookrightarrow L^\infty$  for  $s > \frac{1}{2}$ , we have  $u \in L^\infty$  if  $u \in X(T)$ . We set  $R = \|\phi\|_{H^s} + \|\psi\|_{H^{s-2}}$  and

$$A_R(T) = \{u \in X(T) : \|u\|_{X(T)} \leq 2CR\}.$$

For  $\phi \in H^s, \psi \in H^{s-2}$  and  $w \in X(T)$ , we consider the linear wave equation

$$(2.3) \quad u_{tt} + u_{xxxx} = \Gamma[f(w) + w] + r\Gamma[w_t].$$

It will be shown that  $\Theta : A_R(T) \rightarrow A_R(T)$  is contractive if  $R$  and  $T$  are well chosen. Define  $\eta(x, t)$  by

$$\eta(x, t) = \Gamma[f(w) + w] + r\Gamma[w_t].$$

Using Lemma 2.1 and Lemma 2.3, it follows easily that

$$\begin{aligned} \|\eta(x, t)\|_{H^{s-2}} &\leq \|f(w)\|_{H^{s-2}} + \|w\|_{H^{s-2}} + |r|\|w_t\|_{H^{s-2}} \\ &\leq G_1(R)\|w\|_{H^s} + |r|\|w_t\|_{H^{s-2}}, \end{aligned}$$

where  $G(R)$  is a constant dependent on  $R$ . From the above inequality, we obtain that  $\eta(x, t) \in L^1([0, T]; H^{s-2})$ . From Lemma 2.1 the solution  $u = \Theta w$  of the problem (1.2)-(2.3) belongs to  $C([0, T]; H^s) \cap C^1([0, T], H^{s-2})$  and

$$\begin{aligned} \|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-2}} &\leq C(1 + T)(\|\phi\|_{H^s} + \|\psi\|_{H^{s-2}} + \int_0^t \|\eta(\tau)\|_{H^{s-2}} d\tau) \\ &\leq CR + C[1 + C(G_1(R) + |r|)(1 + T)]RT. \end{aligned}$$

Choosing  $T$  small enough such that

$$(2.4) \quad [1 + C(G_1(R) + |r|)(1 + T)]T \leq 1,$$

then  $\|\Theta w\|_{X(T)} \leq 2CR$ . Therefore,  $\Theta$  maps  $A_R(T)$  into  $A_R(T)$ .

Now, we prove that for  $T$  small enough,  $\Theta$  is a contractive mapping of  $A_R(T)$ . Let  $w, \bar{w} \in A_R(T)$ . Then for  $w$  and  $\bar{w}$  there are the corresponding solutions  $u = \Theta w$  and  $\bar{u} = \Theta \bar{w}$  for problems (2.3) and (1.2). Set  $U = u - \bar{u}, W = w - \bar{w}$ , then  $U$  satisfies

$$(2.5) \quad U_{tt} + U_{xxxx} = Q(x, t), (x, t) \in R \times (0, T),$$

$$(2.6) \quad U(x, 0) = U_t(x, 0) = 0,$$

where  $Q(x, t)$  is defined by

$$(2.7) \quad Q(x, t) = \Gamma[f(w) - f(\bar{w})] + \Gamma W + r\Gamma[W_t].$$

It is observed that  $\Theta$  has the smoothness required to apply Lemma 2.1 to the problem (2.5)-(2.6). Using Lemmas 2.1, 2.2 and 2.3, we get from equation (2.7)

that

$$\begin{aligned} & \|U(t)\|_{H^s} + \|U_t\|_{H^{s-2}} \\ & \leq C(1+T) \int_0^t [\|f(w(\tau)) - f(\bar{w}(\tau))\|_{H^{s-2}} + \|W\|_{H^{s-2}} + |r|\|W_t\|_{H^{s-2}}] d\tau \\ & \leq C(1+T)[G_2(R) \max_{0 \leq t \leq T} \|W(t)\|_{H^s} + |r| \max_{0 \leq t \leq T} \|W_t(t)\|_{H^{s-2}}]T. \end{aligned}$$

Thus, we get

$$\|U(t)\|_{X(T)} \leq C(1+T)[G_2(R) + |r|]T\|W(t)\|_{X(T)}.$$

By choosing  $T$  small enough equation (2.4) holds and

$$(2.8) \quad (1+T)[G_2(R) + |r|]T < \frac{1}{C},$$

which leads to

$$\|\Theta w - \Theta \bar{w}\|_{X(T)} < \|w - \bar{w}\|_{X(T)}.$$

This shows that  $\Theta : A_R(T) \rightarrow A_R(T)$  is strictly contractive.

From contraction mapping principle, it follows that for appropriately chosen  $T > 0$ ,  $\Theta$  has a unique fixed point  $u(x, t) \in A_R(T)$ , which is a strong solution of the problem (1.1)-(1.2). It is easy to prove the uniqueness of the solution which belongs to  $X(T')$  for each  $T' > 0$ .

In fact, let  $u_1, u_2 \in X(T')$  be two solutions of the problem (1.1)-(1.2). Let  $u = u_1 - u_2$ , then we have

$$u_{tt} - u_{ttxx} - u_{xx} + u_{xxxx} - u_{xxxxx} - ru_{txx} = (f(u_1) - f(u_2))_{xx}$$

Multiplying the above equation by  $(-\partial_x^2)^{-1}u_t$  and integrating the product with respect to  $x$ , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|u_{xx}\|_2^2 + \|u_x\|_2^2] + r\|u_t\|_2^2 \\ (2.9) \quad & = \int_{R^n} [f(u_1) - f(u_2)]u_t dx. \end{aligned}$$

From the definition of the space  $X(T')$ ,  $s > 1/2$  and Sobolev imbedding theorem we have  $\|u_1(t)\|_\infty \leq C(T')$ ,  $\|u_2(t)\|_\infty \leq C(T')$  and  $0 \leq t \leq T' < T$ , where  $C(T')$  is a constant dependent on  $T'$ . From Cauchy inequality, we obtain

$$\left| \int_{R^n} [f(u_1) - f(u_2)]u_t dx \right| \leq \|f(u_1) - f(u_2)\|_2 \|u_t\|_2 \leq C(T') \|u\|_2 \|u_t\|_2.$$

We get from Young inequality that

$$\begin{aligned} & \|(-\partial_x^2)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|u_{xx}\|_2^2 + \|u_x\|_2^2 + r \int_0^t \|u_\tau\|_2^2 d\tau \\ & \leq C(T') \int_0^t [\|u\|_2^2 + \|u_t\|_2^2] d\tau. \end{aligned}$$



So, we obtain

$$(2.10) \quad \|u\|_2^2 + \|u_t\|_2^2 \leq [C(T') + 2|r|] \int_0^{T'} [\|u\|_2^2 + \|u_t\|_2^2] d\tau.$$

Making using of Gronwall's inequality, we get from equation (2.10) that  $\|u\|_2^2 + \|u_t\|_2^2 \equiv 0$  for  $0 \leq t \leq T'$ . Hence  $u \equiv 0$ , the problem (1.1)-(1.2) has at most one solution which belongs to  $X(T')$ .

Now, let  $[0, T_0]$  be the maximal time interval of existence for  $u \in X(T_0)$ . It remains only to show that if equation (1.7) is satisfied,  $T_0 = \infty$ .

Suppose that equation (1.7) holds and  $T_0 < \infty$ . For each  $T' \in [0, T_0]$ , we consider the Cauchy problem

$$(2.11) \quad v_{tt} + v_{xxxx} = \Gamma[f(v) + rv_t + v],$$

$$(2.12) \quad v(x, 0) = u(x, T'), v_t(x, 0) = u_t(x, T').$$

By virtue of equation (1.7),

$$\|u(\cdot, t)\|_{2,p} + \|u_t(\cdot, t)\|_{2,p} + \|u(\cdot, t)\|_\infty + \|u_t(\cdot, t)\|_\infty \leq K,$$

is uniformly bounded about  $T' \in [0, T_0]$ , which allows us to choose  $T^* \in (0, T_0)$  such that for each  $T' \in [0, T_0]$ , the problem (2.11)-(2.12) has a unique solution  $v(x, t) \in X(T^*)$ . The existence of such a  $T^*$  follows from the contraction mapping principle. In particular, equation (2.4) and equation (2.8) reveal that  $T^*$  can be selected independently of  $T' \in [0, T_0]$ . Set  $T' = T_0 - \frac{T^*}{2}$ , let  $v$  denote the corresponding solution of the problem (2.11)-(2.12) and define

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & t \in [0, T'], \\ v(x, t - T'), & t \in [T', T_0 + T^*/2], \end{cases}$$

then  $\tilde{u}(x, t)$  is a solution of the problem (1.1)-(1.2) on interval  $[0, T_0 + T^*/2]$ , and by the uniqueness,  $\tilde{u}(x, t)$  extends  $u$ . This violates the maximality to  $[0, T_0]$ . Therefore, if equation (1.7) holds, then  $T_0 = \infty$ . This completes the proof of the theorem.  $\square$

### 3. Existence of global solutions for general nonlinear function $f(u)$

In this section, we study the existence of global solutions to the problem (1.1)-(1.2).

**Lemma 3.1.** *Suppose that  $f(u) \in C(R)$ ,  $G(u) = \int_0^u g(s)ds$ ,  $\phi \in H^2$ ,  $(-\partial_x^2)^{-\frac{1}{2}}\psi \in L^2$  and  $F(\phi) \in L^1$ . Then for the solution  $u(x, t)$  of the Cauchy problem (1.1) and (1.2), it follows that*

$$(3.1) \quad \begin{aligned} E(t) = & \frac{1}{2} [\|(-\partial_x^2)^{-\frac{1}{2}}u_t\|^2 + \|u_t\|^2 + \|u_x\|^2 + \|u\|^2 + \|u_{xx}\|^2] \\ & + r \int_0^t \|u_\tau\|^2 d\tau + \int_R F(u) dx = E(0), \forall t \in (0, T_0), \end{aligned}$$

where  $(-\partial_x^2)^{-\frac{1}{2}}u = \mathcal{F}^{-1}[|\xi|^{-1}\mathcal{F}u(\xi)]$ ,  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote Fourier transformation and inverse transformation in  $R$ .

*Proof.* It follows from equation (1.1) that

$$\begin{aligned} \frac{d}{dt}E(t) &= ((-\partial_x^2)^{-\frac{1}{2}}u_{tt}, (-\partial_x^2)^{-\frac{1}{2}}u_t) + (u_t, u_{tt}) \\ &\quad + (u, u_t) + (u_x, u_{xt}) + ((u_{xx}, u_{xxt}) + 2r(u_t, u_t) + (f(u), u_t)) \\ &= \langle (-\partial_x^2)^{-1}u_{tt} + u_{tt} + ru_t + u - u_{xx} + u_{xxxx} + f(u), u_t \rangle_{X^*X} = 0, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{X^*X}$  means the usual duality of  $X$  and  $X$  with  $X = H^1$ . Integrating the above equality with respect to  $t$ , we have (3.1). The lemma is proved.  $\square$

*Proof of Theorem 1.2.* From Theorem 1.1, let us prove that if

$$(3.2) \quad \limsup_{t \in [0, T_0)} \|u(\cdot, t)\|_{L^\infty} = M < \infty,$$

then  $T_0 = \infty$ . Using equation (1.1), it follows that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|u\|_{H^s}^2 + \|u_t\|_{H^{s-2}}^2) \\ &= ((I - \partial_x^2)^{\frac{s-2}{2}}u_{tt}, (I - \partial_x^2)^{\frac{s-2}{2}}u_t) + ((I - \partial_x^2)^{\frac{s}{2}}u, (I - \partial_x^2)^{\frac{s}{2}}u_t) \\ &= ((I - \partial_x^2)^{s-2}u_{tt} + (I - \partial_x^2)^s u, u_t) \\ &= ((I - \partial_x^2)^{s-2}u_{tt} + (I - \partial_x^2)^{s-2}(I - 2\partial_x^2 + \partial_x^4)u, u_t) \\ &= ((I - \partial_x^2)^{s-2}(u_{tt} + u_{xxxx}), u_t) + ((I - \partial_x^2)^{s-2}2u_{xx}, u_t) + ((I - \partial_x^2)^{s-2}u, u_t) \\ &= ((I - \partial_x^2)^{\frac{s-2}{2}}\partial_x^2(I - \partial_x^2)^{-1}(f(u) + u + ru_t), (I - \partial_x^2)^{\frac{s-2}{2}}u_t) \\ &\quad + ((I - \partial_x^2)^{s-2}u_{xx}, u_t) + ((I - \partial_x^2)^{s-2}u, u_t) \\ &\leq \|f(u) + u + ru_t\|_{H^{s-2}}\|u_t\|_{H^s} + \|u\|_{H^s}\|u_t\|_{H^{s-2}} + \|u\|_{H^{s-2}}\|u_t\|_{H^{s-2}}. \end{aligned}$$

From Lemma 2.2 and (3.1), we obtain

$$\|f(u) + u + ru_t\|_{H^{s-2}} \leq G(R)\|u\|_s + |r|\|u_t\|_{s-2}.$$

It follows from the Cauchy inequality that

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{H^s}^2 + \|u_t\|_{H^{s-2}}^2) \leq (G(R)^2 + C)\|u\|_s + C(|r| + 1)\|u_t\|_{s-2}, t \in (0, T).$$

From Gronwall' inequality, it holds that  $\|u(t)\|_{H^s}^2 + \|u_t(t)\|_{H^{s-2}}^2$  do not blow-up in finite time. The theorem is proved.  $\square$

*Proof of Theorem 1.3.* According to Theorem 1.2, it is enough to ensure that the  $L^\infty$ -norm of the solution  $u(t)$  to the problem (1.1)-(1.2) does not blow up in finite time. We denote by  $T_0 > 0$  the maximal existence time of this solution.

If  $F(u) \geq 0$ , then from (3.1), we get

(3.3)

$$\|(-\partial_x^2)^{-\frac{1}{2}}u_t\|^2 + \|u_{xx}\|^2 + \|u_x\|^2 + \|u\|^2 + \|u_t\|^2 \leq 2E(0) + 2|r| \int_0^t \|u_\tau\|_2^2 d\tau.$$

It follows from Gronwall's inequality and the above inequality that

$$(3.4) \quad \|(-\partial_x^2)^{-\frac{1}{2}}u_t\|^2 + \|u_{xx}\|^2 + \|u_x\|^2 + \|u\|^2 + \|u_t\|^2 \leq 2E(0)e^{2|r|T}.$$

If  $f'(u)$  is bounded below. Let  $f_0(u) = f(u) - k_0u$ , where  $k_0 = \min\{A_0, 0\} (\leq 0)$ , then  $f_0(0) = 0, f_0'(u) = f'(u) - k_0 \geq 0$  and  $f_0(u)$  is a monotonically increasing function. Thus  $F_0(u) = \int_0^u f_0(s)ds \geq 0$ . From (3.1) and the following equality

$$F(u) = \int_0^u f(s)ds = \int_0^u (f_0(s) + k_0s)ds = F_0(u) + \frac{k_0}{2}u^2,$$

we obtain

$$\begin{aligned} & \|(-\partial_x^2)^{-\frac{1}{2}}u_t\|^2 + \|u_{xx}\|^2 + \|u_x\|^2 + \|u\|^2 + \|u_t\|^2 + 2 \int_{R^n} F_0(u)dx \\ &= 2E(0) - 2r \int_0^t \|u_\tau\|_2^2 d\tau - k_0\|u\|_2^2 \\ &= 2E(0) - 2r \int_0^t \|u_\tau\|_2^2 d\tau - k_0\|u_0\|_2^2 + \int_0^t (k_0^2\|u\|_2^2 + \|u_\tau\|_2^2) d\tau \\ &\leq 2E(0) - k_0\|u_0\|_2^2 + (2|r| + 1 + k_0^2) \int_0^t (\|u\|_2^2 + \|u_\tau\|_2^2) d\tau. \end{aligned}$$

It follows from Gronwall's inequality and the above inequality that

$$(3.5) \quad \begin{aligned} & \|(-\partial_x^2)^{-\frac{1}{2}}u_t\|^2 + \|u_{xx}\|^2 + \|u_x\|^2 + \|u\|^2 + \|u_t\|^2 \\ & \leq (2E(0) - k_0\|u_0\|_2^2) \exp[(2|r| + 1 + k_0^2)T]. \end{aligned}$$

Inequality (3.4) and (3.5) ensures that  $H^s$ - norm of the solution  $u(t)$  does not blow up in finite time. We conclude from the Sobolev embedding theorem that the  $L^\infty$ -norm of the solution  $u(t)$  to the problem (1.1)-(1.2) does not blow up in finite time and  $T_0 = \infty$ . The theorem is proved.  $\square$

#### 4. Nonexistence of global solutions for general nonlinear function $f(u)$

In this section, we discuss the blow-up of the solution for the problem (1.1)-(1.2) by the concavity method. Firstly, we give the following lemma [2] which is a generalization of Levine's result [5, 6].

**Lemma 4.1.** *Suppose that for  $t \geq 0$ , a positive, twice differential function  $I(t)$  satisfies the inequality*

$$I''(t)I(t) - (1 + \varepsilon)(I'(t))^2 \geq -2MI(t)I'(t) - M(I(t))^2,$$

where  $\varepsilon > 0$  and  $M_1, M_2$  are constants. If  $I(0) > 0$ ,  $I'(0) > \gamma_2 \nu^{-1} I(0)$  and  $M_1 + M_2 > 0$ , then  $I(t)$  tends to infinity as

$$t \rightarrow t_1 \leq t_2 = \frac{1}{2\sqrt{M_1^2 + \nu M_2}} \ln \frac{\gamma_1 I(0) + \nu I'(0)}{\gamma_1 I(0) + \nu I'(0)},$$

where  $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \nu M_2}$ . If  $I(0) > 0$ ,  $I'(0) > 0$  and  $M_1 = M_2 = 0$ , then  $I(t) \rightarrow \infty$  as  $t \rightarrow t_1 \leq t_2 = I(0)/\nu I'(0)$ .

*Proof of Theorem 1.4.* Suppose  $T = +\infty$ , and let

$$(4.1) \quad I(t) = \|(-\partial_x^2)^{-1/2} u\|_2^2 + \|u\|_2^2 + \beta(t + \tau)^2,$$

where  $\beta, \tau \geq 0$  to be defined later. Then we have

$$(4.2) \quad I'(t) = 2\langle (-\partial_x^2)^{-1/2} u_t, (-\partial_x^2)^{-1/2} u \rangle + 2\beta(t + \tau) + 2(u, u_t).$$

So,

$$(4.3) \quad \begin{aligned} (I'(t))^2 &\leq 4[\|(-\partial_x^2)^{-1/2} u\|_2^2 + \|u\|_2^2 + \beta(t + \tau)^2][\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + \|u_t\|_2^2 + \beta] \\ &= 4I(t)[\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + \|u_t\|_2^2 + \beta]. \end{aligned}$$

By equation (1.1), we get

$$(4.4) \quad \begin{aligned} I''(t) &= 2\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + 2\langle (-\partial_x^2)^{-1/2} u, (-\partial_x^2)^{-1/2} u_{tt} \rangle + 2\|u_t\|_2^2 + 2(u, u_{tt}) + 2\beta \\ &= 2\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta + 2(u, (-\partial_x^2)^{-1} u_{tt} + u_{tt}) \\ &= 2\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2(u, u - u_{xx} + u_{xxxx} + ru_t + f(u)) \\ &= 2\|(-\partial_x^2)^{-1/2} u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta \\ &\quad - 2\|u\|_2^2 - 2\|u_x\|_2^2 - 2\|u_{xx}\|_2^2 - 2r(u, u_t) - 2 \int_{R^n} u f(u) dx. \end{aligned}$$

By the aid of the Cauchy inequality, we obtain

$$(4.5) \quad \begin{aligned} 2r(u, u_t) &\leq r(\|u\|_2^2 + \|u_t\|_2^2) = r[E(0) - \|(-\partial_x^2)^{-1/2} u_t\|_2^2 - \|u_x\|_2^2 - \|u_{xx}\|_2^2 \\ &\quad - 2r \int_0^t \|u_\tau\|_2^2 d\tau - 2 \int_{R^n} F(u) dx]. \end{aligned}$$

It follows from relations (4.1)–(4.5) that

$$\begin{aligned}
 & I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \\
 & \geq I(t)I''(t) - (4 + \alpha)I(t)[\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + \|u_{xx}\|_2^2 + \|u_t\|_2^2 + \beta] \\
 & \geq I(t)\{2\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2\|u_{xx}\|_2^2 - 2\|u\|_2^2 - 2\|u_x\|_2^2 \\
 & \quad - 2r(u, u_t) - 2 \int u f(u) dx - (4 + \alpha)[\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + \|u_t\|_2^2 + \beta]\} \\
 & \geq I(t)\{(r - \alpha - 2)\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + (-2 - \alpha)\|u_t\|_2^2 + (-4 - \alpha)\beta \\
 & \quad + (r - 2)(\|u_x\|_2^2 + \|u_{xx}\|_2^2) + \int [2rF(u) - 2uf(u) - 2u^2] dx \\
 (4.6) \quad & + 2r^2 \int_0^t \|u_\tau\|_2^2 d\tau - rE(0)\}.
 \end{aligned}$$

From equality (3.1), we have

$$\begin{aligned}
 & (r - \alpha - 2)\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + (-2 - \alpha)\|u_t\|_2^2 + (r - 2)(\|u_x\|_2^2 + \|u_{xx}\|_2^2) \\
 & \geq (-\alpha - 2)(\|(-\partial_x^2)^{-1/2}u_t\|_2^2 + \|u_x\|_2^2 + \|u_{xx}\|_2^2 + \|u_t\|_2^2) \\
 & = (\alpha + 2)(\|u\|_2^2 + 2r \int_0^t \|u_\tau\|_2^2 d\tau + 2 \int_{R^n} F(u) dx - E(0)).
 \end{aligned}$$

Thus, from the above inequality, (1.9) and (4.6), we have

$$\begin{aligned}
 & I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \\
 & \geq I(t)\{- (4 + \alpha)\beta - (2 + \alpha + r)E(0) \\
 & \quad + \int_{R^n} [2(2 + \alpha + r)F(u) + \alpha u^2 - 2uf(u)] dx + (2r(2 + \alpha) + 2r^2) \int_0^t \|u_\tau\|_2^2 d\tau\} \\
 & \geq - [(4 + \alpha)\beta + (2 + \alpha + r)E(0)]I(t).
 \end{aligned}$$

(4.7)

If  $E(0) < 0$ , taking  $\beta = -\frac{2+\alpha+r}{4+\alpha}E(0) > 0$ , then we get

$$I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \geq 0.$$

We may choose  $\tau$  so large that  $I'(t) > 0$ . From Lemma 4.1 we know that  $I(t)$  becomes infinite at a time  $T_1$  at most equal to

$$T_1 = \frac{4I(0)}{\alpha I'(t)} < \infty.$$

If  $E(0) = 0$ , taking  $\beta = 0$ , from equation (4.7), we get

$$I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \geq 0.$$

Also  $I'(t) > 0$  by assumption (ii). Thus, we obtain from Lemma 4.1 that  $I(t)$  becomes infinite at a time  $T_2$  at most equal to

$$T_2 = \frac{4I(0)}{\alpha I'(0)} < \infty.$$

If  $E(0) > 0$ , then taking  $\beta = 0$ , inequality (4.7) becomes

$$(4.8) \quad I(t)I''(t) - (1 + \frac{\alpha}{4})(I'(t))^2 \geq -(2 + \alpha + r)E(0)I(t).$$

Define  $J(t) = (I(t))^{-\lambda}$ , where  $\lambda = \alpha/4$ . Then we obtain

$$(4.9) \quad \begin{aligned} J'(t) &= -\lambda(I(t))^{-\lambda-1}I'(t), \\ J''(t) &= -\lambda(I(t))^{-\lambda-2}[I(t)I''(t) - (1 + \lambda)(I'(t))^2] \\ &\leq \lambda(2 + r + 4\lambda)E(0)(I(t))^{-\lambda-1}, \end{aligned}$$

where inequality (4.8) is used. Assumption (iii) implies  $J'(0) < 0$ . Let

$$(4.10) \quad t^* = \sup\{t | J'(\tau) < 0, \tau \in (0, t)\}.$$

By the continuity of  $J'(t)$ ,  $t^*$  is positive. Multiplying (4.9) by  $2J'(t)$  yields

$$(4.11) \quad \begin{aligned} [(J'(t))^2]' &\geq -2\lambda^2(2 + r + 4\lambda)E(0)(I(t))^{-2\lambda-2}I'(t) \\ &= 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)[I(t)^{-2\lambda-1}]'. \end{aligned}$$

Integrate (4.11) with respect to  $t$  over  $[0, t]$  to get

$$\begin{aligned} (J'(t))^2 &\geq 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(t))^{-2\lambda-1} + (J'(0))^2 \\ &\quad - 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(0))^{-2\lambda-1} \\ &\geq (J'(0))^2 - 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(0))^{-2\lambda-1}. \end{aligned}$$

By assumption (iii), we get

$$(J'(0))^2 - 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(0))^{-2\lambda-1} > 0.$$

Hence by continuity of  $J'(t)$ , we have

$$(4.12) \quad J'(t) \leq -[(J'(0))^2 - 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(0))^{-2\lambda-1}]^{1/2}$$

for  $0 \leq t < t^*$ . By the continuity of  $t^*$ , it follows that inequality (4.12) holds for all  $t \geq 0$ . Therefore,

$$J(t) \leq J(0) - [(J'(0))^2 - 2\lambda^2 \frac{2 + r + 4\lambda}{2\lambda + 1} E(0)(I(0))^{-2\lambda-1}]^{1/2} t, \forall t > 0.$$

So  $J(T_1) = 0$  for some  $T_1$  and

$$0 < T_1 \leq T_2 = J(0) / [(J'(0))^2 - [\lambda^2(2 + \lambda + r)/(4\lambda + 8)]E(0)(I(0))^{-(\lambda+2)/2}]^{1/2}.$$

Thus,  $I(t)$  becomes infinite at a time  $T_1$ .

Therefore,  $I(t)$  becomes infinite at a time  $T_1$  under either assumptions. We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof.  $\square$

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