APPLICATIONS OF EPI-RETRACTABLE MODULES

B. M. PANDEYA, A. K. CHATURVEDI* AND A. J. GUPTA

Communicated by Siamak Yassemi

Abstract. An $R$-module $M$ is called epi-retractable if every submodule of $M_R$ is a homomorphic image of $M$. It is shown that if $R$ is a right perfect ring, then every projective slightly compressible module $M_R$ is epi-retractable. If $R$ is a Noetherian ring, then every epi-retractable right $R$-module has direct sum of uniform submodules. If endomorphism ring of a module $M_R$ is von-Neumann regular, then $M$ is semi-simple if and only if $M$ is epi-retractable. If $R$ is a quasi Frobenius ring, then $R$ is a right hereditary ring if and only if every injective right $R$-module is semi-simple. A ring $R$ is semi-simple if and only if $R$ is pri and von-Neumann regular.

1. Introduction

All rings are associative with unit elements and all modules are unitary right modules. Let $R$ be a ring. The ring $R$ is said to be a principal right ideal (pri) ring if every right ideal of $R$ is principal. Ghorbani and Vedadi [3] generalized this concept to modules. An $R$-module $M$ is called epi-retractable if every submodule of $M_R$ is a homomorphic image of $M$. Therefore, $R$ is a pri ring if and only if $R_R$ is epi-retractable. An
$R$-module $N$ is called \textit{M-cyclic} if it is isomorphic to $M/L$, for some submodule $L$ of $M$ (see [10]). Note that $M_R$ is epi-retractable if and only if every submodule of $M$ is $M$-cyclic. Here, we shall investigate epi-retractable modules in terms of $M$-cyclic submodules and also provide those properties of epi-retractable modules which have not been studied earlier.

By [2, 6.9.3], an $R$-module $M$ is called \textit{compressible} if for every non-zero submodule $N$ of $M$ there exists a monomorphism from $M$ to $N$. The concept of epi-retractable modules is dual to the concept of compressible modules. There exist some epi-retractable modules which are not compressible. For example, semi-simple modules are epi-retractable but not compressible.

In Section 2, we study two important properties of epi-retractable modules. We observe that every epi-retractable module is a \textit{slightly compressible module} (see [6]), but the converse need not be true. In Theorem 2.2, we provide a sufficient condition for slightly compressible modules to be epi-retractable. We show that if $R$ is a right perfect ring, then every projective slightly compressible module $M_R$ is epi-retractable. This is a well known problem in the theory of rings and modules when a module has direct sum of uniform submodules. In Theorem 2.3, we show that if $R$ is a Noetherian ring, then every epi-retractable right $R$-module has direct sum of uniform submodules.

In Section 3, we study the semi-simplicity of epi-retractable modules and pri rings. Note that every semi-simple module is epi-retractable, but the converse need not be true. In some results of that section, we provide sufficient conditions for the epi-retractable modules to be semi-simple by injective modules, projective modules, right hereditary rings, von-Neumann regular rings. We show that if endomorphism ring of a module $M$ is von-Neumann regular, then $M$ is semi-simple if and only if $M$ is an epi-retractable module. If $R$ is a quasi Frobenius ring, then $R$ is a right hereditary ring if and only if every injective $R$-module is semi-simple. We characterize semi-simple rings by epi-retractable modules so that a ring $R$ is semi-simple if and only if $R$ is right hereditary and every epi-retractable $R$-module is projective. We end up with a result that states: A ring $R$ is semi-simple if and only if $R$ is pri and von-Neumann regular.

We refer to [10] and [1] for all undefined notions used in the text.
2. Epi-retractable modules

Let \( R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \) and \( M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}, P_R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \) are right \( R \)-modules. It is clear that \( R_R, N_R, P_R \) and \((M/P)_R\) are epi-retractable \( R \)-modules. \( M_R \) is not an epi-retractable module. Moreover, submodules of an epi-retractable module need not be epi-retractable and also factors of an epi-retractable module need not be epi-retractable. We begin with the observation that the class of epi-retractable modules is closed under direct sums.

\[ \text{Proposition 2.1.} \text{ Let } \{M_i\}_{i \in I} \text{ be a family of epi-retractable modules. Then, } M = \bigoplus_{i \in I} M_i \text{ is an epi-retractable module.} \]

\[ \text{Proof.} \text{ Let } K \text{ be a submodule of } M. \text{ Then, } K \cap M_i \text{ is a submodule of } M_i, \text{ for each } i \in I. \text{ Since each } M_i \text{ is an epi-retractable module, there exists an epimorphism } \alpha_i : M_i \to K \cap M_i. \text{ Define } \alpha = \sum_{i \in I} \alpha_i : M \to K. \text{ Then, clearly } \alpha \text{ is a surjective homomorphism. Hence, } \bigoplus_{i \in I} M_i \text{ is an epi-retractable module.} \]

A projective \( R \)-module \( P \) together with a small epimorphism \( \pi : P \to M \) is called a \textit{projective cover of } \( M \). A ring \( R \) is said to be \textit{right perfect} if every \( R \)-module has a projective cover. In [6], Smith calls an \( R \)-module \( M \) \textit{slightly compressible} if, for every non-zero submodule \( N \) of \( M \), there exists a non-zero homomorphism from \( M \) to \( N \). An \( R \)-module \( M \) is called to be \textit{self-generator} if, for each submodule \( N \) of \( M \), there exists an index set \( J \) and an epimorphism \( \theta : M^{(J)} \to N \). It is clear that every epi-retractable module is self-generator. Moreover, every self-generator is slightly compressible. Then, epi-retractable modules are slightly compressible. In general, every slightly compressible module is not a self-generator (see [6, Proposition 3.1]). Therefore, every slightly compressible module need not be epi-retractable.

The following result shows a sufficient condition for slightly compressible modules to be epi-retractable.

\[ \text{Theorem 2.2.} \text{ Let } R \text{ be a right perfect ring. Then, every projective slightly compressible } R \text{-module is epi-retractable.} \]

\[ \text{Proof.} \text{ Assume that } M \text{ is a projective and slightly compressible module. Let } K \text{ be a submodule of } M. \text{ Since } R \text{ is right perfect, there is a projective cover } P \text{ of } K \text{ with a small } Ker(\pi), \text{ where } \pi : P \to K \text{ is an epimorphism.} \]
Then, there exists a non-zero homomorphism \( f : M \to K \). Consider the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{h} & P \\
\downarrow f & & \downarrow \pi \\
K & & \\
\end{array}
\]

Since \( M \) is projective, \( f \) can be lifted to a homomorphism \( h \) from \( M \) to \( P \) such that the above diagram is commutative, that is, \( f = \pi h \). It follows that \( P = \text{Im}(h) + \text{Ker}(\pi) \). Then, \( P = \text{Im}(h) \), because \( \text{Ker}(\pi) \) is small. This implies that \( h \) is surjective. Therefore, \( f \) is also surjective, and hence \( M \) is an epi-retractable module.

A ring \( R \) is called a right \( V \)-ring if every simple \( R \)-module is injective. Moreover, if \( R \) is a right \( V \)-ring, then every projective \( R \)-module is slightly compressible (see [6, Theorem 1.5]). Theorem 2.2 has the following consequence.

**Corollary 2.3.** Let \( R \) be a right perfect and right \( V \)-ring. Then, every projective \( R \)-module is epi-retractable.

**Proof.** This follows from [6, Theorem 1.5] and Theorem 2.2.

Following [9], an \( R \)-module \( M \) is called quasi-polysimple if every non-zero submodule of \( M \) contains a uniform submodule of \( M \). Note that over a Noetherian ring \( R \), every \( R \)-module is quasi-polysimple (see [5, Theorem 2.2]).

We shall now investigate when a epi-retractable module has direct sum of uniform submodules.

**Theorem 2.4.** Let \( R \) be a Noetherian ring. If \( M \) is an epi-retractable \( R \)-module, then \( M \) has direct sum of uniform submodules of \( M \).

**Proof.** It is clear that \( M \) is quasi-polysimple. Therefore, \( M \) is an essential extension of the direct sum \( \oplus_{i \in J} K_i \), where each \( K_i \) is the uniform submodule of \( M \) and \( J \) is some index set (see [5, Lemma 2.1]). Since \( M \) is epi-retractable, there exists an endomorphism \( f \in S \) such that \( f(M) = \oplus_{i \in J} K_i \).

3. **Semi-simplicity of epi-retractable modules**

A ring \( R \) is called right hereditary if every right ideal is projective. Moreover, \( R \) is right hereditary if and only if every submodule of every
projective $R$-module is projective and if and only if quotients of injective, $R$-modules are injective (see [4, Corollary 2.26] and [4, Theorem 3.22]). There are some modules which are injective, but not epi-retractable. For example, the set of rational numbers $\mathbb{Q}$ is an injective module, but is not epi-retractable. Note that every semi-simple module is epi-retractable, but in general the converse is not true.

In the following, we investigate when an epi-retractable module is semi-simple.

**Proposition 3.1.** Let $R$ be a right hereditary ring. Then, the followings hold:

1. Every injective epi-retractable $R$-module is semi-simple.
2. Every projective epi-retractable $R$-module is semi-simple.

**Proof.** (1). Assume that $R$ is a right hereditary ring and $K$ is submodule of an epi-retractable injective $R$-module $M$. Since $M$ is epi-retractable, $K \cong M/L$, for some submodule $L$ of $M$. It follows that $K$ is injective. Suppose $I$ is the identity map from $K$ to $K$. Therefore, $I$ can be extended to a homomorphism from $M$ to $K$. Hence, $K$ is a direct summand of $M$. This implies that $M$ is semi-simple.

(2). This is clear. □

Recall that a ring $R$ is said to be a quasi Frobenius ring if it is a (left) right self injective Noetherian ring. Note that if $R$ is a ring such that every injective $R$-module is epi-retractable, then $R$ is a quasi Frobenius ring (see [3, Proposition 3.2]). In the following, we characterize right hereditary rings.

**Proposition 3.2.** Let $R$ be a quasi Frobenius ring. Then, $R$ is a right hereditary ring if and only if every injective $R$-module is semi-simple.

**Proof.** Assume $R$ is a right hereditary ring. Let $M$ be an injective $R$-module. By [3, Proposition 3.2], $M$ is an epi-retractable module. By Proposition 3.1, it is clear that $M$ is a semi-simple module.

Conversely, assume that every injective $R$-module is semi-simple. Suppose that $K$ is the homomorphic image of an injective $R$-module $M$. Then, $K$ is a direct summand of $M$, because $M$ is semi-simple. Therefore, $K$ is also injective. This implies that quotients of injective $R$-modules are injective. This proves that $R$ is a right hereditary ring. □

**Theorem 3.3.** If the endomorphism ring $S$ of a module $M$ is von-Neumann regular, then $M$ is semi-simple if and only if $M$ is an epi-retractable module.
Proof. Suppose $M$ is an epi-retractable module and $K$ is a submodule of $M$. Then, there is an epimorphism $f$ from $M$ to $K$. Since $S = \text{End}(M)$ is von-Neumann regular, $f(M) = K$ is a direct summand of $M$. Hence, $M$ is a semi-simple module. The converse is obvious. \qed

Let $R$ be a ring and $M$ be an $R$-module. We denote $r(x) = \{s \in R : xs = 0\}$, for some $x \in M$. Note that $r(x)$ is a right ideal of $R$ and $R/r(x) \cong xR$, for all $x \in M$. In the following, we characterize semi-simple ring.

**Theorem 3.4.** A ring $R$ is semi-simple if and only if $R$ is right hereditary and every epi-retractable $R$-module is projective.

**Proof.** Assume that $R$ is a right hereditary ring and every epi-retractable $R$-module is projective. Let $M$ be a simple $R$-module. It follows that $M$ is epi-retractable and projective. For any $x \in M$, $xR \cong R/r(x)$. Then, $xR$ (and hence $R/r(x)$) is projective, because $R$ is a right hereditary ring. Therefore, the exact sequence $0 \to r(x) \to R \to R/r(x) \to 0$ splits. This implies that $r(x)$ is a direct summand of $M$. Since $r(x)$ is a maximal right ideal, $R$ is a semi-simple ring. The converse is obvious. \qed

An $R$-module $M$ is said to satisfy (**)-property if every non-zero endomorphism of $M$ is an epimorphism (see [11]). In general, epi-retractable modules do not satisfy (**)-property. For example, $Z$ as $Z$-module is epi-retractable, but it does not satisfy (**)-property. The following result shows that epi-retractable module with (**)-property is simple.

**Proposition 3.5.** An $R$-module $M$ is simple if and only if $M$ is epi-retractable with (**)-property.

**Proof.** Assume that $M$ is epi-retractable with (**)-property. Let $K$ be a proper submodule of $M$. Then, there is an epimorphism $f : M \to K$. This implies that $f$ is a non-zero endomorphism from $M$ to $M$. Since $M$ satisfies (**)-property, $f(M) = M = K$. Hence, $M$ is simple. The converse is obvious. \qed

**Corollary 3.6.** If an $R$-module $M$ is epi-retractable with (**)-property, then $\text{End}(M_R)$ is a division ring.

An $R$-module $M$ is said to satisfy (*)-property if every non-zero endomorphism of $M$ is a monomorphism (see [7]). This is dual to the concept of (**)-property defined earlier.

**Proposition 3.7.** Every epi-retractable module with (*)-property is a co-Hopfian module.
Proof. Straightforward.

**Theorem 3.8.** A ring $R$ is semi-simple if and only if $R$ is a pri and von-Neumann regular ring.

*Proof.* Assume that $R$ is a pri ring. Then, every right ideal of ring $R$ is a principal right ideal. This implies that every right ideal is a direct summand of $R$, because $R$ is von-Neumann regular. It follows by [10, 20.7] that $R$ is a semi-simple ring.

**Proposition 3.9.** Let $R$ be a ring such that every slightly compressible $R$-module is pseudo-projective. Then, $R$ is a right $V$-ring if and only if $R$ is a semi-simple ring.

*Proof.* Let $M$ be a slightly compressible $R$-module. Suppose there is a free $R$-module $F$ with an epimorphism $g : F \to M$. By [6, Theorem 1.5], $F$ is a slightly compressible module. Then, $F \oplus M$ is a slightly compressible module by [6, Proposition 1.4]. Consider the exact sequence $0 \to \text{Ker}(g) \xrightarrow{i} F \xrightarrow{g} M \to 0$. This sequence splits by [8, Lemma 1.3]. Therefore, $M$ is a direct summand of $F$. Hence, $M$ is projective. In particular, every simple $R$-module is projective. It follows by [10, 20.7] that $R$ is a semi-simple ring.

**Corollary 3.10.** Over a right $V$-ring $R$, if every slightly compressible $R$-module is pseudo-projective, then every $R$-module is epi-retractable.

A ring $R$ is called right semi-artinian if every non-zero $R$-module has non-zero socle.

**Proposition 3.11.** Let $R$ be a right semi-artinian right $V$-ring. Then, $R$ is semi-simple if and only if every $R$-module is pseudo-projective.

*Proof.* Assume that over a right semi-artinian right $V$-ring $R$, every $R$-module is pseudo-projective. By [6, Proposition 1.18], every right $R$-module is slightly compressible. It follows by Proposition 3.9 that $R$ is semi-simple.

**Corollary 3.12.** Over a right semi-artinian right $V$-ring, every pseudo-projective module is an epi-retractable module.

Recall that a ring $R$ is right PP-ring if every cyclic right ideal of $R$ is projective. A ring $R$ is called a regular if for any $a \in R$ there is an element $b \in R$ with $aba = a$. Note that $R$ is regular if and only if every right principal ideal is a direct summand in $R$ (see [10, 3.10]).
Proposition 3.13. The followings are equivalent for a pri ring $R$.

1. $R$ is a right PP-ring.
2. $R$ is a right hereditary ring.
3. $R$ is a von-Neumann regular ring.

Proof. (1) $\Rightarrow$ (2). Straightforward.

(2) $\Rightarrow$ (3). Assume the condition (2). Let $L$ be a principal right ideal of $R$. Then, $L$ is projective, because $R$ is a right hereditary ring. Suppose $\pi : R \to L$ is an epimorphism and $I : L \to L$ is the identity map. This implies that $I$ can be lifted to a homomorphism $f$ from $L$ to $R$, that is, $I = \pi f$. It follows that $L$ is a direct summand of $R$. Hence, $R$ is a von-Neumann regular ring.

(3) $\Rightarrow$ (1). Obvious. $\square$

Acknowledgments

The authors thank the referee for his/her careful considerations.

References

Applications of epi-retractable modules

Bashishth Muni Pandeya
Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, P.O. Box 221005, Varanasi, India
Email: bmpandeya@bhu.ac.in; bmpandeya@gmail.com

Avanish Kumar Chaturvedi
Department of Mathematics, Jaypee Institute of Information Technology, P.O. Box 201307, Noida(UP), India
Email: akc99@rediffmail.com; akchaturvedi.math@gmail.com

Ashok Ji Gupta
Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, P.O. Box 221005, Varanasi, India
Email: ashokg@bhu.ac.in