ON SKEW ARMENDARIZ AND SKEW QUASI-ARMENDARIZ MODULES

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ABSTRACT. Let $\alpha$ be an endomorphism and $\delta$ an $\alpha$-derivation of a ring $R$. In this paper we study the relationship between an $R$-module $M_R$ and the general polynomial module $M[x]$ over the skew polynomial ring $R[x; \alpha, \delta]$. We introduce the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of $\alpha$-Armendariz modules and extend the classes of non-reduced skew-Armendariz modules. An equivalent characterization of an $\alpha$-skew Armendariz module is given. Some properties of this generalization are established, and connections of properties of a skew-Armendariz module $M_R$ with those of $M[x]_{R[x; \alpha, \delta]}$ are investigated. As a consequence we extend and unify several known results related to Armendariz modules.

1. Introduction

Throughout this paper $R$ denotes an associative ring with unity, $\alpha$ is a ring endomorphism and $\delta$ an $\alpha$-derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote $R[x; \alpha, \delta]$ the Ore extension (skew polynomial ring) whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. 

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© 2012 Iranian Mathematical Society.
A ring $R$ is called Baer (respectively, quasi-Baer) if the right annihilator of every nonempty subset (respectively, right ideal) of $R$ is generated, as a right ideal, by an idempotent of $R$. Kaplansky [23], introduced the Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [13] introduced the quasi-Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. All modules are assumed to be unitary right modules. Let $ann_R(X) = \{ r \in R \mid Xr = 0 \}$, where $X$ is a subset of a module $M_R$.

In [29], Lee and Zhou introduced Baer, quasi-Baer and p.p.-modules as follows:

1. $M_R$ is called Baer (respectively, quasi-Baer) if, for any subset (respectively, submodule) $X$ of $M$, $ann_R(X) = eR$ where $e^2 = e \in R$.
2. $M_R$ is called principally projective (or simply p.p.) module (respectively, principally quasi-Baer (or simply p.q.-Baer) module) if, for any element $m \in M$, $ann_R(m) = eR$ (respectively, $ann_R(mR) = eR$) where $e^2 = e \in R$.

Clearly, a ring $R$ is Baer (respectively, p.p. or quasi-Baer) if and only if $R_R$ is Baer (respectively, p.p. or quasi-Baer) module. If $R$ is a Baer (respectively, p.p. or quasi-Baer) ring, then for any right ideal $I$ of $R$, $I_R$ is Baer (respectively, p.p. or quasi- Baer) module. It is clear that $R$ is a right p.q.-Baer ring if and only if $R_R$ is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

A ring is called reduced if it has no nonzero nilpotent elements and $M_R$ is called reduced by Lee and Zhou [29] if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Lee and Zhou have extended various results of reduced rings to reduced modules and Agayev et al. [1] introduced and studied abelian modules as a generalization of abelian rings.

Zhang and Chen [43] introduced the notion of $\alpha$-skew Armendariz modules. Namely, an $R$-module $M_R$ is called $\alpha$-skew Armendariz, if for polynomials $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^j(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$. According to Lee and Zhou [29], a module $M_R$ is called $\alpha$-Armendariz if $M_R$ is $\alpha$-compatible and $\alpha$-skew-Armendariz. If $\alpha$ is equal to the identity, then the above definition boils down to the standard notion of Armendariz module. Moreover, they proved that $R$ is an $\alpha$-skew Armendariz ring if and only if every...
flat right \( R \)-module is \( \alpha \)-skew Armendariz. By [29], a module \( M_R \) is \( \alpha \)-reduced if \( M_R \) is \( \alpha \)-compatible and reduced.

The polynomial extensions of Baer, quasi-Baer, right p.q.-Baer and p.p.-rings and modules have been investigated by many authors [5-10, 15-21, 34-43]. Most of these have worked either with the case \( \delta = 0 \) and \( \alpha \) an automorphism or the case where \( \alpha \) is the identity. With the impetus of quantized derivations, renewed interest in the general Ore extension \( R[x; \alpha, \delta] \) has arisen during the last few years.

In this paper, we study the relationship between an \( R \)-module \( M_R \) and the general polynomial module \( M[x] \) over the skew polynomial ring \( R[x; \alpha, \delta] \). We introduce the notions of skew-Armendariz modules and skew quasi-Armendariz modules which are generalizations of \( \alpha \)-skew Armendariz modules [43] and \( \alpha \)-reduced modules [29]. An equivalent characterization of an \( \alpha \)-skew-Armendariz module is given, which is useful to simplify the proofs. Also new families of non-reduced skew-Armendariz modules are presented. Among other results, we show that there is a strong connection of the Baer, quasi-Baer and the p.p.-property of the two modules, respectively.

Furthermore, we show that for an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \) of a ring \( R \), (1) A right \( R \)-module \( M_R \) is \( \alpha \)-skew-Armendariz if and only if for polynomials \( m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x] \) and \( f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha] \), \( m(x)f(x) = 0 \) implies \( m_0b_j = 0 \) for each \( 0 \leq j \leq n \); (2) An \( \alpha \)-compatible module \( M_R \) is reduced if and only if \( M[x]/M[x](x^n) \) is an \( \alpha \)-skew Armendariz module over \( R[x]/(x^n) \) for any integer \( n \geq 2 \). This result shows that \( \alpha \)-compatible reduced modules play so important roles in the study of skew-Armendariz modules (and hence skew-Armendariz rings) as that of reduced modules in the study of Armendariz modules. (3) An \( (\alpha, \delta) \)-compatible module \( M_R \) is quasi-Baer (respectively, p.q.-Baer) if and only if \( M[x] \) is a quasi-Baer (respectively, p.q.-Baer) module over \( R[x; \alpha, \delta] \); (4) If \( M_R \) is skew-Armendariz with \( R \subseteq M \), then \( M_R \) is Baer (respectively, p.p.) if and only if \( M[x] \) is a Baer (respectively, p.p.-) module over \( R[x; \alpha, \delta] \); (5) A necessary and sufficient condition for the trivial extension \( T(R, R) \) to be skew quasi-Armendariz is obtained. Examples to illustrate the concepts and results are included.

We also study the relations between the set of annihilators in \( M \) and the set of annihilators in \( M[x]_{R[x; \alpha, \delta]} \). We give a sufficient condition for a module to be skew quasi-Armendariz and study the structure of the skew quasi-Armendariz modules. This work extends and unifies several
known results related to Armendariz rings and modules, in particular the landmark results of Hong et al. [20, 21], parallels results of the second author and A.R. Nasr-Isfahani [35] on Ore extensions, and complements later results of E. Hashemi [16] and Zhang and Chen [43] to general polynomial modules over Ore polynomial extension $R[x; \alpha, \delta]$.

2. Skew-Armendariz Modules

In this section the notion of a skew-Armendariz module is introduced as a generalization of skew-Armendariz rings to modules and its properties are studied. We prove that many results of skew-Armendariz rings can be extended to modules with this general settings. We show that the notion of skew-Armendariz module generalizes that of $\alpha$-skew Armendariz modules of Zhang and Chen [43] as well as $\alpha$-Armendariz modules and $\alpha$-reduced modules of Lee and Zhou [29]. Moreover we extend the classes of skew-Armendariz modules.

We will be working here with general right modules $M_R$ rather than just $R_R$, and the restrictions on $\alpha$ and $\delta$ we require are best phrased as conditions on the module $M_R$ that arise from the use of general $\alpha$ and $\delta$. Let us formally define these conditions here:

From the Ore commutation law, an inductive argument can be made to calculate an expression for $x^j a$, for all $j \in \mathbb{N}$ and $a \in R$. To record this result, we shall use some convenient notation introduced in [3, 27]:

**Notation.** Given $\alpha$ and $\delta$ as above and integers $j \geq i \geq 0$, let us write $f^j_i$ for the sum of all "words" in $\alpha$ and $\delta$ in which there are $i$ factors of $\alpha$ and $j - i$ factors of $\delta$. For instance, $f^j_j = \alpha^j$, $f^0_0 = \delta^0$, and $f^j_{j-1} = \alpha^{j-1} \delta + \alpha^{j-2} \delta \alpha + \cdots + \delta \alpha^{j-1}$.

Using recursive formulas for the $f^j_i$'s and induction, as done in [27], one can show with a routine computation that

\[ x^j a = \sum_{i=0}^{j} f^j_i(a)x^i, \tag{2.1} \]

for all $a \in R$, where $j \geq i \geq 0$. This formula uniquely determines a general product of (left) polynomials in $S = R[x; \alpha, \delta]$ and will be used freely in what follows. More generally, given a right $R$-module $M_R$, we
can form the polynomial module $M[x]_S$ over $S$ as follows. Elements of $M[x]$ have the form $\sum m_i x^i$ ($m_i \in M$), and the action of $S$ on such elements is basically dictated by (2.1), since it suffices to define the action of monomials of $S$ on monomials in $M[x]_S$ via

$$(mx^j)(ax^l) = m \sum_{i=0}^j f_i^j (a)x^{i+l}$$

for all $a \in R$ and $j, l \in \mathbb{N}$. It is readily verified that this makes $M[x]$ into an $S$-module.

A ring $R$ is called Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $i,j$. Following Anderson and Camillo [2], a module $M_R$ is called Armendariz if, whenever $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^s m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^t a_j x^j \in R[x]$, we have $m_ia_j = 0$ for all $i,j$.

The term Armendariz was introduced by Rege and Chhawchharia [41]. This nomenclature was used by them since it was Armendariz [5], who initially showed that a reduced ring always satisfies this condition.

The more comprehensive study of Armendariz rings was carried out recently (see, e.g., [1,2,5-6,11-12,15-22,28-29]). The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring $R$ and the annihilators of the polynomial ring $R[x]$. The reason behind these is the fact that there is a natural bijection between the set of annihilators of $R$ and the set of annihilators of $R[x]$ (see Hirano, [19]).

In [21], C.Y. Hong, N.K. Kim and T.K. Kwak extended the Armendariz property of rings to skew polynomial rings $R[x; \alpha]$: For an endomorphism $\alpha$ of a ring $R$, $R$ is called an $\alpha$-skew Armendariz ring (or, a skew-Armendariz ring with the endomorphism $\alpha$) if for polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m$ in $R[x; \alpha]$, $f(x)g(x) = 0$ implies $a_i \alpha^i(b_j) = 0$ for each $0 \leq i \leq n$ and $0 \leq j \leq m$.

M. Başer in [6] studied relations between the set of annihilators in $M_R$ and the set of annihilators in $M[x]$. In [43], Zhang and Chen extended a result of [42] and they showed that, a ring $R$ is $\alpha$-skew Armendariz if and only if every flat right $R$-module is $\alpha$-skew Armendariz. Some other properties of Armendariz rings and modules have been studied in Armendariz [5], Rege and Chhawchharia [41], Rege and Buhphang [42], Anderson and Camillo [2], Hong et al. [20, 21], Kim and Lee
According to Krempa [26], an endomorphism $\alpha$ of a ring $R$ is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is said to be $\alpha$-rigid if there exists a rigid endomorphism $\alpha$ of $R$. Hong et al. [20], studied Ore extensions of Baer rings over $\alpha$-rigid rings, and show that a ring $R$ is $\alpha$-rigid if and only if $R[x; \alpha, \delta]$ is reduced. Clearly a reduced ring is Baer if and only if it is quasi-Baer.

In [35], the second author and A.R. Nasr-Isfahani, introduced the concept of a skew-Armendariz ring and studied its properties. Our focus in this section is to introduce the concept of a skew-Armendariz module and study its properties. We prove that the notion of skew-Armendariz module generalizes that of $\alpha$-skew Armendariz rings of Hong et al. [21] and Krempa’s $\alpha$-rigid rings [26] as well as that of the second author and A.R. Nasr-Isfahani’s skew-Armendariz rings [35] to general polynomial modules over Ore polynomial extension $R[x; \alpha, \delta]$.

**Definition 2.1.** (Zhang and Chen [43]) Let $R$ be a ring with an endomorphism $\alpha$ and $M_R$ an $R$-module. A module $M_R$ is called an $\alpha$-skew Armendariz module, if for polynomials $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]$ and $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i \alpha^i(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$.

**Definition 2.2.** Let $R$ be a ring with an endomorphism $\alpha$ and $\alpha$-derivation $\delta$. Let $M_R$ be an $R$-module. We say that $M_R$ is an $(\alpha, \delta)$-skew Armendariz module if, for polynomials $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]$ and $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha, \delta]$, $m(x)f(x) = 0$ implies $m_i x^i b_j x^j = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$.

Notice that in the case when $\delta = 0$, the above definition boils down to the notion of $\alpha$-skew Armendariz of Zhang and Chen [43].

**Definition 2.3.** Let $R$ be a ring with an endomorphism $\alpha$ and $\alpha$-derivation $\delta$. Let $M_R$ be an $R$-module. We say that $M_R$ is a skew-Armendariz module, if for polynomials $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]$ and $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha, \delta]$, $m(x)f(x) = 0$ implies $m_0 b_j = 0$ for each $0 \leq j \leq n$. 
It is clear that $(\alpha, \delta)$-skew Armendariz modules are skew-Armendariz, and each Armendariz module is $\alpha$-skew Armendariz, where $\alpha = id_R$, and every submodule of a skew-Armendariz module is skew-Armendariz. It is also clear that $R$ is a skew-Armendariz ring if $R_R$ is an skew-Armendariz module. In [35], the second author and A.R. Nasr-Isfahani provided numerous examples of non-semiprime (and hence non-reduced) skew-Armendariz rings.

The following equivalent characterization of an $\alpha$-skew-Armendariz module is useful to simplify the proofs of results in the context of Armendariz rings and modules. It is shown that our definition of a skew-Armendariz module is a generalization of Hong et al.’s $\alpha$-skew Armendariz ring [21] and Zhang and Chen’s $\alpha$-skew Armendariz module [43], for the more general setting.

The following result shows that our definition of a skew-Armendariz module is a generalization of the notion of an $\alpha$-skew-Armendariz module for the more general setting:

**Theorem 2.4.** Let $M_R$ be a module and $\alpha$ an endomorphism of $R$. Then $M_R$ is $\alpha$-skew Armendariz if and only if for every polynomials $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ and $f(x) = b_0+b_1x+\cdots+b_nx^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$.

**Proof.** The forward direction is clear that if $M_R$ is an $\alpha$-skew Armendariz, then for every polynomials $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ and $f(x) = b_0+b_1x+\cdots+b_nx^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$. For the backward direction, suppose that for every polynomials $m(x) = m_0+m_1x+\cdots+m_kx^k \in M[x]$ and $f(x) = b_0+b_1x+\cdots+b_nx^n \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_0b_j = 0$ for each $0 \leq j \leq n$. We show that $M_R$ is $\alpha$-skew Armendariz. We have, $0 = (m_0 + m_1x + \cdots + m_kx^k)(b_0 + b_1x + \cdots + b_nx^n) = m_0(b_0+b_1x+\cdots+b_nx^n)+(m_1+m_2x+\cdots+m_kx^{k-1})x(b_0+b_1x+\cdots+b_nx^n)$. So $(m_1+m_2x+\cdots+m_kx^{k-1})(\alpha(b_0)x+\alpha(b_1)x^2+\cdots+\alpha(b_n)x^{n+1}) = 0$. Hence $m_1\alpha(b_j) = 0$ for each $0 \leq j \leq n$. Inductively, we can see that $m_i\alpha^i(b_j) = 0$ for each $0 \leq i \leq k$ and $0 \leq j \leq n$ and the result follows.

**Corollary 2.5.** A ring $R$ with an endomorphism $\alpha$ is $\alpha$-skew Armendariz if and only if for every polynomials $f(x) = a_0 + a_1x + \cdots +
If we take \( \alpha = id_R \), we deduce the following equivalent condition for a module to be Armendariz.

**Corollary 2.6.** A module \( M_R \) is Armendariz if and only if for every polynomials \( m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x] \) and \( f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x] \), \( m(x)f(x) = 0 \) implies \( m_0b_j = 0 \) for each \( 0 \leq j \leq n \).

**Corollary 2.7.** A ring \( R \) is Armendariz if and only if for every polynomials \( f(x) = a_0 + a_1x + \cdots + a_nx^n \), \( g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \), \( f(x)g(x) = 0 \) implies \( a_0b_j = 0 \) for each \( 0 \leq j \leq m \).

**Definition 2.8.** Let \( R \) be a ring with an endomorphism \( \alpha \) and an \( \alpha \)-derivation \( \delta \). We say that \( M_R \) is a linearly skew-Armendariz module, if for linear polynomials \( m(x) = m_0 + m_1x \in M[x] \) and \( g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x] \), \( m(x)g(x) = 0 \) implies \( m_0b_0 = m_0b_1 = 0 \).

It is clear that each skew-Armendariz module is linearly skew-Armendariz and that every submodule of a linearly skew-Armendariz module is also linearly skew-Armendariz.

By [12, Example 2.2], there exists an \( \alpha \)-skew Armendariz ring \( R \) such that \( \alpha \) is not a monomorphism and \( R \) is not a reduced ring.

**Example 2.9.** Let \( D \) be a domain and \( R_n(D) \) a subring of \( M_n(D) \), where \( n \geq 2 \) and

\[
R_n(D) := \left\{ \begin{pmatrix}
a & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & a & a_{22} & \cdots & a_{2n} \\
0 & 0 & a & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a
\end{pmatrix} \mid a, a_{ij} \in D \right\}.
\]

Let \( \alpha \) be an endomorphism of \( R_n(D) \) such that
\[ \alpha \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix}. \]

Clearly, \( \alpha \) is not a monomorphism and \( R_n(D) \) is not a reduced ring. In [12, Example 2.2] it is proved that \( R_n(D) \) is an \( \alpha \)-skew Armendariz ring.

Let \( R \) be a subring of a ring \( S \) with \( 1_S \in R \) and \( M_R \subseteq L_S \). Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of \( S \) such that \( \alpha(R) \subseteq R \) and \( \delta(R) \subseteq R \). If \( L_S \) is \( (\alpha, \delta) \)-skew Armendariz, then \( M_R \) is also \( (\alpha, \delta) \)-skew Armendariz.

We can deduce the following result, using the definition of skew-Armendariz modules.

**Proposition 2.10.** Let \( \alpha \) be an endomorphism and \( \delta \) an \( \alpha \)-derivation of a ring \( R \). The class of skew-Armendariz modules is closed under submodules, direct products and direct sums.

\section*{Definition 2.11.} (Annin, [3]) Given a module \( M_R \), an endomorphism \( \alpha : R \to R \) and an \( \alpha \)-derivation \( \delta : R \to R \), we say that \( M_R \) is \( \alpha \)-compatible if for each \( m \in M \) and \( r \in R \), we have \( mr = 0 \iff m\alpha(r) = 0 \). Moreover, we say \( M_R \) is \( \delta \)-compatible if for each \( m \in M \) and \( r \in R \), we have \( mr = 0 \Rightarrow m\delta(r) = 0 \). If \( M_R \) is both \( \alpha \)-compatible and \( \delta \)-compatible, we say that \( M_R \) is \( (\alpha, \delta) \)-compatible.

The \( (\alpha, \delta) \)-compatibility condition on \( M_R \) is a natural, independently interesting condition from which we can derive a number of interesting properties, and it will be of invaluable service in the proof of our main results. After a few quick remarks about Definition 2.11, we will present some results on modules and annihilators in Ore extension rings that can be deduced for these \( (\alpha, \delta) \)-compatible modules. These fundamental properties of \( (\alpha, \delta) \)-compatible modules will lay the groundwork for our main results.
Remark 2.12. (a) It is important to note that the $\alpha$-compatibility assumption requires an “if and only if” while the $\delta$-compatibility assumption is only a one-sided implication. The reason for the stronger assumption on $\alpha$ is that we will often need to consider the leading coefficient of an expression $m(x)r$, where $m(x) \in M[x]$ and $r \in R$, where by (2.1) will involve powers of $\alpha$ but will be free of $\delta$. Finally, observe that in the classical case where $\delta = 0$, one never has the reverse implication to the $\delta$-compatibility condition for a nonzero module $M_R$, so we certainly do not expect a two-sided implication for the condition on $\delta$.

(b) If $M_R$ is $\alpha$-compatible (respectively, $\delta$-compatible), then so is any submodule of $M_R$.

(c) If $M_R$ is $\alpha$-compatible (respectively, $\delta$-compatible), then for all $i \geq 1$, $M_R$ is $\alpha^i$-compatible (respectively, $\delta^i$-compatible).

The following lemma shows that the $(\alpha, \delta)$-compatibility property on a module $M_R$ is inherited by the polynomial module $M[x]$.

Lemma 2.13. [3, Lemma 2.16] A module $M_R$ is $(\alpha, \delta)$-compatible if and only if the polynomial extension $M[x]_R$ is $(\alpha, \delta)$-compatible.

Lemma 2.14. The following are equivalent for a module $M_R$.

(i) $M_R$ is reduced and $(\alpha, \delta)$-compatible;

(ii) The following conditions hold. For any $m \in M$ and $a \in R$,

(a) $ma = 0$ implies $mRa = 0$,

(b) $ma = 0$ implies $m\delta(a) = 0$,

(c) $ma = 0$ if and only if $ma(a) = 0$,

(d) $ma^2 = 0$ implies $ma = 0$.

Proof. The proof is straightforward.

Lemma 2.15. Let $M_R$ be an $(\alpha, \delta)$-compatible module. Let $m \in M$ and $a, b \in R$. Then we have the following:

(i) If $ma = 0$, then $ma^j(a) = 0 = m\delta^j(a)$ for any positive integer $j$;

(ii) If $mab = 0$, then $ma(\delta^j(a)\delta(b) = 0 = ma(\delta(a))\delta^j(b)$, and hence $ma\delta^j(b) = 0 = m\delta^j(a)b$ for any positive integer $i, j$;

(iii) $\operatorname{ann}_R(ma) = \operatorname{ann}_R(ma(a)) \subseteq \operatorname{ann}_R(m\delta(a))$.

Proof. (i) This follows from section (c) of Remark 2.12.

(ii) Suppose that $mab = 0$. Since $M_R$ is $\delta$-compatible, $ma\delta^j(b) = 0$ for each $j$. 

Let \( \alpha \)-compatibility of \( M_R \), \( m\alpha(ab) = 0 \), so \( m\alpha(a)b = 0 \). Since \( M_R \) is \( \delta \)-compatible, \( m\alpha(a)\delta(b) = 0 \).

Since \( M_R \) is \( \delta \)-compatible, \( mab = 0 \) implies \( 0 = m\delta(a)b + m\alpha(a)\delta(b) \).

By above, we deduce \( m\delta(a)b = 0 \).

Using \( \alpha \)-compatibility of \( M_R \), \( m\alpha(\delta(a)b) = 0 \) if and only if \( m\alpha(\delta(a))\alpha(b) = 0 \) if and only if \( m\alpha(\delta(a))b = 0 \). By \( \delta \)-compatibility of \( M_R \), we have \( m\alpha(\delta(a))\delta(b) = 0 \).

By above calculations, \( m\delta(a)b = 0 \) and by \( \delta \)-compatibility of \( M_R \), \( 0 = m\delta(\delta(a)b) = m\delta^2(a)b + m\alpha(\delta(a))\delta(b) \). So, \( m\delta^2(a)b = 0 \).

Therefore, inductively we get \( m\delta^j(a)b = 0 \) for each \( j \). So, \( m\delta^j(b) = 0 \). Also, we can similarly deduce that \( m\alpha(\delta^j(a))\delta(b) = 0 \).

Now we show that \( mab = 0 \) implies that \( m\alpha(\delta(a))\delta^j(b) = 0 \). By above, \( m\delta(a)b = 0 \), and then \( \alpha \)-compatibility of \( M_R \) implies \( m\alpha(\delta(a)b) = 0 \) and hence \( m\alpha^j(\delta(a))\alpha^j(b) = 0 \). Also using \( \alpha \)-compatibility of \( M_R \), it implies \( m\alpha^j(\delta(a))b = 0 \). Since \( M_R \) is \( \delta \)-compatible, \( m\alpha^j(\delta(a))\delta^j(b) = 0 \).

These computations imply the result.

(iii) Note that \( \alpha \)-compatibility of \( M_R \) yields \( m\alpha(ab) = 0 \) \( \iff \) \( m\alpha(a)\alpha(b) = 0 \) \( \iff \) \( m\alpha(ab) = 0 \) \( \iff \) \( mab = 0 \) for all \( a, b \in R \). It remains only to show that \( \text{ann}_R(m\alpha) \subseteq \text{ann}_R(m\delta(a)) \). To see this, let \( mab = 0 \) for some \( b \in R \). Using \( \delta \)-compatibility, we get \( 0 = m\delta(ab) = m(\delta(a)b + \alpha(a)\delta(b)) = 0 \).

Since we have already concluded that \( m\alpha(ab) = 0 \), \( \delta \)-compatibility implies that \( m\alpha(a)\delta(b) = 0 \), and hence \( m\delta(a)b = 0 \), as desired.

\( \Box \)

**Lemma 2.16.** Let \( M_R \) be an \( (\alpha, \delta) \)-compatible module and \( m(x) = m_0 + \cdots + m_k x^k \in M[x] \) and \( r \in R \). Then \( m(x)r = 0 \) if and only if \( m_ir = 0 \) for all \( 0 \leq i \leq k \).

**Proof.** Assume \( m_ir = 0 \) for all \( 0 \leq i \leq k \). An easy calculation using (2.1) shows that

\[
(2.2) \quad m(x)r = \sum_{i=0}^{k} \left( \sum_{j=i}^{k} m_{j} f_{i}^{j}(r) \right) x^i.
\]

By \( (\alpha, \delta) \)-compatibility of \( M_R \), we have \( m_j f_{i}^{j}(r) = 0 \), for all \( i, j \). Thus (2.2) yields \( m(x)r = 0 \). Conversely, assume that \( m(x)r = 0 \). We deduce from (2.2) that,
(2.3) \[ \sum_{j=i}^{k} m_j f_i^j(r) = 0, \]

for each \( i \leq k \).

Starting with \( i = k \), Eq. (2.3) yields \( m_k \alpha^k(r) = 0 \) and hence \( m_j f_i^j(r) = 0 \), for each \( j > i \), by \((\alpha, \delta)\)-compatibility of \( M_R \). Using (2.3) again, we deduce that \( m_i \alpha^i(r) = 0 \), and that \( m_i r = 0 \) as desired. \( \square \)

**Proposition 2.17.** A module \( M_R \) is \( \alpha \)-reduced if and only if the polynomial extension \( M[x]_R \) is an \( \alpha \)-reduced module.

**Proof.** It is enough to prove the forward direction. By Lemma 2.13, \( M_R \) is \( \alpha \)-compatible if and only if \( M[x]_R \) is \( \alpha \)-compatible. Now assume that, \( M_R \) is reduced, to show that \( M[x]_R \) is reduced, using Lemma 2.14, we only need to show that \( m(x) a = 0 \) implies \( m(x) Ra = 0 \) and \( m(x) a^2 = 0 \) implies \( m(x) a = 0 \), where \( m(x) = \sum_{i=0}^{k} m_i x^i \in M[x] \) and \( a \in R \). First let \( m(x) a = 0 \). Since \( M_R \) is reduced and \( m_i a = 0 \) for each \( i \), \( m_i Ra = 0 \) for each \( i \) and hence \( m(x) Ra = 0 \). Now suppose \( m(x) a^2 = 0 \). Since \( M_R \) is reduced and \( m_i a^2 = 0 \) for each \( i \), \( m_i a = 0 \) for each \( i \) and hence \( m(x) a = 0 \). Thus \( M[x]_R \) is reduced and the result follows by Lemma 2.14. \( \square \)

Notice that, the concept of \( \alpha \)-reduced for the regular module \( R_R \) coincides with that of reduced and \( \alpha \)-compatible ring \( R \), which in this case \( R \) is indeed an \( \alpha \)-rigid ring; and note also that, a ring \( R \) is \( \alpha \)-rigid if and only if \( R \) is reduced and \((\alpha, \delta)\)-compatible. So we deduce the following:

**Corollary 2.18.** A ring \( R \) is \( \alpha \)-rigid if and only if \( R[x]_R \) (\( R[x; \alpha] \) or \( R[x; \alpha, \delta] \)) is an \( \alpha \)-reduced \( R \)-module.

**Theorem 2.19.** Every \((\alpha, \delta)\)-compatible and reduced module is skew-Armendariz.

**Proof.** Let \( m(x) = m_0 + \cdots + m_k x^k \in M[x] \), \( f(x) = a_0 + \cdots + a_n x^n \in R[x; \alpha, \delta] \) and \( m(x) f(x) = 0 \). So \( m_k \alpha^k(a_n) = 0 \), because it is the leading coefficient of \( m(x) f(x) \). By \( \alpha \)-compatibility of \( M_R \), we have \( m_k a_n = 0 \). By Lemma 2.14, \( m_k Ra_n = 0 \), and by \((\alpha, \delta)\)-compatibility of \( M_R \), \( m_k f_i^j(a_n) = 0 \). Thus the coefficient of \( x^{k+n-1} \) in the equation \( m(x) f(x) = 0 \) is \( m_k \alpha^k(a_{n-1}) + m_{k-1} \alpha^{k-1}(a_n) = 0 \). Multiplying by \( a_n \) from right we
get \( m_{k-1} \alpha^{k-1} (a_n) a_n = 0 \). Using \( \alpha \)-compatibility repeatedly we obtain \( m_{k-1} a_n^2 = 0 \). Hence \( m_{k-1} a_n = 0 \), by Lemma 2.14. So \( m_{k-1} R a_n = 0 \), by Lemma 2.14 and by \((\alpha, \delta)\)-compatibility of \( M_R \), \( m_{k-1} f^j_i (a_n) = 0 \). Therefore \( m_{k-1} a_{n-1} = 0 \). Continuing this process and using \((\alpha, \delta)\)-compatibility of \( M_R \), we obtain \( m_i x^j a_j x^j = 0 \) for each \( 0 \leq i \leq k \) and \( 0 \leq j \leq n \). Since \((\alpha, \delta)\)-skew Armendariz modules are skew Armendariz, the result follows. \( \square \)

Zhang and Chen [43] proved that, for an endomorphism \( \alpha \) of a ring \( R \) and \( \alpha^\ell = id_R \) for some positive integer \( \ell \), \( M_R \) is \( \alpha \)-reduced if and only if \( M[x]/M[x](x^n) \) is an \( \alpha \)-skew Armendariz module over \( R[x]/(x^n) \) for integer \( n \geq 2 \). They also asked if the condition \( \alpha^\ell = id_R \) superfluous.

For a right \( R \)-module \( M_R \) and \( A = (a_{ij}) \in M_n(R) \), let \( MA = \{ (ma_{ij}) \mid m \in M \} \). For \( n \geq 2 \), let \( V = \sum_{i=1}^{n-1} E_{i,i+1} \) where \( \{ E_{i,j} \mid 1 \leq i,j \leq n \} \) are the matrix units, and set \( T(R, n) = R I_n + RV + \cdots + R V^{n-1} \), \( T(M, n) = M I_n + MV + \cdots + M V^{n-1} \). Then \( T(R, n) \) is a ring and \( T(M, n) \) becomes a right module over \( T(R, n) \) under usual addition and multiplication of matrices. There is a ring isomorphism \( \psi : T(R, n) \rightarrow R[x]/(x^n) \) given by \( \psi(r_0 I_n + r_1 V + \cdots + r_{n-1} V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1} x^{n-1} + (x^n) \) and an Abelian group isomorphism \( \phi : T(M, n) \rightarrow M[x]/M[x](x^n) \) given by \( \phi(m_0 I_n + m_1 V + \cdots + m_{n-1} V^{n-1}) = m_0 + m_1 x + \cdots + m_{n-1} x^{n-1} + M[x](x^n) \) such that \( \phi(W A) = \phi(W) \psi(A) \) for all \( W \in T(M, n) \) and \( A \in T(R, n) \).

Notice that

\[
T(R, n) := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \mid a_i \in R \right\},
\]

with \( n \geq 2 \), is a ring with point-wise addition and usual matrix multiplication. We can denote elements of \( T(R, n) \) by \( (a_0, a_1, \ldots, a_{n-1}) \).

Lee and Zhou [29] proved that for each integer \( n \geq 2 \), \( M[x]/M[x](x^n) \) is an Armendariz right module over \( R[x]/(x^n) \) if and only if \( M_R \) is reduced. In the following we generalize this to \( \alpha \)-reduced modules.

Let \( \alpha \) be an endomorphism of a ring \( R \). Then the map \( T(R, n) \rightarrow T(R, n) \) defined by \( a_0 I_n + a_1 V + \cdots + a_{n-1} V^{n-1} \rightarrow \alpha(a_0) I_n + \alpha(a_1) V + \cdots + \alpha(a_{n-1}) V^{n-1} \) is an endomorphism of \( T(R, n) \). Similarly it is easy to see that the map \( R[x]/(x^n) \rightarrow R[x]/(x^n) \) defined by \( a_0 + a_1 x + \cdots + \)
\[ a_{n-1}x^{n-1} + (x^n) \to \alpha(a_0) + \alpha(a_1)x + \cdots + \alpha(a_{n-1})x^{n-1} + (x^n) \] is an endomorphism of \( R[x]/(x^n) \). We will also denote the two maps above by \( \alpha \).

The following result shows that \( \alpha \)-compatible reduced modules play so important roles in the study of skew-Armendariz modules (and hence skew-Armendariz rings) as that of reduced rings in the study of Armendariz rings.

**Theorem 2.20.** An \( \alpha \)-compatible module \( M_R \) is reduced if and only if \( M[x]/M[x](x^n) \) is an \( \alpha \)-skew Armendariz module over \( R[x]/(x^n) \) for integer \( n \geq 2 \).

**Proof.** First assume that \( T(M, n) \) is an \( \alpha \)-skew Armendariz module over \( T(R, n) \) and let \( ma = 0 \) for \( a \in R \) and \( m \in M \). Let \( p(x) = (m, 0, \ldots, 0) + (0, 0, \ldots, m \alpha)x \in T(M, n)[x; \alpha] \), \( q(x) = (a, 0, \ldots, 0) - (0, 0, \ldots, r\alpha(a))x \in T(R, n)[x; \alpha] \) with \( p(x)q(x) = 0 \). Since \( T(M, n) \) is \( \alpha \)-skew Armendariz, \( (m, 0, \ldots, 0)(0, 0, \ldots, r\alpha(a)) = 0 \) implies \( m\alpha(a) = 0 \) for each \( r \in R \). Hence \( m\alpha(a) = 0 \) yields \( m\alpha = 0 \), because \( M_R \) is \( \alpha \)-compatible. Thus \( M_R \) is reduced. Conversely, assume that \( M_R \) is reduced. Consider the following mapping

\[
\varphi_1 : T(M, n)[x; \alpha] \to T(M[x; \alpha], n), \text{ given by } \varphi_1(A_0 + A_1x + \cdots + A_kx^k) = (f_1, f_2, \ldots, f_n), \text{ where } A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in T(M, n), f_i' = a_{i0} + a_{i1}x + \cdots + a_{ik}x^k \in M[x], 0 \leq i \leq k \text{ and } 1 \leq i' \leq n. \]

\[
\varphi_2 : T(R, n)[x; \alpha] \to T(R[x; \alpha], n), \text{ given by } \varphi_2(B_0 + B_1x + \cdots + B_lx^l) = (g_1, g_2, \ldots, g_n), \text{ where } B_j = (b_{j1}, b_{j2}, \ldots, b_{jn}) \in T(R, n), f_j' = b_{j0} + b_{j1}x + \cdots + b_{jl}x^l \in R[x; \alpha], 0 \leq j \leq l \text{ and } 1 \leq j' \leq n. \]

It is easy to see that \( \varphi_1, \varphi_2 \) are isomorphisms. Suppose that \( p = A_0 + A_1x + \cdots + A_tx^t \in T(M, n)[x; \alpha] \) and \( q = B_0 + B_1x + \cdots + B_mx^m \in T(R, n)[x; \alpha] \), where \( A_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in T(M, n) \), for each \( 0 \leq i \leq t \) and \( B_j = (b_{j1}, b_{j2}, \ldots, b_{jn}) \in T(R, n) \) for each \( 0 \leq j \leq m \) and let \( p(x)q(x) = 0 \). Suppose that \( p_i = a_{i0} + a_{i1}x + \cdots + a_{it}x^t \in M[x; \alpha] \) and \( q_j = b_{0j} + b_{1j}x + \cdots + b_{mj}x^m \in R[x; \alpha] \), then \( p_iq_j = 0 \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - i + 1 \). We then have the system of equations

\[
\begin{align*}
(A_0) & \quad a_{00}b_{0j} = 0, \\
(A_1) & \quad a_{00}b_{1j} + a_{1j} \alpha(b_{0j}) = 0, \\
(A_2) & \quad a_{00}b_{2j} + a_{1j} \alpha(b_{1j}) + a_{2j} \alpha^2(b_{2j}) = 0, \\
& \quad \vdots \\
(A_{t+m-1}) & \quad a_{(t-1)j}b_{mj} + a_{tj} \alpha(b_{(m-1)j}) = 0, \\
(A_{t+m}) & \quad a_{tj} \alpha^t(b_{mj}) = 0.
\end{align*}
\]
Let (A_{t+m}), we have \(a_t \alpha^t(b_{mj}) = 0\), which implies \(a_t b_{mj} = 0\), by \(\alpha\)-compatibility of \(M_R\). Hence \(a_t R b_{mj} = 0\). Multiplying \((A_{t+m-1})\) by \(b_{mj}\) from the right, \((A_{t+m-1})\) becomes \(a_{(t-1)} b_{mj}^2 + a_t \alpha^t(b_{(m-1)j}) b_{mj} = 0\).

Since \(a_t R b_{mj} = 0\), we get \(a_{(t-1)} b_{mj}^2 = 0\). But \(M_R\) is reduced, so \(a_{(t-1)} b_{mj} = 0\). Continuing this process, we have \(a_0 b_{lj} = 0\), where \(0 \leq l \leq m\), \(1 \leq i \leq n\) and \(1 \leq j \leq n - i + 1\). This shows that \(A_0 B_s = 0\) for \(0 \leq s \leq m\), proving that \(T(M, n)\) is \(\alpha\)-skew Armendariz module over \(T(R, n)\).

**Corollary 2.21.** [29, Theorem 1.9] A module \(M_R\) is reduced if and only if \(M[x]/M[x](x^n)\) is an Armendariz module over \(R[x]/(x^n)\) for an integer \(n \geq 2\).

Next we recall a well-known result.

**Proposition 2.22.** Suppose that \(M\) is a flat right \(R\)-module. Then for every exact sequence \(0 \to K \to F \to M \to 0\), where \(F\) is \(R\)-free, we have \((FI) \cap K = KI\) for each left ideal \(I\) of \(R\); in particular, we have \(Fa \cap K = Ka\) for each element \(a\) of \(R\).

**Proposition 2.23.** Let \(\alpha\) be an endomorphism of a ring \(R\) and \(\delta\) an \(\alpha\)-derivation. Then \(R\) is a skew-Armendariz ring if and only if every flat \(R\) module \(M\) is skew-Armendariz.

**Proof.** Let \(M\) be a flat \(R\)-module. Suppose \(0 \to K \to F \to M \to 0\) is an exact sequence with \(F\) free over \(R\). For an element \(y \in F\), we denote \(\bar{y} = y + K\) in \(M\). Suppose that \(f(x) = \sum_{i=0}^{t} \bar{y}_i x^i \in M[x]\) and \(g(x) = \sum_{j=0}^{n} a_j x^j \in R[x; \alpha, \delta]\) with \(f(x)g(x) = 0\). We show that \(\bar{y}_0 a_j = 0\) for \(0 \leq j \leq n\). We have \(f(x)g(x) = 0\), so we get,

The constant term: \(\bar{y}_0 a_0 + \bar{y}_1 \delta(a_0) + \bar{y}_2 \delta^2(a_0) + \cdots = 0\);

The coefficient of \(x\): \(\bar{y}_0 a_1 + \bar{y}_1 \alpha(a_0) + \bar{y}_1 \delta(a_1) + \cdots = 0\);

\(\vdots\)

The coefficient of \(x^{t+n}\): \(\bar{y}_t \alpha^t(a_n) = 0\).

Since \(M\) is a flat \(R\)-module, there exists an \(R\)-module homomorphism \(\beta: F \to K\) such that \(\beta\) fixes these coefficients. Write \(w_i := \beta(y_i) - y_i\) for \(i = 0, 1, \ldots, t\). Each \(w_i\) is an element of \(F\), therefore the polynomial \(h(x) = \sum_{j=0}^{t} w_j x^j \in F[x]\) and \(h(x)g(x) = 0\). Since \(R\) is skew-Armendariz and \(F\) is a free \(R\)-module, \(F\) is skew-Armendariz by Proposition 2.10. Thus, we have \(w_0 a_j = 0\) for all \(j\). It follows that \(y_0 a_j \in K\) for all \(j\), so \(\bar{y}_0 a_j = 0\).
in $M$, proving that $M$ is skew-Armendariz.

\[ \square \]

Put $\text{Ann}_R(2^{M_R}) = \{\text{Ann}_R(U) \mid U \subseteq M_R\}$, where $M_R$ is an $R$-module.

**Theorem 2.24.** Let $M_R$ be an $(\alpha, \delta)$-compatible module and $S = R[x; \alpha, \delta]$. Then the following statements are equivalent:

1. $M_R$ is a skew-Armendariz module;
2. The map $\psi : \text{Ann}_R(2^{M_R}) \rightarrow \text{Ann}_S(2^{M[x]_S})$, defined by $A \mapsto AS$ for all $A \in \text{Ann}_R(2^{M_R})$, is bijective.

**Proof.** (1) $\Rightarrow$ (2). Consider the maps $\psi : \{\text{Ann}_R(U) \mid U \subseteq M_R\} \rightarrow \{\text{Ann}_S(U) \mid U \subseteq M[x]_S\}$ defined by $A \mapsto AS$ for every $A \in \{\text{Ann}_R(U) \mid U \subseteq M_R\}$, and $\psi' : \{\text{Ann}_S(U) \mid U \subseteq M[x]_S\} \rightarrow \{\text{Ann}_R(U) \mid U \subseteq M_R\}$ defined by $B \mapsto B \cap R$. It is clear that $\psi$ is well defined, because $\text{Ann}_R(U)S = \text{Ann}_S(U)$ for each $U \subseteq M_R$. Since $M_R$ is $(\alpha, \delta)$-compatible, we see that $\text{Ann}_S(V) \cap R = \text{Ann}_R(V_0)$ for each $V \subseteq M[x]_S$, where $V_0$ is the set of coefficients of all elements of $V$. Hence $\psi'$ is also well defined. Since $\psi' \psi = \text{id}$, $\psi$ is injective. Assume that $B \in \{\text{Ann}_S(U) \mid U \subseteq M[x]_S\}$, then $B = \text{Ann}_S(J)$ for some $J \subseteq M[x]_S$. Let $B_1$ and $J_1$ denote the set of coefficients of elements of $B$ and $J$, respectively. We claim that $\text{Ann}_R(J_1) = B_1R$. Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in J$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in B$. Then $m(x)f(x) = 0$. Since $M_R$ is skew-Armendariz and $(\alpha, \delta)$-compatible, $m_ib_j = 0$ for all $i$ and $j$. Thus $J_1B_1 = 0$, hence $B_1R \subseteq \text{Ann}_R(J_1)$. Since $M_R$ is $(\alpha, \delta)$-compatible, $\text{Ann}_R(J_1) \subseteq B_1R$. Thus $\text{Ann}_R(J_1) = B_1R$, and hence $\text{Ann}_S(J) = B_1RS$. Therefore $\psi$ is surjective.

(2) $\Rightarrow$ (1). Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]_S$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in S = R[x; \alpha, \delta]$ satisfy $m(x)f(x) = 0$. Then $f(x) \in \text{Ann}_S(m(x)) = AS$, where $A = \text{Ann}_R(U)$ and $U \subseteq M_R$. Hence $b_0, \ldots, b_n \in A$ and so $m(x)b_j = 0$ for $0 \leq j \leq n$. Hence $m_0b_j = 0$ for each $0 \leq j \leq n$, and the result follows.

\[ \square \]

**Theorem 2.25.** If $M_R$ is a linearly skew-Armendariz module with $R \subseteq M$, then for each idempotent $e \in R$, $\alpha(e) = e$ and $\delta(e) = 0$.

**Proof.** Since $M_R$ is a linearly skew-Armendariz module with $R \subseteq M_R$, then $R_R$ is also linearly skew-Armendariz. Hence by [35, Theorem 3.1], the result follows.

\[ \square \]
N. Agayev et al. [1] introduced and studied the notion of abelian modules:
A module $M_R$ is called abelian if, for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $mae = mea$. It is proved in [1] that every Armendariz module and hence every reduced module is abelian. The class of abelian modules is closed under direct sums, and a ring $R$ is abelian if and only if every flat $R$-module is abelian.

**Theorem 2.26.** If $M_R$ is a linearly skew-Armendariz module with $R \subseteq M$, then $M_R$ is an abelian module.

**Proof.** Let $M_R$ be a linearly skew-Armendariz module. Consider the polynomials $m_1(x) = me - mer(1-e)x$ and $m_2(x) = m(1-e) - m(1-e)rex \in M[x]_{R[x;\alpha,\delta]}$ and $f_1(x) = (1-e) + er(1-e)x$ and $f_2(x) = e + (1-e)rex \in R[x;\alpha,\delta]$, where $e$ is an idempotent in $R$, $r \in R$ and $m \in M$. Since $\alpha(e) = e$ and $\delta(e) = 0$, we have $m_1(x)f_1(x) = 0$ and $m_2(x)f_2(x) = 0$. Since $M_R$ is linearly skew-Armendariz, we get $mere = mer$ and $mere = mre$. Thus $mere = mre$ for each $r \in R$, and hence $M_R$ is an abelian module.

**Corollary 2.27.** If $M_R$ is a skew-Armendariz module with $R \subseteq M$, then $M_R$ is an abelian module.

**Theorem 2.28.** Let $M_R$ be a reduced module. Then $M_R$ is a p.p.-module if and only if $M_R$ is a p.q.-Baer module.

**Proof.** Since $M_R$ is reduced, by Lemma 2.14, for each $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$. So $ann_R(m) \subseteq ann_R(mR)$ and hence $ann_R(m) = ann_R(mR)$.

**Theorem 2.29.** Let $M_R$ be an $(\alpha,\delta)$-compatible and skew-Armendariz module with $R \subseteq M$. Then $M_R$ is p.p. if and only if $M[x]_{R[x;\alpha,\delta]}$ is p.p.

**Proof.** Suppose that $M_R$ is a p.p.-module and $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$. So $ann_R(m_i) = e_iR$ for idempotents $e_i \in R$ with $0 \leq i \leq k$. Set $e = e_0e_1\cdots e_k$, then $e$ is an idempotent, this is because $M_R$ is abelian by Corollary 2.27. Hence $eR = \cap_{i=0}^{k}ann_R(m_i)$. By Theorem 2.25, $\alpha(e) = e$ and $\delta(e) = 0$. Thus $m(x)e = 0$ and hence $eS \subseteq ann_S(m(x))$, where $S = R[x;\alpha,\delta]$. Next, assume that $q(x) =$
\[ \sum_{j=0}^{n} b_j x^j \in \text{ann}_S(m(x)) \]. Since \( M_R \) is skew-Armendariz, \( m_0b_j = 0 \) for \( 0 \leq j \leq n \). So \( b_j \in eR \) and hence \( q(x) \in eS \), so \( \text{ann}_S(m(x)) = eS \). This shows that \( M[x] \) is a p.p.-module over \( R[x; \alpha, \delta] \).

Conversely, suppose that \( M[x] \) is a p.p.-module over \( R[x; \alpha, \delta] \) and \( m \in M \). Let \( e(x) = e_0 + e_1 x + \cdots + e_n x^n \) be an idempotent in \( R[x; \alpha, \delta] \). Then from \( e(1-e) = 0 = (1-e)e \), we get \( (e_0 + e_1 x + \cdots + e_n x^n)(1-e_0 - e_1 x - \cdots - e_n x^n) = 0 \) and \( (1-e_0 - e_1 x - \cdots - e_n x^n)(e_0 + e_1 x + \cdots + e_n x^n) = 0 \). Since \( M_R \) is skew-Armendariz, \( e_0(1-e_0) = 0 \), \( (1-e_0)e_i = 0 \). So \( e_0e_i = 0 \), \( e_i = e_0e_i \), and hence \( e_i = 0 \). Thus \( e(x) = e_0^2 = e_0 \in R \), and \( \text{ann}_S(m) = eS \), which yields \( \text{ann}_R(m) = eR \) and the result follows.

\[ \text{Theorem 2.30.} \text{ Let } M_R \text{ be an } (\alpha, \delta) \)-compatible skew-Armendariz module with } R \subseteq M. \text{ Then } M_R \text{ is Baer if and only if } M[x]_{R[x; \alpha, \delta]} \text{ is Baer.} \]

\[ \text{Proof.} \text{ Assume that } M_R \text{ is a Baer module and } J \subseteq M[x]. \text{ First suppose } J_0 = \{ m \in M | m \text{ is a leading coefficient of some non-zero element of } J \}. \text{ Clearly, } J_0 \text{ is a subset of } M. \text{ Since } M_R \text{ is Baer, there exists } e^2 = e \in R \text{ such that } \text{ann}_R(J_0) = eR. \text{ Hence } eS \subseteq \text{ann}_S(J) \text{ by Lemma 2.15. Let } f(x) = b_0 + b_1 x + \cdots + b_n x^n \in \text{ann}_S(J). \text{ Then } J_0b_j = 0 \text{ for each } j = 0, \ldots, n, \text{ because } M_R \text{ is skew-Armendariz. Hence } b_j = eb_j \text{ for each } j = 0, \ldots, n \text{ and } f(x) = ef(x) \in eS. \text{ Thus } \text{ann}_S(J) = eS \text{ and } M[x]_S \text{ is a Baer module. Conversely, assume that } M[x]_S \text{ is a Baer module and } A \subseteq M. \text{ Then } A[x] \subseteq M[x]. \text{ Since } M[x] \text{ is Baer, there exists an idempotent } e(x) = e_0 + \cdots + e_n x^n \in S \text{ such that } \text{ann}_S(A[x]) = e(x)S. \text{ Hence } Ae_0 = 0 \text{ and } e_0 R \subseteq \text{ann}_R(A). \text{ Next, let } t \in \text{ann}_R(A). \text{ Then } A[x]t = 0 \text{ by Lemma 2.16. Hence } t = e(x)t \text{ and so } t = e_0 t \in e_0 R. \text{ Thus } \text{ann}_R(A) = e_0 R \text{ and } M_R \text{ is a Baer module.} \]

\[ \text{Example 2.31.} \text{ Let } F \text{ be a field and } R = \left( \begin{array}{cc} F & 0 \\ 0 & F \end{array} \right) \text{ and let } M_R = \left( \begin{array}{cc} F & 0 \\ F & 0 \end{array} \right) \text{ be a right } R \text{-module. Let } \alpha : R \rightarrow R \text{ be the automorphism given by } \alpha \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = \left( \begin{array}{cc} b & 0 \\ 0 & a \end{array} \right), \text{ for each } a,b \in F. \text{ Note that } R \text{ is an abelian ring and } M_R \text{ is an abelian module. But we see that } M_R \text{ is not } \alpha\text{-skew Armendariz. For this let } m(x) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} -2 & 0 \\ 0 & 0 \end{array} \right) x \in \text{ann}_R(m(x)). \]
Let $M[x]$ and $f(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} x \in R[x; \alpha]$. Then, we can easily see that $m(x)f(x) = 0$. But we have, $m_0a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$.

McCoy [31, Theorem 2] proved that if $R$ is a commutative ring, then whenever $g(x)$ is a zero-divisor in $R[x]$ there exists a nonzero $c \in R$ such that $cg(x) = 0$. We shall extend this result as follows.

**Proposition 2.32.** Let $M_R$ be an $(\alpha, \delta)$-compatible and reduced module. If $m(x)$ is a torsion element in $M[x]$ (i.e., $m(x)h(x) = 0$ for some $0 \neq h(x) \in R[x; \alpha, \delta]$), then there exists a non-zero element $c$ of $R$ such that $m(x)c = 0$.

**Proof.** Let $m(x) = \sum_{i=0}^n m_i x^i \in M[x]$ and $h(x) = \sum_{j=0}^s h_j x^j \in R[x; \alpha, \delta]$ and $m(x)h(x) = 0$. Then $m_n \alpha^n(h_s) = 0$, and since $M$ is $\alpha$-compatible, we have $m_n h_s = 0$. By Lemma 2.14, we get $m_n Rh_s = 0$. Since $M_R$ is $(\alpha, \delta)$-compatible, it is $(\alpha^i, \delta^j)$-compatible for each $i, j$ and hence $m_n f_j^i(h_s) = 0$ for each $j \geq i \geq 0$. Hence the coefficient of $x^{n+i-1}$ in $m(x)h(x) = 0$ is $m_n \alpha^n(h_{s-1}) + m_{n-1} \alpha^{n-1}(h_s) = 0$.

Multiply the above equation from right by $h_s$, we get $m_{n-1} \alpha^{n-1}(h_s)h_s = 0$. Using $\alpha$-compatibility repeatedly, we obtain $m_{n-1} h_s^2 = 0$, and then by Lemma 2.14, we have $m_{n-1} h_s = 0$. Using Lemma 2.14 again, we have $m_{n-1} Rh_s = 0$, and by $(\alpha, \delta)$-compatibility of $M_R$, $m_{n-1} f_j^i(h_s) = 0$ for each $j \geq i \geq 0$. Hence the coefficient of $x^{n+i-2}$ in $m(x)h(x) = 0$ is $m_n \alpha^n(h_{s-2}) + m_{n-1} \alpha^{n-1}(h_{s-1}) + m_n f_{n-1}^n(h_{s-1}) + m_{n-2} \alpha^{n-2}(h_s) = 0$.

Multiplying the above equation from right by $h_s$, we get $m_{n-2} \alpha^{n-2}(h_s)h_s = 0$. Using $\alpha$-compatibility repeatedly we obtain $m_{n-2} h_s^2 = 0$, and then by Lemma 2.14, we have $m_{n-2} h_s = 0$. Continuing this process we deduce that $m_j h_s = 0$ for each $j$. Since $h(x) \neq 0$ we may assume that $c = h_s \neq 0$. Then by Lemma 2.16, we get $m(x)c = 0$.

\[ \square \]

**Corollary 2.33.** Let $M_R$ be an $(\alpha, \delta)$-compatible and reduced module. Then $M_R$ is Baer (respectively, p.p.) if and only if so is $M[x]_{R[x; \alpha, \delta]}$.

**Proof.** This follows from Theorems 2.19, 2.29 and 2.30. \[ \square \]
Corollary 2.34. Let \( R \) be an \( \alpha \)-compatible and reduced ring. Then \( R \) is Baer (respectively, p.p.) if and only if \( R[x; \alpha, \delta] \) is Baer (respectively, p.p.).

Proof. Since \( R_R \) is \( \alpha \)-compatible and reduced, by definition, \( R \) is an \( \alpha \)-rigid ring. Hence the result follows by Theorems 11 and 14 of [20].

Example 2.35. Let \( R_0 \) be a domain with characteristic 0 and let \( R \) be the polynomial ring \( R_0[t] \). Let \( \alpha \) be the automorphism of \( R \) which is invariant on \( R_0 \) and \( \alpha(t) = -t \). For each fixed element \( a \in R_0 \), let \( \delta \) be the derivation on \( R \) given by \( \delta(at^n) = \begin{cases} at^{n-1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \)

Assume that \( M := R_0 \oplus R_0 \oplus \cdots \). Then \( M \) is a right \( R \) module given by \((m_0, m_1, \cdots)^r = (0, m_0 k_0, m_1 k_1, \cdots) \) for each \((m_0, m_1, \cdots) \in M \) and \( r \in R \) and fixed non-zero integers \( k_0, k_1, k_2, \cdots \). First we show that \( M_R \) is \((\alpha, \delta)\)-compatible. It is enough to show that for each \( 0 \neq m \in M \), \( \text{ann}(m) = 0 \). Suppose that \((a_0, a_1, a_2, \cdots)(b_0 t^r + b_{r+1} t^{r+1} + \cdots) = 0 \), where \( a_i, b_i \in R_0 \) for each \( i \in \mathbb{N}_0 \) and \( b_r \neq 0 \). So we have \((0, 0, \cdots, 0, a_0 k_0 k_1 \cdots k_{r-1}, a_1 k_1 k_2 \cdots k_r, \cdots)(b_0 + b_r t + \cdots) = 0 \).

This implies that \( a_0 k_0 k_1 \cdots k_{r-1} b_r = 0 \). Since \( R_0 \) is of characteristic 0, \( R \) is a domain. Since \( b_r \neq 0 \) and hence \( k_0 k_1 \cdots k_{r-1} b_r \neq 0 \), we get \( a_0 = 0 \). By induction we can see that \( a_i = 0 \) for each \( i \). Now we show that \( M_R \) is \((\alpha, \delta)\)-skew Armendariz. To see this let \( m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x] \) and \( f(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha, \delta] \) with \( 0 = m(x) f(x) = \sum_{p=0}^{k+n} \left( \sum_{i+l=p} \sum_{j=i}^k \alpha^j f_i^j(b_l) \right) x^p \). So \( m_k \alpha^k(a_n) = 0 \). By \( \alpha \)-compatibility of \( M_R \), we have \( m_k a_n = 0 \). Since \( M_R \) is reduced module, \( m_k R a_n = 0 \). On the other hand, by \((\alpha, \delta)\)-compatibility of \( M_R \), \( m_k f_i^j(a_n) = 0 \). Thus the coefficient of \( x^{k+n-1} \) in equation \( m(x) f(x) = 0 \) is \( m_k \alpha^k(a_{n-1}) + m_{k-1} \alpha^{k-1}(a_n) = 0 \). Multiplying by \( a_n \) from right we get \( m_k-1 \alpha^{k-1}(a_n) a_n = 0 \). Using \( \alpha \)-compatibility repeatedly we obtain \( m_k-1 a_n^2 = 0 \). Hence \( m_k-1 a_n = 0 \). Since \( M_R \) is reduced, \( m_k R a_n = 0 \), and by \((\alpha, \delta)\)-compatibility of \( M_R \), \( m_k f_i^j(a_n) = 0 \). Therefore \( m_k a_n = 0 \). Continuing this process and using \((\alpha, \delta)\)-compatibility of \( M_R \), we obtain \( m_i x^i a_j x^j = 0 \) for each \( 0 \leq i \leq k \) and \( 0 \leq j \leq n \), as desired.

In the following, we show by an example that the “\((\alpha, \delta)\)-compatibility condition” in Lemma 2.16, is not superfluous.
Example 2.36. Let $R_0$ be a domain and $R = R_0[t_1, t_2]$, where $t_1, t_2$ are commuting indeterminates. Let $\alpha$ be the $R_0$-automorphism defined by $\alpha(t_1) = t_2$ and $\alpha(t_2) = t_1$. Let $M$ be the polynomial ring $R_0[t_1]$. Consider $M$ to be a right $R$-module given by ordinary polynomial multiplication subject to the condition $Mt_2 = 0$. Then it is easy to see that $M_R$ is not $\alpha$-compatible. Now take $0 \neq m(x) = g_0(t_1) + g_1(t_1)x + \cdots + g_r(t_1)x^r \in M[x]$ and $t_2 \in R$. Then $0 = m(x)t_2 = g_0(t_1)t_2 + g_1(t_1)xt_2 + \cdots + g_r(t_1)x^rt_2 = g_1(t_1)t_1x + g_3(t_1)t_1x^3 + \cdots$. Thus for odd integers $i$, $g_i(t_1)t_1 = 0$ which implies that $g_i(t_1) = 0$, as $R_0$ is a domain. But $0 \neq m(x)$, so for some even number $j$, $0 \neq g_j(t_1)$ and hence $g_j(t_1)t_2 \neq 0$ for some $j$.

3. Skew Quasi-Armendariz Modules

Following Hirano [19], a module $M_R$ is called quasi-Armendariz if, whenever $m(x)R[x]f(x) = 0$, where $m(x) = \sum_{i=0}^{t} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{l} a_j x^j \in R[x]$, we have $m_i Ra_j = 0$ for all $i, j$.

In this section, we generalize the notions of quasi-Armendariz rings and quasi-Armendariz modules and consider the relations between the set of annihilators in $M_R$ and the set of annihilators in $M[x]_{R[x, \alpha, \delta]}$.

We give a sufficient condition for a module to be skew quasi-Armendariz and study the structure of the skew quasi-Armendariz modules.

By Hirano in [19], a ring $R$ is called a quasi-Armendariz ring if, whenever $f(x)R[x]g(x) = 0$ where $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in R[x]$ and $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$, it implies that $a_i Rb_j = 0$ for all $i$ and $j$. Every semiprime ring is a quasi-Armendariz ring, by [19].

In [19], a module $M_R$ is called a quasi-Armendariz module if whenever $m(x)R[x]f(x) = 0$, where $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]$ and $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$, it implies that $m_i Rb_j = 0$ for all $i$ and $j$.

Definition 3.1. Let $M_R$ be a module, $\alpha$ an endomorphism of $R$ and $\delta$ an $\alpha$-derivation. We say $M_R$ is skew quasi-Armendariz, if whenever $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$, $f(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ satisfy $m(x)R[x; \alpha, \delta]f(x) = 0$, we have $m_i R x^t b_j x^j = 0$ for $t \geq 0$, $i = 0, 1, \ldots, k$ and $j = 0, 1, \ldots, n$. 
Theorem 3.2. Let $M_R$ be an $\alpha$-compatible module and $S = R[x;\alpha]$. Then,
(1) The following statements are equivalent:
(a) for any $m(x) \in M[x]_S$, $(\text{ann}_S(m(x)S) \cap R)[x;\alpha] = \text{ann}_S(m(x)S)$.
(b) for any $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]_S$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in S$, $m(x)Sf(x) = 0$ implies $m_i R a_j = 0$, for each $i, j$.
(2) Let $M_R$ be an skew quasi-Armendariz module and $m(x) \in M[x]_S$. If $\text{ann}_S(m(x)S) \neq 0$, then $\text{ann}_S(m(x)S) \cap R \neq 0$.

Proof. (1). (a) $\Rightarrow$ (b) Let $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]_S$, $f(x) = \sum_{j=0}^{t} a_j x^j \in S$ and assume that $m(x)Sf(x) = 0$. By (a), $f(x) \in (\text{ann}_S(m(x)S) \cap R)[x;\alpha]$, and we deduce that $a_j \in \text{ann}_S(m(x)S) \cap R$ for each $0 \leq j \leq t$. So $m(x)Sa_j = 0$ and then by $\alpha$-compatibility of $M_R$, we obtain $m_i R a_j = 0$ for each $i, j$.
(b) $\Rightarrow$ (a) Let $g(x) = \sum_{j=0}^{s} b_j x^j \in (\text{ann}_S(m(x)S) \cap R)[x;\alpha]$, so $b_j \in \text{ann}_S(m(x)S) \cap R$. So $m(x)Sb_j = 0$ for each $j$ and hence $m(x)Sg(x) = 0$. Thus $g(x) \in \text{ann}_S(m(x)S)$. Now assume that $h(x) = \sum_{j=0}^{k} c_j x^j \in \text{ann}_S(m(x)S)$. So $m(x)Sh(x) = 0$ and by (b) we get $m_i R c_j = 0$. By $\alpha$-compatibility of $M_R$, $m(x)R c_j = 0$. So $c_j \in \text{ann}_S(m(x)S) \cap R$ for each $j$ and hence $h(x) \in (\text{ann}_S(m(x)S) \cap R)[x;\alpha]$. So $\text{ann}_S(m(x)S) = (\text{ann}_S(m(x)S) \cap R)[x;\alpha]$.
(2). The proof follows by Lemma 2.15 and (1) (b) $\Rightarrow$ (a).

In the following result, we give relations between the set of annihilators in $M_R$ and the set of annihilators in $M[x]_{R[x;\alpha]}$.

Theorem 3.3. Let $M_R$ be an $\alpha$-compatible module and $S = R[x;\alpha]$. Then the following statements are equivalent:
(1) $M_R$ is a skew quasi-Armendariz module;
(2) The map $\psi : \text{Ann}_R(\text{sub}(M_R)) \to \text{Ann}_S(\text{sub}(M[x]_S))$, defined by $\psi(\text{ann}_R(N)) = \text{ann}_S(N) = \text{ann}_S(N[x])$ for all $N \in \text{sub}(M_R)$, is bijective, where $\text{sub}(M_R)$ and $\text{sub}(M[x]_S)$ denote the sets of submodules.

Proof. (1) $\Rightarrow$ (2) Assume that $M_R$ is skew quasi-Armendariz. Obviously $\psi$ is injective. Therefore, it is enough to show $\psi$ is surjective. Let $V \in \text{sub}(M[x]_S)$ and $C_V$ denotes the set of all coefficients of elements of $V$. Then for $\text{ann}_R(C_V R) \in \text{Ann}_R(\text{sub}(M))$, we have $\psi(\text{ann}_R(C_V R)) = \text{ann}_S(C_V R) = \text{ann}_S(V)$. In fact, let $f(x) \in \text{ann}_S(C_V R)$. Then $C_V R f(x) = 0$ and hence $V f(x) = 0$. So $f(x) \in \text{ann}_S(V)$. Conversely, let $g(x) = b_0 + \cdots + b_k x^k \in \text{ann}_S(V)$. Then $V g(x) = 0$. Since $V$ is a submodule of $M[x]_S$, $V S g(x) = 0$. So $v(x) S g(x) = 0$ for all $v(x) =
\[ v_0 + v_1 x + \cdots + v_t x^t \in V. \] Since \( M_R \) is \( \alpha \)-compatible and skew Armendariz, \( v_i R b_j = 0 \) for all \( i,j \). Hence \( C_V R g(x) = 0 \) and therefore \( g(x) \in \text{ann}_S(C_V R) \). Consequently \( \psi \) is surjective.

(2) \( \Rightarrow \) (1) Assume \( m(x)S f(x) = 0 \), where \( m(x) = m_0 + m_1 x + \cdots + m_t x^t \in M[x] \) and \( f(x) = a_0 + a_1 x + \cdots + a_k x^k \in S \). By hypothesis, \( \text{ann}_S(m(x)S) = \text{ann}_R(N)[x;\alpha] \) for some submodule \( N \) of \( M \). Then \( f(x) \in \text{ann}_R(N)[x;\alpha] \) and hence \( a_j \in \text{ann}_R(N) \) for all \( j \). So \( a_j \in \text{ann}_R(N) \subseteq \text{ann}_R(N)[x;\alpha] = \text{ann}_S(m(x)S) \) and then \( m(x)S a_j = 0 \).

In particular \( m(x)R a_j = 0 \) and hence \( m_i R a_j = 0 \) for all \( i,j \). Since \( M_R \) is \( \alpha \)-compatible, \( m_i x^i Rx^t a_j x^j = 0 \), for \( t \geq 0, i = 0,1,\ldots,t \) and \( j = 0,1,\ldots,k \). Therefore \( M_R \) is skew quasi-Armendariz.

Let \( R \) be a ring. The trivial extension of \( R \) is given by:

\[
T(R, R) = \left\{ \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} \mid a, r \in R \right\}.
\]

Clearly, \( T(R, R) \) is a subring of the ring of \( 2 \times 2 \) matrices over \( R \). The endomorphism \( \alpha \) of \( R \) and the \( \alpha \)-derivation \( \delta \) on \( R \) are extended to \( \bar{\alpha} : T(R, R) \to T(R, R) \) by:

\[
\bar{\alpha} \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha(a) & \alpha(r) \\ 0 & \alpha(a) \end{pmatrix}, \quad \bar{\delta} \begin{pmatrix} a & r \\ 0 & a \end{pmatrix} = \begin{pmatrix} \delta(a) & \delta(r) \\ 0 & \delta(a) \end{pmatrix}.
\]

One can show that \( \bar{\delta} \) is an \( \bar{\alpha} \)-derivation on \( T(R, R) \) and also we can see \( T(R, R)[x;\alpha, \delta] \cong T(R[x;\alpha, \delta], R[x;\alpha, \delta]) \).

**Proposition 3.4.** If the trivial extension of \( R \), \( T(R, R) \), is skew-quasi Armendariz, then so is \( R \).

**Proof.** Let \( f(x) = a_0 + \cdots + a_n x^n, g(x) = b_0 + \cdots + b_m x^m \in R[x;\alpha, \delta] \) and \( f(x)R(x;\alpha, \delta) g(x) = 0 \). For each \( a, r \in R \) and \( t \geq 0 \), we have the following equation:

\[
0 = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} a x^t & r x^t \\ 0 & a x^t \end{pmatrix} \begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f(x) a x^t g(x) \\ 0 & 0 \end{pmatrix}.
\]

Since \( T(R, R) \) is skew quasi-Armendariz, it implies that \( a_i x^i ax^t b_j x^j = 0 \), for each \( i, j, t \). Therefore \( R \) is skew quasi-Armendariz.

When the trivial extension \( T(R, R) \) is skew quasi-Armendariz?

**Theorem 3.5.** Let \( R \) be a ring such that

(i) \( R \) is skew quasi-Armendariz;

(ii) \( f(x)R(x;\alpha, \delta) g(x) = 0 \), then \( f(x)R(x;\alpha, \delta) \cap R(x;\alpha, \delta) g(x) = 0 \).

Then the trivial extension \( T = T(R, R) \) is skew quasi-Armendariz.

**Proof.** Suppose that \( \alpha(x)T[x;\bar{\alpha}, \bar{\delta}] \beta(x) = 0 \), where
\[ \alpha(x) = \left( \begin{array}{cc} a_0 & r_0 \\ 0 & a_0 \end{array} \right) + \left( \begin{array}{cc} a_1 & r_1 \\ 0 & a_1 \end{array} \right) x + \ldots + \left( \begin{array}{cc} a_n & r_n \\ 0 & a_n \end{array} \right) x^n \]

and

\[ \beta(x) = \left( \begin{array}{cc} b_0 & s_0 \\ 0 & b_0 \end{array} \right) + \left( \begin{array}{cc} b_1 & s_1 \\ 0 & b_1 \end{array} \right) x + \ldots + \left( \begin{array}{cc} b_m & s_m \\ 0 & b_m \end{array} \right) x^m \in T[x; \bar{\alpha}, \bar{\delta}] . \]

Let \( f(x) = a_0 + a_1 x + \ldots + a_n x^n, r(x) = r_0 + r_1 x + \ldots + r_n x^n, \)
\( g(x) = b_0 + b_1 x + \ldots + b_m x^m \) and \( s(x) = s_0 + s_1 x + \ldots + s_m x^m \in R[x; \alpha, \delta] . \)

For each \( \left( \begin{array}{ccc} a & r \\ 0 & a \end{array} \right) x^t \in T[x; \bar{\alpha}, \bar{\delta}] , \) it follows that

\[
0 = \left( \begin{array}{ccc} f(x) & r(x) & ax^t \\ f(x) & 0 & 0 \\ 0 & ax^t & 0 \end{array} \right) \left( \begin{array}{ccc} g(x) & 0 \\ 0 & g(x) \\ g(x) & 0 \end{array} \right) = \left( \begin{array}{ccc} f(x)ax^tg(x) & f(x)ax^ts(x) + f(x)rx^tg(x) + r(x)ax^tg(x) \\ 0 & f(x)ax^tg(x) \end{array} \right),
\]

Hence

\[ (3.1) \quad f(x)ax^tg(x) = 0, \]

and

\[ (3.2) \quad f(x)ax^ts(x) + f(x)rx^tg(x) + r(x)ax^tg(x) = 0. \]

Since \( \left( \begin{array}{ccc} a & r \\ 0 & a \end{array} \right) x^t \) is an arbitrary element of \( T(R, R)[x; \bar{\alpha}, \bar{\delta}] \) and
\( T(R, R)[x; \bar{\alpha}, \bar{\delta}] \cong T(R[x; \alpha, \delta], R[x; \alpha, \delta]) , \) by (3.1) we get

\[ (3.3) \quad f(x)R[x; \alpha, \delta]g(x) = 0. \]

Since \( R \) is skew quasi-Armendariz, \( a_i x^i R x^t b_j x^j = 0, \) for all \( i, j, t. \) Thus
by (3.2), \( f(x)[ax^ts(x) + rx^tg(x)] + [r(x)ax^t]g(x) = 0. \) Hence by (3.2) and (3.3), we have
\( f(x)[ax^ts(x) + rx^tg(x)] = -[r(x)ax^t]g(x) \in f(x)R[x; \alpha, \delta] \cap R[x; \alpha, \delta]g(x) = 0. \) So \( f(x)[ax^ts(x) + rx^tg(x)] = 0 = r(x)ax^tg(x), \) and hence we have \( r(x)R[x; \alpha, \delta]g(x) = 0, \) since \( ax^t \) is an arbitrary element. Thus
\( r_i x^i R x^t b_j x^j = 0 \) for all \( i, j, t, \) since \( R \) is skew quasi-Armendariz. Also we have \( f(x)[ax^ts(x)] = -[f(x)rx^t]g(x) \in f(x)R[x; \alpha, \delta] \cap R[x; \alpha, \delta]g(x) = 0. \) Thus \( f(x)ax^ts(x) = 0. \) So we have \( f(x)R[x; \alpha, \delta]s(x) = 0. \) Since \( R \) is skew quasi-Armendariz, we deduce \( a_i x^i R x^t s_j x^j = 0 \) for all \( i, j, t. \) Hence
\[
\left( \begin{array}{cc} a_i & r_i \\ 0 & a_i \end{array} \right) x^t \left( \begin{array}{ccc} a & r \\ 0 & a \end{array} \right) x^t \left( \begin{array}{ccc} b_j & s_j \\ 0 & b_j \end{array} \right) x^j = 0.
\]
Assume that \( M \) is Armendariz, \( M \) Let \( a \) such that 
\[
\begin{pmatrix}
  a_{i}x^{i}ax^{j}b_{j}x^{j} & a_{i}x^{i}rx^{t}b_{j}x^{j} + a_{i}x^{i}ax^{t}b_{j}x^{j} \\
  0 & a_{i}x^{i}ax^{t}b_{j}x^{j}
\end{pmatrix}
\]

= 0 for all \( i, j \) and each \( \begin{pmatrix}
  a \\
  0
\end{pmatrix} \)
\( x^{i} \in T(R, R) \). Therefore the trivial extension \( T(R, R) \) is skew quasi-Armendariz.

Kerr [24] constructed an example of a commutative Goldie ring \( R \) whose polynomial ring \( R[x] \) has an infinite ascending chain of annihilator ideals.

**Theorem 3.6.** Let \( M_{R} \) be an skew quasi-Armendariz module. If \( M_{R} \) is \((\alpha, \delta)\)-compatible, then \( M_{R} \) satisfies the ascending chain condition on annihilator of submodules if and only if so does \( M[x]_{S} \), where \( S = R[x; \alpha, \delta] \).

**Proof.** Assume that \( M_{R} \) satisfies the ascending chain condition on annihilator of submodules. Let \( I_{1} \subseteq I_{2} \subseteq I_{3} \subseteq \ldots \) be a chain of annihilator of submodules of \( M[x]_{S} \). Then there exist submodules \( K_{i} \) of \( M[x]_{S} \) such that \( \text{ann}S(K_{i}) = I_{i} \), for all \( i \geq 1 \) and \( K_{1} \supseteq K_{2} \supseteq K_{3} \supseteq \cdots \). Let \( M_{i} = \{ \text{all coefficients of elements of } K_{i} \} \). Since \( M \) is skew quasi-Armendariz, \( M_{i} \) is submodule of \( M \) for all \( i \geq 1 \). Clearly \( M_{i} \supseteq M_{i+1} \) for all \( i \geq 1 \). Thus \( \text{ann}_{R}(M_{1}) \subseteq \text{ann}_{R}(M_{2}) \subseteq \text{ann}_{R}(M_{3}) \subseteq \cdots \). Since \( M_{R} \) satisfies the ascending chain condition on annihilator of submodules, there exists \( n \geq 1 \) such that \( \text{ann}_{R}(M_{i}) = \text{ann}_{R}(M_{n}) \) for all \( i \geq n \). We show that \( \text{ann}_{S}(K_{i}) = \text{ann}_{S}(K_{n}) \) for all \( i \geq n \). Let \( f(x) = a_{0} + a_{1}x + \cdots + a_{m}x^{m} \in \text{ann}_{S}(K_{i}) \). Then \( M_{i}a_{j} = 0 \) for \( j = 0, \ldots, m \), because \( M \) is skew quasi-Armendariz. Thus \( M_{n}a_{j} = 0 \) for \( j = 0, \ldots, m \) and so \( K_{n}f(x) = 0 \) by Lemma 2.16. Therefore \( \text{ann}_{S}(K_{i}) = \text{ann}_{S}(K_{n}) \) for all \( i \geq n \) and \( M[x]_{S} \) satisfies the ascending chain condition on annihilator of submodules. Now assume \( M[x]_{S} \) satisfies the ascending chain condition on annihilator of submodules. Let \( J_{1} \subseteq J_{2} \subseteq J_{3} \subseteq \ldots \) be a chain of annihilator of submodules of \( M_{R} \). Then there exist submodules \( M_{i} \) of \( M \) such that \( \text{ann}_{R}(M_{i}) = J_{i} \) and \( M_{1} \supseteq M_{2} \supseteq M_{3} \supseteq \cdots \) for all \( i \geq 1 \). Hence \( M_{i}[x] \) is a submodule of \( M[x] \) and \( M_{i}[x] \supseteq M_{i+1}[x] \) and \( \text{ann}_{S}(M_{i}[x]) \subseteq \text{ann}_{S}(M_{i+1}[x]) \) for all \( i \geq 1 \). Since \( M[x]_{S} \) satisfies the ascending chain condition on annihilator of submodules, there exists \( n \geq 1 \) such that \( \text{ann}_{S}(M_{i}[x]) = \text{ann}_{S}(M_{n}[x]) \) for all \( i \geq n \). Since \( M \) is skew quasi-Armendariz, by a similar argument as used in the previous paragraph, one can show that \( \text{ann}_{R}(M_{i}) = \text{ann}_{R}(M_{n}) \) for all \( i \geq n \).

\( \square \)
Following [3], the second author and E. Hashemi [17] introduced \((\alpha, \delta)\)-compatible rings and studied its properties. A ring \(R\) is \(\alpha\)-compatible if for each \(a, b \in R\), \(ab = 0\) if and only if \(a\alpha(b) = 0\). Moreover, \(R\) is said to be \(\delta\)-compatible if for each \(a, b \in R\), \(ab = 0\) implies \(\alpha\delta(b) = 0\). A ring \(R\) is \((\alpha, \delta)\)-compatible if it is both \(\alpha\)-compatible and \(\delta\)-compatible. In this case, clearly the endomorphism \(\alpha\) is injective. Also by [17, Lemma 2.2], a ring \(R\) is \((\alpha, \delta)\)-compatible and reduced if and only if \(R\) is \(\alpha\)-rigid in the sense of Krempa [26]. Thus the \(\alpha\)-compatible ring is a generalization of \(\alpha\)-rigid ring to the more general case where \(R\) is not assumed to be reduced.

**Corollary 3.7.** Let \(R\) be an \((\alpha, \delta)\)-compatible and skew quasi-Armendariz ring. Then \(R\) satisfies the ascending chain condition on right annihilators if and only if so does \(R[x; \alpha, \delta]\).

**Corollary 3.8.** [19, Corollary 3.3] Let \(R\) be an Armendariz ring. Then \(R\) satisfies the ascending chain condition on right annihilators if and only if so does \(R[x]\).

**Theorem 3.9.** Let \(M_R\) be an \((\alpha, \delta)\)-compatible module. Then \(M_R\) is quasi-Baer (respectively, p.q.-Baer) if and only if \(M[x]_{R[x; \alpha, \delta]}\) is quasi-Baer (respectively, p.q.-Baer). In this case \(M_R\) is skew quasi-Armendariz.

**Proof.** Assume \(M_R\) is quasi-Baer. First we shall prove that \(M_R\) is skew quasi-Armendariz. Suppose that \((m_0 + m_1 x + \cdots + m_k x^k)R[x; \alpha, \delta](b_0 + b_1 x + \cdots + b_n x^n) = 0\), with \(m_i \in M, b_j \in R\). In particular case we have

\[(3.4) \quad (m_0 + m_1 x + \cdots + m_k x^k)R(b_0 + b_1 x + \cdots + b_n x^n) = 0.\]

Thus \(m_k Rb_n = 0\) and \(b_n \in \text{ann}_R(m_k R)\). Then \(m_k x^k R x^i b_n x^n = 0\), by Lemma 2.15. Since \(M_R\) is quasi-Baer, there exists \(e_k^2 = e_k \in R\) such that \(\text{ann}_R(m_k R) = e_k R\) and so \(b_n = e_k b_n\). Replacing \(R\) by \(Re_k\) in \(3.4)\) and using Lemma 2.15, we obtain \((m_0 + m_1 x + \cdots + m_{k-1} x^{k-1})Re_k(b_0 + b_1 x + \cdots + b_n x^n) = 0\). Hence \(m_{k-1} Re_k b_n = m_{k-1} R b_n = 0\) and \(b_n \in \text{ann}_R(m_{k-1} R)\). Then \(m_{k-1} x^{k-1} Rx b_n x^n = 0\), by Lemma 2.15. Hence \(b_n \in \text{ann}_R(m_k R) \cap \text{ann}_R(m_{k-1} R)\). Since \(M_R\) is quasi-Baer, there exists \(f^2 = f \in R\) such that \(\text{ann}_R(m_k R) = fR\) and so \(b_n = fb_n\). If we put \(e_{k-1} = e_k f\), then \(e_{k-1} b_n = e_k f b_n = e_k b_n = b_n\) and \(e_{k-1} \in \text{ann}_R(m_k R) \cap \text{ann}_R(m_{k-1} R)\). Next, replacing \(R\) by \(Re_{k-1}\) in \(3.4)\), and using Lemma 2.15, we obtain \((m_0 + m_1 x + \cdots + m_{k-2} x^{k-2})Re_{k-1}(b_0 +\)
$b_1x + \cdots + b_nx^n = 0$. Hence we have $m_{k-2}Re_{k-1}b_n = m_{k-2}Rb_n = 0$
and that $b_n \in \text{ann}_R(m_{k-2}R)$ and so $m_{k-2}x^{k-2}Rx^ib_nx^n = 0$, by Lemma
2.15. Continuing this process, we get $m_ix^iRx^jb_nx^n = 0$ for $i = 0, \ldots, k$.
Using induction on $k+n$, we obtain $m_ix^iRx^jb_jx^j = 0$ for all $i, j, t$. Therefore
$M_R$ is skew quasi-Armendariz. Let $J$ be a $S$-submodule of $M[x]$. Let $N = \{m \in M \mid m$ is a leading coefficient of some non-zero element of $J\}$
$\cup \{0\}$. Clearly, $N$ is a submodule of $M$. Since $M_R$ is quasi-Baer, there exists $e^2 = e \in R$ such that $\text{ann}_R(N) = eR$. Hence $eS \subseteq \text{ann}_S(J)$ by Lemma 2.15. Let $f(x) = b_0 + b_1x + \cdots + b_nx^n \in \text{ann}_S(J)$. Then $Nb_j = 0$
for each $j = 0, \ldots, n$, because $M_R$ is skew quasi-Armendariz. Hence
$b_j = eb_j$ for each $j = 0, \ldots, n$ and $f(x) = ef(x) \in eS$. Thus $\text{ann}_S(J) = eS$ and $M[x]_S$ is quasi-Baer. Now assume that $M[x]_S$ is quasi-Baer and
$I$ is a submodule of $M$. Then $I[x]$ is a submodule of $M[x]$. Since $M[x]$ is quasi-Baer, there exists an idempotent $e(x) = e_0 + \cdots + e_nx^n \in S$
such that $\text{ann}_S(I[x]) = e(x)S$. Hence $Ie_0 = 0$ and $e_0R \subseteq \text{ann}_R(I)$. Let
t $t \in \text{ann}_R(I)$. Then $I[x]t = 0$, by Lemma 2.16. Hence $t = e(x)t$ and so
t $t = e_0t \in e_0R$. Thus $\text{ann}_R(I) = e_0R$ and $M_R$ is quasi-Baer.

It is clear that $R$ is a right p.q.-Baer ring if and only if $R_R$ is a p.q.-
Baer module. But, there exists a p.q.-Baer right $R$-module such that $R$
is not right p.q.-Baer.

**Example 3.10.** Let $R = \mathbb{Z}_2[x]/(x^2)$, where $\mathbb{Z}_2[x]$ is the polynomial ring
over the field $\mathbb{Z}_2$ of two elements and $(x^2)$ is the ideal of $\mathbb{Z}_2[x]$ generated
by $x^2$. It is easy to see that $R$ is a quasi-Armendariz ring. Since right
annihilator of $x + (x^2)$ is not generated by any idempotent, $R$ is not a
right p.q.-Baer ring. Now let $e = 1 + (x^2)$ and $I = ReR$. Then $e^2 = e$,
and for each $a \in R$, $\text{ann}_R((a + I)R) = eR$. Therefore $R/I$ is p.q.-Baer
right $R$-module.

**Corollary 3.11.** [17, Corollary 2.8] Let $R$ be an $(\alpha, \delta)$-compatible ring.
Then $R$ is quasi-Baer (respectively, right p.q.-Baer) if and only if $R[x; \alpha, \delta]$
is quasi-Baer (respectively, right p.q.-Baer). In this case $R$ is a skew
quasi-Armendariz ring.

**Corollary 3.12.** [9, Corollary 2.8] A ring $R$ is quasi-Baer (respectively,
right p.q.-Baer) if and only if $R[x]$ is quasi-Baer (respectively, right p.q.-
Baer).
Corollary 3.13. [20, Theorems 12, 15] Let $R$ be an $\alpha$-rigid ring. Then $R$ is quasi-Baer (respectively, right p.q.-Baer) if and only if $R[x;\alpha,\delta]$ is quasi-Baer (respectively, right p.q.-Baer).

The following example shows that “$(\alpha,\delta)$-compatibility condition” on $M_R$ in Theorem 3.9 is not superfluous.

Example 3.14. [5, Example 11] There is a ring $R$ and a derivation $\delta$ of $R$ such that $R[x;\delta]$ is a Baer (hence quasi-Baer) ring, but $R$ is not quasi-Baer. In fact let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation $\delta$ such that $\delta(t) = 1$ where $t = t + (t^2)$ in $R$ and $\mathbb{Z}_2[t]$ is the polynomial ring over the field $\mathbb{Z}_2$ of two elements. Consider the Ore extension $R[x;\delta]$. If we set $e_{11} = tx, e_{12} = \bar{t}, e_{21} = \bar{tx}^2 + x$, and $e_{22} = 1 + \bar{tx}$ in $R[x;\delta]$, then they form a system of matrix units in $R[x;\delta]$. Now the centralizer of these matrix units in $R[x;\delta]$ is $\mathbb{Z}_2[x^2]$. Therefore $R[x;\delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So the ring $R[x;\delta]$ is a Baer ring, but $R$ is not quasi-Baer.

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References

On skew Armendariz and skew quasi-Armendariz modules


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