EPI-RETRACTABLE MODULES AND SOME APPLICATIONS

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Abstract. Generalizing concepts “right Bezout” and “principal right ideal” of a ring $R$ to modules, an $R$-module $M$ is called $n$-epi-retractable (resp. epi-retractable) if every $n$-generated submodule (resp. submodule) of $M_R$ is a homomorphic image of $M$. It is shown that if $M_R$ is finitely generated quasi-projective 1-epi-retractable, then $\text{End}_R(M)$ is a right Bezout (resp. principal right ideal) ring if and only if $M_R$ is $n$-epi-retractable for all $n \geq 1$ (resp. epi-retractable). For a ring $R$ and an infinite ordinal $\beta \geq |R|$, the $R$-module $M = F \oplus N$ is epi-retractable where $F$ is a free $R$-module with a basis set of cardinality $\beta$ and $N$ is a $\gamma$-generated $R$-module with $\gamma \leq \beta$. A ring $R$ is quasi Frobenius if every injective $R$-module is epi-retractable. Injective modules in $\sigma[M_R]$ are epi-retractable for every $N \in \sigma[M_R]$ if and only if every non-zero factor ring of $S$ is a quasi Frobenius ring where $S$ is an endomorphism ring of a progenerator in $\sigma[M_R]$.

1. Introduction

All rings are associative with unit elements and all modules are unitary right modules. Let $R$ be a ring. The ring $R$ is said to be a principal right ideal ring (pri) if every right ideal of $R$ is principal. Also, $R$ is said to be a right Bezout ring if every finitely generated right ideal of

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$R$ is principal. Generalizing these concepts to modules, an $R$-module $M$ is called epi-retractable (resp. $n$-epi-retractable) if every submodule (resp. $n$-generated submodule) of $M$ is a homomorphic image of $M_R$. Therefore, $R$ is a pri (resp. right Bezout) ring if and only if $R_R$ is epi-retractable (resp. $n$-retractable $\forall n \geq 1$). In [5], morphic modules are introduced and shown to have internal cancellation property by investigating the epi-retractable condition for such modules ([5], Theorem 15). If $M_R$ is $n$-epi-retractable for some $n \geq 1$, then it is retractable (i.e., $\text{Hom}_R(M,N) \neq 0$ for any non-zero submodule $N \leq M_R$). Retractable modules have been investigated by several authors; see for example, [2], [3], [6], [8]. Here, we reveal some applications of projective, nonsingular, injective epi-retractable modules regarding the characterization of Bezout, pri, quasi Frobenius, rings. Examples are also given where ($n$)-epi-retractable modules appear. A brief description of the content of the paper will now follow.

In Section 2, it is proved in Theorem 2.2 that if $M$ is a non-zero finitely generated quasi-projective 1-epi-retractable $R$-module, then:

(i) $\text{End}_R(M)$ is a right Bezout ring if and only if $M_R$ is $n$-epi-retractable, for all $n \geq 1$.

(ii) $\text{End}_R(M)$ is a pri ring if and only if $M_R$ is epi-retractable.

In particular, the matrix ring $\text{Mat}_{m \times m}(R)$ is pri (resp. right Bezout) if and only if $R_R^{(m)}$ is epi-retractable (resp. $R_R^{(m)}$ is $n$-epi-retractable $\forall n \geq m$).

Projective epi-retractable modules are investigated and it is shown that for a ring $R$ and an infinite ordinal $\beta \geq |R|$, the $R$-module $M = F \oplus N$ is epi-retractable where $F$ is a free $R$-module with a basic set of cardinality $\beta$ and $N$ is a $\gamma$-generated $R$-module with $\gamma \leq \beta$ (Theorem 2.8). Over a ring in which principal right ideals are projective, finite dimensional torsionfree epi-retractable modules are characterized and shown to be projective. It also shown that a ring $R$ is a pri domain if and only if $R_R$ is uniform and there exists a uniform nonsingular epi-retractable $R$-module (Proposition 2.16). If every injective $R$-module is epi-retractable, then $R$ is a quasi Frobenius ring (Proposition 3.2). We end up with Theorem 3.5 that states: For any module $M_R$ with a progenerator $P \in \sigma[M_R]$, any non-zero factor ring of $\text{End}_R(P)$ is a quasi Frobenius ring if and only if for any $N \in \sigma[M_R]$, every injective module in $\sigma[N_R]$ is epi-retractable. Recall that a ring $R$ is said to be a quasi Frobenius ring if it is a (left) right self injective Noetherian ring. Any unexplained terminology, and
all the basic results on rings and modules that are used in the sequel can be found in [4] and [7].

2. Epi-Retractable condition for projective modules

We begin with the following observation.

Proposition 2.1. Let $M$ be a non-zero quasi-projective 1-epi-retractable $R$-module with $\text{End}_R(M) = S$. If $S$ is an $n$-epi-retractable right $S$-module (resp. a pri ring), then $M_R$ is $n$-epi-retractable (resp. epi-retractable).

Proof. Let $X = x_1R + \cdots + x_nR$ be an $n$-generated submodule of $M$. Then, by hypothesis, $M$ is $(x_1R \oplus \cdots \oplus x_nR)$-projective. Thus, for every $h \in \text{Hom}_R(M, X)$, there exists $\bar{h} : M \rightarrow (x_1R \oplus \cdots \oplus x_nR)$ such that $h = \mu \bar{h}$ where $\mu : (x_1R \oplus \cdots \oplus x_nR) \rightarrow X$ is the natural surjective homomorphism. It follows that $\text{Hom}_R(M, X) = \sum_{i=1}^n \text{Hom}_R(M, x_iR)$. Now, since $M_R$ is 1-epi-retractable, then each $x_iR$ is equal to $f_i(M)$ for some $f_i \in S$ and we have $\text{Hom}_R(M, x_iR) = f_iS$ ($1 \leq i \leq n$), by the quasi-projective condition on $M_R$. Hence, by the hypothesis on $S$, $\text{Hom}_R(M, X) = \sum_{i=1}^n f_iS = gS$ for some $g \in S$. We now show that $X = g(M)$. Clearly, $g(M) \subseteq X$. Let $x$ be an element in $X$. Then, there exists $\theta \in S$ such that $\theta(M) = xR$ and we have $\theta S = \text{Hom}_R(M, \theta(M)) \subseteq \text{Hom}_R(M, X) = gS$. Hence, $\theta = gt$ for some $t \in S$ and so $\theta(M) \subseteq g(M)$. It follows that $x \in g(M)$. Consequently, $X = g(M)$ is a homomorphic image of $M_R$, proving that $M_R$ is $n$-epi-retractable. The other case of the result is proved similarly. \qed

Theorem 2.2. If $M$ is a non-zero quasi-projective 1-epi-retractable finitely generated $R$-module, then:

(i) $\text{End}_R(M)$ is a right Bezout ring if and only if $M_R$ is $n$-epi-retractable, $\forall n \geq 1$.

(ii) $\text{End}_R(M)$ is a pri ring if and only if $M_R$ is epi-retractable.

Proof. We prove (i). One direction follows from Proposition 2.1. Conversely, set $\text{End}_R(M) = S$ and let $I$ be a finitely generated right ideal of $S$. Because $M_R$ is finitely generated, then $IM$ is a finitely generated $R$-submodule of $M$. Since $M_R$ is $n$-epi-retractable for all $n \geq 1$, then
we must have \( IM = f(M) \) for some \( f \in S \). Hence, by hypothesis, 
\[
    fS = \text{Hom}_R(M, f(M)) = I.
\]

**Corollary 2.3.** Let \( m \) be a positive integer. Then, the following statements are equivalent on a ring \( R \).

(i) \( R^{(m)}_R \) is epi-retractable (resp. \( R^{(m)}_R \) is \( n \)-epi-retractable \( \forall n \geq m \)).

(ii) The matrix ring \( \text{Mat}_{m \times m}(R) \) is pri (resp. right Bezout).

**Proof.** Apply Theorem 2.2 for \( M = R^{(m)} \), and note that \( M_R \) is a \( k \)-epi-retractable \( R \)-module for any \( k < m \).

**Examples 2.4.** (1) Let \( I \) be a right ideal in a regular ring \( R \). Then \( I_R \) is \( n \)-epi-retractable for every \( n \geq 1 \). In fact, if \( I \) contains a finitely generated right ideal \( J \) of \( R \), then \( J \) is a direct summand of \( I_R \). It follows that \( J \) is a homomorphic image of \( I_R \).

(2) For any non-zero \( R \)-module \( X \), the \( R \)-module \( M = R/\text{ann}_R(X) \oplus X \) is 1-epi-retractable. Note that for any \( m \in M \), we have \( \text{ann}_R(X) \subseteq \text{ann}_R(m) \). Hence, \( mR \simeq R/\text{ann}_R(m) \) is a homomorphic image of \( R/\text{ann}_R(X) \) and so of \( M_R \).

(3) Over a PID, every finitely generated module is epi-retractable. To see this, let \( R \) be a PID and let \( M \) be a finitely generated \( R \)-module, and \( N \leq M \). Then, by a well known result, \( M \simeq (\bigoplus_{i=1}^n R_i) \oplus (\bigoplus_{i=1}^m R/(p^i_k)) \) and \( N \simeq (\bigoplus_{i=1}^t R_i) \oplus (\bigoplus_{s=1}^r R/(p^s_i)) \), where \( R_i = R, 0 \leq t \leq n, 0 \leq s \leq m \), and \( 0 \leq s_i \leq k_i \). From this, we see that \( N \) is a homomorphic image of \( M_R \).

We shall now investigate when a projective module is epi-retractable and vice versa.

**Proposition 2.5.** Let \( R \) be a right hereditary ring. Then, \( R \) is a pri ring if and only if every free right \( R \)-module is epi-retractable.

**Proof.** \((\Leftarrow)\). This is obtained from the definitions.

\((\Rightarrow)\). By Kaplansky’s Theorem ([4], Theorem 2.24), any submodule \( N \) of a free right \( R \)-module \( F = \bigoplus_{\alpha \in \Omega} e_\alpha R \), is isomorphic to \( \bigoplus_{\alpha \in \Omega} J_\alpha \), where the \( J_\alpha \) are right ideals of \( R \). Thus, the result is proved by the fact that if \( f_\alpha : e_\alpha R \rightarrow J_\alpha \) is surjective \( R \)-homomorphism for all \( \alpha \in \Omega \), then the homomorphism \( \bigoplus_{\alpha \in \Omega} f_\alpha \) is also surjective. \(\square\)
The following Lemmas are needed.

Lemma 2.6. Let $R$ be a ring of cardinality $|R| = \alpha$. Then any free $R$-module $F$ with an infinite basis set $X$ of cardinality $\beta \geq \alpha$ is an epi-retractable $R$-module.

Proof. We first show that $|F| = \beta$. Let $X_n = \{\sum_{i=1}^{n} x_i r_i \mid x_i \in X, r_i \in R\}$ where $n \geq 1$. Then, there naturally exist a surjective map $Y_n := (X \times R)^{(n)} \to X_n$ and an injective map $X \to X_n$. Consequently, $|X| \leq |X_n| \leq |Y_n| = \beta$ for any $n \geq 1$. It follows from $F = \bigcup_{n \in \mathbb{N}} X_n$ that $|F| = \beta$, as desired. Now, any submodule $N$ of $F_R$ is a homomorphic image of a free $R$-module $G$ with a basic set of cardinality $\gamma \leq \beta$. Thus, $G$ and hence $N$ is a homomorphic image of $F_R$, proving that $F_R$ is epi-retractable.

Lemma 2.7. The following statements are equivalent for a module $M$.

(i) $M$ is epi-retractable.

(ii) There exist surjective homomorphisms $M \to N$ and $N \to M$ for some epi-retractable module $N$.

(iii) There exists a surjective homomorphism $M/K \to M$ for some epi-retractable factor module $M/K$.

Proof. (i)⇒(ii). This is clear.

(ii)⇒(iii). Suppose that there exist an epi-retractable module $N$ and surjective homomorphisms $\alpha : M \to N$, $\beta : N \to M$. Let $K = \ker \alpha$. Then, $\alpha$ induces an isomorphism $\tilde{\alpha} : M/K \to N$. Thus, $M/K$ is an epi-retractable module.

(iii)⇒(i). Let $L$ be any submodule of $M$. By our assumption, there exists an isomorphism $\tilde{\phi} : M/K' \to M$ for some submodule $K'$ of $M$ with $K \subseteq K'$. Let $\tilde{\phi}(N/K') = L$ for some submodule $N$ of $M$. Since $M/K$ is assumed epi-retractable, then there exists a surjective homomorphism $\theta : M/K \to N/K$. Consider $\alpha : N/K \to N/K'$ with $\alpha(n + K) = n + K'$, and the canonical epimorphism $\pi : M \to M/K$. Then, $\tilde{\phi} \alpha \theta \pi : M \to L$ is a surjective homomorphism, proving that $M$ is epi-retractable.

Theorem 2.8. Let $R$ be a ring and $\beta$ be an infinite ordinal $\geq |R|$. Suppose that $M = F \oplus N$ where $F$ is a free $R$-module with a basic set of
cardinality $\beta$ and $N$ is a $\gamma$-generated $R$-module with $\gamma \leq \beta$. Then, $M_R$ is epi-retractable.

**Proof.** By Lemma 2.6, $F_R$ is epi-retractable. By hypothesis, $N$ is a homomorphic image of $F_R$. Since $F \oplus F \simeq F$, there exist surjective homomorphisms $M \to F$ and $F \to M$. Thus, the result holds by Lemma 2.7.

**Corollary 2.9.** Let $R$ be an infinite ring. Then, every free $R$-module with a basic set of cardinality $\geq |R|$ is epi-retractable.

**Proof.** It follows from Theorem 2.8.

**Proposition 2.10.** Let $R$ be a countable semiprime pri ring. Then, any free $R$-module is epi-retractable.

**Proof.** In view of Theorem 2.8, we need to show that any finitely generated free $R$-module is epi-retractable. Because $R$ is a pri semiprime ring, then by a well known result $R = \bigoplus_{i=1}^{t} S_i$ is a finite product of prime pri rings. Hence, $\text{Mat}_{n \times n}(R) \simeq \bigoplus_{i=1}^{t} \text{Mat}_{n \times n}(S_i)$ and each $\text{Mat}_{n \times n}(S_i)$ is a pri ring; see [1]. The result now follows from Corollary 2.3.

We now investigate when a direct summand of an epi-retractable module is epi-retractable.

**Proposition 2.11.** Let $M$ be an epi-retractable $R$-module. Then:

(i) $M/N$ is epi-retractable for any fully invariant submodule $N$ of $M_R$.

(ii) If $M = L \oplus N$ such that $\text{Hom}_R(L, N) = 0$, then $N_R$ is epi-retractable.

**Proof.** (i) Let $N$ be a fully invariant submodule of $M$, and let $K/N$ be any submodule of $M/N$. There is a surjective homomorphism $\varphi : M \to K$. Now, $\varphi(N) \subseteq N$ by our assumption, and so $\bar{\varphi} : M/N \to K/N$, with $\bar{\varphi}(m + N) = \varphi(m) + N$ is a surjective homomorphism.

(ii) Note that $\text{End}_R(M) = \begin{bmatrix} \text{End}_R(L) & \text{Hom}_R(N, L) \\ 0 & \text{End}_R(N) \end{bmatrix}$. Hence, $\text{End}_R(M) \begin{bmatrix} L \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} L \\ 0 \end{bmatrix}$. It follows that $(L \oplus 0)$ is a fully invariant submodule of $M_R$. Now, apply (i).
Remark 2.12. Let $G$ be a free $\mathbb{Z}$-module with an infinite countable basic set and $X$ be any countable $\mathbb{Z}$-module which is not epi-retractable (e.g., $X = \mathbb{Q}$). Then, the $\mathbb{Z}$-module $M = X \oplus G$ is epi-retractable by Theorem 2.8. This shows that a direct summand (and hence a submodule or a factor module) of an epi-retractable module need not be epi-retractable.

We are now going to investigate when an epi-retractable module is projective. A ring $R$ is called right Rickart or right principally projective if every principal right ideal in $R$ is projective (as a right $R$-module). Semi-hereditary rings and domains are clearly Rickart. A module $M$ has a finite uniform dimension (or finite rank) if $M$ contains no infinite direct sum of non-zero submodules or equivalently there exist independent uniform submodules $U_1, \ldots, U_n$ in $M$ such that $\bigoplus_{i=1}^n U_i$ is an essential submodule of $M$. In this case, it is written $u\dim(M) = n$.

Proposition 2.13. Let $R$ be a right Rickart ring. Then, every finite dimensional nonsingular epi-retractable $R$-module $M$ is isomorphic to a direct sum of principal right ideals of $R$. In particular, $M_R$ is projective.

Proof. Let $m$ be any non-zero element of $M$. Since $M$ is nonsingular, then the right annihilator $m$ in $R$ is not an essential right ideal. Let $B = r.\text{ann}_R(m)$ and $B \cap A = 0$ for some non-zero principal right ideal $A$ of $R$. Thus, $mA \simeq A$. It follows that every non-zero submodule of $M_R$ isomorphically contains a non-zero principal right ideal of $R$. Now, let $\mathcal{P}$ be a maximal independent family of elements in the set $\{N \leq M_R \mid N$ is isomorphic to a principal right ideal of $R\}$ and let $L = \bigoplus \mathcal{P}$. Then, $L$ is an essential submodule of $M_R$. By the epi-retractable condition on $M_R$, there exists a surjective homomorphism from $M$ to $L$. Also, by hypothesis on $R$, $L_R$ is projective. Hence, $L$ is isomorphic to a direct summand $K$ of $M_R$. Now, $u\dim(K) = u\dim(L) = u\dim(M)$. It follows that $L \simeq K = M$, as desired. \qed

Corollary 2.14. Let $R$ be a right and left Ore domain. Then, every finitely generated torsion free epi-retractable $R$-module is a free $R$-module.

Proof. This is obtained by Proposition 2.13 and the well known result from Gentile and Levy which states that over a semiprime right and left Goldie ring, finitely generated torsion free modules have finite ranks. \qed
We end this section with an application of nonsingular epi-retractable modules.

**Lemma 2.15.** A non-zero module $M_R$ is uniform nonsingular epi-retractable if and only if $Z(M) \neq M$ and $M \simeq N$, for all non-zero submodules $N$ of $M$.

**Proof.** ($\Rightarrow$). We have $Z(M) \neq M$ and for each non-zero submodule $N$ of $M$, there exists a surjective homomorphism $f : M \rightarrow N$, and so $M/\ker f \hookrightarrow M$. Since $M_R$ is nonsingular, then $M/\ker f$ is nonsingular and hence $\ker f = 0$, because $M$ is uniform. Thus, $M \simeq N$.

($\Leftarrow$) By hypothesis, $M_R$ is epi-retractable and if $Z(M) \neq 0$, then we must have $M \simeq Z(M)$. Consequently, $M = Z(M)$, which is a contradiction. Therefore $Z(M) = 0$. Also, for all $0 \neq x \in M$, $M \simeq xR$, and so $M_R$ is Noetherian. Therefore, $M_R$ has a uniform submodule $U$ ([4], Proposition 6.4). It follows that $M \simeq U$ is uniform. □

**Proposition 2.16.** The following statement are equivalent for a ring $R$.

(i) $R$ is a principal right ideal domain.

(ii) $R_R$ is uniform and there exists a uniform nonsingular epi-retractable $R$-module.

**Proof.** (i)$\Rightarrow$(ii). Apply Lemma 2.15 for $M = R$.

(ii)$\Rightarrow$(i). Let $M$ be a uniform nonsingular epi-retractable $R$-module. Since $R_R$ is uniform and $Z(M) \neq M$, then there exists $x \in M$ such that $\ann_R(x) = 0$. Thus, $R$ can be embedded in $M_R$ and hence $M \simeq R$, by Lemma 2.15. It follows that $R$ is a right nonsingular principal right ideal ring with uniform dimension 1. Now, if $ab = 0$ and $0 \neq b$ for some $a, b \in R$, then $0 \neq r.\ann_R(a)$ is an essential right ideal of $R$, which implies that $a \in Z(R_R) = 0$. The proof is now complete. □

3. Epi-Retractable condition for injective objects

By a class $C$ of $R$-modules we mean a collection $C$ of $R$-modules which contains a non-zero module and which is closed under taking isomorphisms. Let $C$ be a class of modules. A module $X \in C$ is called an injective module in $C$, if every exact sequence $0 \rightarrow X \rightarrow A \rightarrow B \rightarrow 0$ with $A, B \in C$ splits. We denote by $\sigma[M_R]$, the full subcategory of mod-$R$ whose objects are all $R$-submodules of $M$-generated modules. In this
section, we observe that every non-zero factor ring of a ring \( R \) is a quasi Frobenius ring if and only if for any \( R \)-module \( N \), in the class \( \sigma[N_R] \), injective \( R \)-modules are epi-retractable. It is well known that injective modules are continuous. We first record that continuous nonsingular epi-retractable modules are semisimple. Recall that an \( R \)-module \( M \) is said to be a continuous module if it satisfies the following conditions:

\((C_1)\) Every submodule of \( M \) is essential inside a direct summand of \( M_R \).

\((C_2)\) Every submodule of \( M \) that is isomorphic to a summand of \( M \) is itself a summand of \( M_R \).

**Proposition 3.1.** The following are equivalent for a nonsingular \( R \)-module \( M \).

(i) \( M_R \) is semisimple.

(ii) \( M_R \) is continuous epi-retractable.

**Proof.** (i) \( \Rightarrow \) (ii). This is clear.

(ii) \( \Rightarrow \) (i). Let \( N \) be a submodule of \( M_R \). By assumption, there exists an \( R \)-epimorphism \( f \) from \( M \) to \( N \). Because \( N \cong M/\ker f \) is nonsingular, then \( \ker f \) is an essentially closed submodule of \( M_R \). Then, by the \( C_1 \)-condition on \( M_R \), \( \ker f \) is a direct summand of \( M_R \). It follows that \( N \) is isomorphic to a direct summand of \( M_R \). Now, the \( C_2 \)-condition implies that \( N \) is a direct summand of \( M_R \), proving that \( M_R \) is semisimple. \( \square \)

**Proposition 3.2.** If \( R \) is a ring such that every injective \( R \)-module is epi-retractable, then \( R \) is a quasi Frobenius ring.

**Proof.** By Remark 15.10 in [4], we need to show that every projective \( R \)-module is injective. Now, let \( X \) be a projective \( R \)-module and let \( E \) be the injective hull of \( X_R \). Then, by our assumption, there exists a surjective \( R \)-homomorphism \( f \) from \( E \) to \( X \). Because \( X_R \) is projective, then the \( \ker f \) is a direct summand of \( E \). It follows that \( X = E \), as desired. \( \square \)

It is known that if \( R \) is a ring such that every non-zero factor ring of \( R \) is a quasi Frobenius ring, then every \( R \)-module is a direct sum of homo-uniserial modules. An \( R \)-module \( M \) is called homo-uniserial if for any non-zero finitely generated submodules \( K, L \subseteq M \), the factor modules \( K/J(K) \) and \( L/J(L) \) are simple and isomorphic. In particular, if \( M \neq \)
J(M) and Soc(M) \neq 0, then M_R is finitely generated and M/J(M) \cong Soc(M); see [7, 56].

**Theorem 3.3.** The following are equivalent on a ring R.

(i) Every non-zero factor ring of R is a quasi Frobenius ring.

(ii) For any R-module N, in the class \( \sigma[N_R] \), injective R-modules are epi-retractable.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that N is a non-zero R-module and set \( B = \text{ann}_R(N) \). Because R is an Artinian ring by (i), then it is well known that there are \( x_1, \ldots, x_t \in N \) such that \( B = \bigcap_{i=1}^t \text{ann}_R(x_i) \). Consequently, \( R/B \) embeds in \( N^{(0)} \). It follows that \( \sigma[N] = \text{Mod}-R/B \). Hence, it is enough to show that every injective \( R/B \)-module is epi-retractable. By hypothesis, if \( W \) is an \( R/B \)-module, then \( W = \bigoplus_{\lambda \in \Lambda} U_\lambda \) is a direct sum of homo-uniserial modules. Because \( R/B \) is an Artinian ring, then it is easy to verify that \( J(U_\lambda) \neq U_\lambda \) for each \( \lambda \in \Lambda \). Thus, by the above remarks \( W/J(W) \cong \text{Soc}(W) \). Now, let \( E \) be any injective \( R/B \)-module and \( Y \) be a submodule of \( E \). Because \( \text{Soc}(Y) \) is a direct summand of \( \text{Soc}(E) \), then there exists a surjective homomorphism \( f \) from \( E \) to \( Y/J(Y) \). On the other hand, the injective \( R/B \)-module \( E \) is also projective as an \( R/B \)-module. Hence, there exists a homomorphism \( g : E \to Y \) such that \( \pi g = f \) where \( \pi : Y \to Y/J(Y) \) is the canonical projection. Because \( \pi g \) is a surjective homomorphism, then we can conclude that \( g(E) + J(Y) = Y \). But, by hypothesis, \( J(Y) \) is a small submodule of \( Y \), and hence \( g(E) = Y \). This shows that \( E \) is an epi-retractable \( R/B \)-module, as desired.

(ii) \( \Rightarrow \) (i). Let \( A \) be a proper ideal of \( R \) and set \( N = R/A \). Then, \( \sigma[N_R] = \text{Mod}-R/A \). The result follows now Proposition 3.2.

**Lemma 3.4.** Being epi-retractable is a Morita invariant property.

**Proof.** In fact, a module \( M_R \) is epi-retractable if and only if for any \( X \in \text{Mod}-R \) with an injective homomorphism \( X_R \to M_R \) there exists a surjective homomorphism \( M_R \to X_R \). Thus, the result follows from the fact that any category equivalence preserves injective and surjective homomorphisms.

**Theorem 3.5.** Let \( M \) be an \( R \)-module and let \( S \) be the endomorphism ring of a progenerator in \( \sigma[M_R] \). Then, the following statements are equivalent.


(i) For any $N \in \sigma[M_R]$, in the class $\sigma[N_R]$, injective $R$-modules are epi-retractable.
(ii) Every non-zero factor ring of $S$ is a quasi Frobenius ring.

Proof. By Theorem 46.2 in [7], there exists a category equivalence between $\sigma[M_R]$ and $\text{Mod-}S$. Hence, for any $X \in \text{Mod-}S$, the class $\sigma[X_S]$ corresponds to the class $\sigma[N_R]$ for some suitable $N \in \sigma[M_R]$ and vice versa. The result is then obtained by Lemma 3.4 and Theorem 3.3. □

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