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EMBEDDING NORMED LINEAR SPACES INTO $C(X)$

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Abstract. It is well known that every (real or complex) normed linear space L is isometrically embeddable into $C(X)$ for some compact Hausdorff space X. Here X is the closed unit ball of L^* (the set of all continuous scalar-valued linear mappings on L) endowed with the weak* topology, which is compact by the Banach–Alaoglu theorem. We prove that the compact Hausdorff space X can indeed be chosen to be the Stone–Čech compactification of $L^* \setminus \{0\}$, where $L^* \setminus \{0\}$ is endowed with the supremum norm topology.
Keywords: Stone–Čech compactification, Banach–Alaoglu theorem, embedding theorem.
MSC(2010): Primary: 46A50; Secondary: 54C35, 54D35, 46B20, 46B50, 46E15.

1. Introduction

Throughout this note by a *space* we will mean a topological space, unless we explicitly state otherwise. The field of scalars (which is fixed throughout discussion) is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and is denoted by $\mathbb{F}$.

For a compact Hausdorff space X, we denote by $C(X)$ the set of all continuous scalar-valued mappings on X. The set $C(X)$ is a normed linear space when equipped with the supremum norm and pointwise addition and scalar multiplication.

It is known that every (real or complex) normed linear space L can be isometrically embedded into $C(X)$ for some compact Hausdorff space X. Here X is the closed unit ball of L^* (the set of all continuous scalar-valued linear mappings on L) endowed with the weak* topology, which is known to be compact by the Banach–Alaoglu theorem. In this note we give a new proof of this well known fact, with X being chosen as the Stone–Čech compactification of

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$L^* \setminus \{0\}$, where $L^* \setminus \{0\}$ is endowed with the supremum norm topology. Our proof is rather topological and makes use of some elementary properties of the Stone–Čech compactification. We conclude with a result which provides an upper bound for the density of X in terms of the density of L.

Recall that a *compactification* of a completely regular space X is a compact Hausdorff space which contains X as a dense subspace. The *Stone–Čech compactification* of a completely regular space X, denoted by βX, is the (unique) compactification of X which is characterized among all compactifications of X by the fact that every continuous bounded mapping $f : X \to \mathbb{F}$ is extendable to a continuous mapping $F : \beta X \to \mathbb{F}$. The Stone–Čech compactification of a completely regular space always exists. For more information on the theory of the Stone–Čech compactification see [4], [5], or [7].

The Stone–Čech compactification was introduced independently by M.H. Stone [8] and E. Čech [3] in 1937, developing an idea of A. Tychonoff [9] (used in the proof of his celebrated result nowadays referred to as the *Tychonoff theorem*). The Banach–Alaoglu theorem was proved by L. Alaoglu [1] in 1940, as a consequence of the Tychonoff theorem; though, a proof of this theorem for separable normed linear spaces had been already published in 1932 by S. Banach [2]. For an interesting proof of the Banach–Alaoglu theorem assuming the existence of the Stone–Čech compactification see the recent paper [6] by H. Hosseini Giv.

2. **The embedding theorem**

Here we prove our embedding theorem. The proof uses only some basic facts from the theory of the Stone–Čech compactification besides an appeal to the Hahn–Banach theorem.

Theorem 2.1. *Let L be a normed linear space. Then L can be isometrically embedded into $C(Y)$ for a compact Hausdorff space Y, namely, for*

$$Y = \beta\big(L^* \setminus \{0\}\big),$$

where $L^* \setminus \{0\}$ is endowed with the supremum norm topology.

Proof. Let $x \in L$. Define

$$\theta_x : L^* \setminus \{0\} \longrightarrow \mathbb{F}$$

by

$$\theta_x(x^*) = \frac{x^*(x)}{\|x^*\|}.$$

It is clear that the mapping θ_x is continuous, when $L^* \setminus \{0\}$ is endowed with the supremum norm topology. We verify that θ_x is bounded. For this purpose we indeed show that

$$\|\theta_x\| = \|x\|. \tag{2.1}$$

Note that
$$\big|\theta_x(x^*)\big| = \frac{|x^*(x)|}{\|x^*\|} \leq \frac{\|x^*\|\|x\|}{\|x^*\|} = \|x\|$$
for any $x^* \in L^* \setminus \{0\}$. Thus $\|\theta_x\| \leq \|x\|$. On the other hand, by the Hahn–Banach theorem, there exists some $z^* \in L^*$ such that $\|z^*\| = 1$ and $|z^*(x)| = \|x\|$. Therefore
$$\big|\theta_x(z^*)\big| = \frac{|z^*(x)|}{\|z^*\|} = \|x\|,$$
which implies that $\|x\| \leq \|\theta_x\|$. This shows (2.1). Observe that the mapping θ_x, being continuous and bounded, can be extended to the continuous mapping
$$\Theta_x : \beta\big(L^* \setminus \{0\}\big) \longrightarrow \mathbb{F}.$$
Define
$$\Theta : L \longrightarrow C\big(\beta\big(L^* \setminus \{0\}\big)\big)$$
such that
$$x \longmapsto \Theta_x.$$
We show that Θ embeds L isometrically into $C(\beta(L^*\setminus\{0\}))$, that is, Θ preserves addition, scalar multiplication and norm. Note that
$$\theta_{x+z} = \theta_x + \theta_z \tag{2.2}$$
by definition. Thus
$$\Theta(x+z) = \Theta(x) + \Theta(z),$$
as $\Theta(x+z)$ and $\Theta(x) + \Theta(z)$ are continuous and agree on the dense subspace $L^* \setminus \{0\}$ of $\beta(L^* \setminus \{0\})$ by (2.2); indeed
$$\Theta_{x+z}|_{L^*\setminus\{0\}} = \theta_{x+z} \quad \text{and} \quad (\Theta_x + \Theta_z)|_{L^*\setminus\{0\}} = \theta_x + \theta_z.$$
Similarly, we can show that
$$\Theta(\alpha x) = \alpha\Theta(x).$$
To conclude the proof we need to show that Θ preserves norm. It is clear that
$$\|\theta_x\| \leq \|\Theta_x\|,$$
as Θ_x extends θ_x. Also
$$\|\Theta_x\| \leq \|\theta_x\|,$$
since
$$\begin{aligned} |\Theta_x|\big(\beta\big(L^* \setminus \{0\}\big)\big) &= |\Theta_x|\big(\overline{L^* \setminus \{0\}}\big) \\ &\subseteq \overline{|\Theta_x|\big(L^* \setminus \{0\}\big)} = \overline{|\theta_x|\big(L^* \setminus \{0\}\big)} \subseteq \big[0, \|\theta_x\|\big], \end{aligned}$$
that is
$$\|\Theta_x\| = \|\theta_x\|.$$
This, together with (2.1), proves that
$$\big\|\Theta(x)\big\| = \|x\|.$$

□

The following corollary should be known. We derive it here, however, as an immediate consequence of the construction given in Theorem 2.1.

Recall that the *density* of a space X, denoted by $\mathrm{d}(X)$, is the minimum cardinality of a dense subset of X; more precisely

$$\mathrm{d}(X) = \min\big\{|D| : D \text{ is dense in } X\big\}.$$

In particular, a space X is separable if and only if $\mathrm{d}(X) \leq \aleph_0$. It is clear that the density of a space is always bounded by its cardinality.

Corollary 2.2. *A non-zero normed linear space L can be isometrically embedded into $C(Y)$ for a compact Hausdorff space Y of density at most $2^{\,\mathrm{d}(L)}$.*

Proof. Observe that

$$\mathrm{d}\big(\beta\big(L^* \setminus \{0\}\big)\big) \leq \mathrm{d}\big(L^* \setminus \{0\}\big),$$

as any dense subset of $L^* \setminus \{0\}$ is also dense in $\beta(L^* \setminus \{0\})$, since $L^* \setminus \{0\}$ is dense in $\beta(L^* \setminus \{0\})$. Let D be a dense subset of L of minimum cardinality. Note that D is infinite, since L is a non-zero linear space. Note that

$$|L^*| = \big|\{x^*|_D : x^* \in L^*\}\big|,$$

as any continuous scalar-valued mapping on L is determined by its value on the dense subset D of L. We have

$$\mathrm{d}\big(L^* \setminus \{0\}\big) \leq \big|L^* \setminus \{0\}\big| \leq |L^*| \leq |\mathbb{R}^D| = |\mathbb{R}|^{|D|} = 2^{\,|D|} = 2^{\,\mathrm{d}(L)}.$$

Theorem 2.1 now concludes the proof. □

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