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Embedding normed linear spaces into $C(X)$

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EMBEDDING NORMED LINEAR SPACES INTO $C(X)$

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Abstract. It is well known that every (real or complex) normed linear space $L$ is isometrically embeddable into $C(X)$ for some compact Hausdorff space $X$. Here $X$ is the closed unit ball of $L^*$ (the set of all continuous scalar-valued linear mappings on $L$) endowed with the weak* topology, which is compact by the Banach–Alaoglu theorem. We prove that the compact Hausdorff space $X$ can indeed be chosen to be the Stone–Čech compactification of $L^* \setminus \{0\}$, where $L^* \setminus \{0\}$ is endowed with the supremum norm topology.

Keywords: Stone–Čech compactification, Banach–Alaoglu theorem, embedding theorem.


1. Introduction

Throughout this note by a space we will mean a topological space, unless we explicitly state otherwise. The field of scalars (which is fixed throughout discussion) is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and is denoted by $\mathbb{F}$.

For a compact Hausdorff space $X$, we denote by $C(X)$ the set of all continuous scalar-valued mappings on $X$. The set $C(X)$ is a normed linear space when equipped with the supremum norm and pointwise addition and scalar multiplication.

It is known that every (real or complex) normed linear space $L$ can be isometrically embedded into $C(X)$ for some compact Hausdorff space $X$. Here $X$ is the closed unit ball of $L^*$ (the set of all continuous scalar-valued linear mappings on $L$) endowed with the weak* topology, which is known to be compact by the Banach–Alaoglu theorem. In this note we give a new proof of this well known fact, with $X$ being chosen as the Stone–Čech compactification of

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$L^* \setminus \{0\}$, where $L^* \setminus \{0\}$ is endowed with the supremum norm topology. Our proof is rather topological and makes use of some elementary properties of the Stone–Čech compactification. We conclude with a result which provides an upper bound for the density of $X$ in terms of the density of $L$.

Recall that a compactification of a completely regular space $X$ is a compact Hausdorff space which contains $X$ as a dense subspace. The Stone–Čech compactification of a completely regular space $X$, denoted by $\beta X$, is the (unique) compactification of $X$ which is characterized among all compactifications of $X$ by the fact that every continuous bounded mapping $f : X \to F$ is extendable to a continuous mapping $F : \beta X \to F$. The Stone–Čech compactification of a completely regular space always exists. For more information on the theory of the Stone–Čech compactification see [4], [5], or [7].


2. The embedding theorem

Here we prove our embedding theorem. The proof uses only some basic facts from the theory of the Stone–Čech compactification besides an appeal to the Hahn–Banach theorem.

**Theorem 2.1.** Let $L$ be a normed linear space. Then $L$ can be isometrically embedded into $C(Y)$ for a compact Hausdorff space $Y$, namely, for

$$Y = \beta(L^* \setminus \{0\}),$$

where $L^* \setminus \{0\}$ is endowed with the supremum norm topology.

**Proof.** Let $x \in L$. Define

$$\theta_x : L^* \setminus \{0\} \to F$$

by

$$\theta_x(x^*) = \frac{x^*(x)}{\|x^*\|}.$$  

It is clear that the mapping $\theta_x$ is continuous, when $L^* \setminus \{0\}$ is endowed with the supremum norm topology. We verify that $\theta_x$ is bounded. For this purpose we indeed show that

$$\|\theta_x\| = \|x\|. \quad (2.1)$$
Note that
\[ |\theta_x(x^*)| = \frac{|x^*(x)|}{\|x^*\|} \leq \frac{\|x^*\| \|x\|}{\|x^*\|} = \|x\| \]
for any \( x^* \in L^* \setminus \{0\} \). Thus \( \|\theta_x\| \leq \|x\| \). On the other hand, by the Hahn–Banach theorem, there exists some \( z^* \in L^* \) such that \( \|z^*\| = 1 \) and \( |z^*(x)| = \|x\| \). Therefore
\[ |\theta_x(z^*)| = \frac{|z^*(x)|}{\|z^*\|} = \|x\|, \]
which implies that \( \|x\| \leq \|\theta_x\| \). This shows (2.1). Observe that the mapping \( \theta_x \), being continuous and bounded, can be extended to the continuous mapping \( \Theta_x : \beta(L^* \setminus \{0\}) \to F \).

Define
\[ \Theta : L \to C(\beta(L^* \setminus \{0\})) \]
such that
\[ x \mapsto \Theta_x. \]
We show that \( \Theta \) embeds \( L \) isometrically into \( C(\beta(L^* \setminus \{0\})) \), that is, \( \Theta \) preserves addition, scalar multiplication and norm. Note that
\[ \theta_x + \theta_z = \theta_x + \theta_z \]
by definition. Thus
\[ \Theta(x + z) = \Theta(x) + \Theta(z), \]
as \( \Theta(x + z) \) and \( \Theta(x) + \Theta(z) \) are continuous and agree on the dense subspace \( L^* \setminus \{0\} \) of \( \beta(L^* \setminus \{0\}) \) by (2.2); indeed
\[ \Theta_{x+z}|_{L^*\setminus\{0\}} = \theta_{x+z} \quad \text{and} \quad (\Theta_x + \Theta_z)|_{L^*\setminus\{0\}} = \theta_x + \theta_z. \]
Similarly, we can show that
\[ \Theta(\alpha x) = \alpha \Theta(x). \]
To conclude the proof we need to show that \( \Theta \) preserves norm. It is clear that
\[ \|\theta_x\| \leq \|\Theta_x\|, \]
as \( \Theta_x \) extends \( \theta_x \). Also
\[ \|\Theta_x\| \leq \|\theta_x\|, \]
since
\[ |\Theta_x|(\beta(L^* \setminus \{0\})) = |\Theta_x|(L^* \setminus \{0\}) \subseteq |\Theta_x|(L^* \setminus \{0\}) = |\Theta_x|(L^* \setminus \{0\}) \subseteq [0, \|\theta_x\|], \]
that is
\[ \|\Theta_x\| = \|\theta_x\|. \]
This, together with (2.1), proves that
\[ \|\Theta(x)\| = \|x\|. \]
The following corollary should be known. We derive it here, however, as an immediate consequence of the construction given in Theorem 2.1.

Recall that the density of a space $X$, denoted by $d(X)$, is the minimum cardinality of a dense subset of $X$; more precisely

$$d(X) = \min \{|D| : D \text{ is dense in } X\}.$$ 

In particular, a space $X$ is separable if and only if $d(X) \leq \aleph_0$. It is clear that the density of a space is always bounded by its cardinality.

**Corollary 2.2.** A non-zero normed linear space $L$ can be isometrically embedded into $C(Y)$ for a compact Hausdorff space $Y$ of density at most $2^{d(L)}$.

**Proof.** Observe that

$$d(\beta(L^* \setminus \{0\})) \leq d(L^* \setminus \{0\}),$$

as any dense subset of $L^* \setminus \{0\}$ is also dense in $\beta(L^* \setminus \{0\})$, since $L^* \setminus \{0\}$ is dense in $\beta(L^* \setminus \{0\})$. Let $D$ be a dense subset of $L$ of minimum cardinality. Note that $D$ is infinite, since $L$ is a non-zero linear space. Note that

$$|L^*| = \left| \{x^*|_D : x^* \in L^* \} \right|,$$

as any continuous scalar-valued mapping on $L$ is determined by its value on the dense subset $D$ of $L$. We have

$$d(L^* \setminus \{0\}) \leq |L^* \setminus \{0\}| \leq |L^*| \leq |\mathbb{R}^D| = |\mathbb{R}|^{|D|} = 2^{|D|} = 2^{d(L)}.$$

Theorem 2.1 now concludes the proof. \qed

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