ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 1, pp. 137–145

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 1, pp. 137–145
Online ISSN: 1735-8515

LOCAL TRACIAL C^*-ALGEBRAS

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. Let Ω be a class of unital C^*-algebras. We introduce the notion of a local tracial Ω-algebra. Let A be an α-simple unital local tracial Ω-algebra. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has a certain non-simple tracial Rokhlin property. Then the crossed product algebra $C^*(G, A, \alpha)$ is a unital local tracial Ω-algebra.

Keywords: C^*-algebra, local tracial algebra, tracial Rokhlin property.
MSC(2010): Primary: 46L05; Secondary: 46L80, 46L35.

1. **Introduction**

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes in [2]. It was adapted by Hermann and Ocneanu for UHF-algebras in [15]. Rordam [25] and Kishimoto [17] introduced the Rokhlin property to a much more general context of C^*-algebras. More recently, Phillips and Osaka studied finite group actions which satisfy a certain type of Rokhlin property on some simple C^*-algebras in [21–23] and [24].

N. C. Phillips raised the question how to introduce an appropriate Rokhlin property for non-simple C^*-algebras. In [16] J. Hua introduced a certain Rokhlin property for non-simple C^*-algebras. When the C^*-algebra is simple, this Rokhlin property is weaker than the Rokhlin property in [21].

Let Ω be a class of separable unital C^*-algebras. In this paper, we introduce the notion of a local tracial Ω-algebra. When A is a simple unital local tracial Ω-algebra, then the definition we give is equivalent to the definition given by Yang and Fang, in [26]. We prove that if A is a unital local tracial local tracial Ω-algebra, then A is a unital local tracial Ω-algebra. Using this result, we show that if A is an α-simple unital local tracial Ω-algebra with the property SP, and if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has a

Article electronically published on February 22, 2017.
Received: 2 May 2015, Accepted: 21 October 2015.
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certain non-simple tracial Rokhlin property, then the crossed product algebra $C^*(G, A, \alpha)$ is a unital local tracial Ω-algebra.

2. Preliminaries and definitions

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leq [b]$(cf Definition 3.5.2 of [19]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her(a)}, \mathrm{v^*c}, \mathrm{cv^*} \in \mathrm{A}, \mathrm{vv^*} = \mathrm{P_a}$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her(b)}$. We write $[a] = [b]$ if $v^*\mathrm{Her(a)v} = \mathrm{Her(b)}$. Let n be a positive integer. We write $n[a] \leq [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \cdots, b_n \in \mathrm{Her(b)}$ such that $[a] \leq [b_i], i = 1, 2, \ldots, n$.

Let $0 < \sigma_1 < \sigma_2 \leq 1$ be two positive numbers. Definie

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & if\ t \geq \sigma_2 \\ \frac{t-\sigma_1}{\sigma_2-\sigma_1} & if\ \sigma_1 \leq t \leq \sigma_2 \\ 0 & if\ 0 < t \leq \sigma_1 \end{cases}$$

We say a C^*-algebra A has the property SP, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Definition 2.1. ([21]) Let Ω be a class of separable unital C^*-algebras. Then Ω is finitely saturated if the following closure conditions holds:

(1) If $A \in \Omega$ and $B \cong A$, then $B \in \Omega$.
(2) If $A_1, A_2, \ldots A_n \in \Omega$, then $\oplus_{k=1}^n A_n \in \Omega$.
(3) If $A \in \Omega$ and $n \in \mathbb{N}$, then $M_n(A) \in \Omega$.
(4) If $A \in \Omega$ and $p \in A$ is a nonzero projection, then $pAp \in \Omega$.

Definition 2.2. ([21]) Let Ω be a class of separable unital C^*-algebras. A unital local Ω-algebra is a separable unital C^*-algebra A such that for every finite set $S \subseteq A$ and every $\varepsilon > 0$, there is a C^*-algebra B in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : B \to A$(not necessarily injective) such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in S$.

Let A be a C^*-algebra, and let F be a subset of A, $a, b, x \in A, \varepsilon > 0$. If $\|a - b\| < \varepsilon$; then we write $a \approx_\varepsilon b$. If there exists an element $y \in F$ such that $\|x - y\| < \varepsilon$, then we write $x \in_\varepsilon F$.

Definition 2.3. ([26]) Let Ω be a class of separable unital C^*-algebras. A unital C^*-algebra A is said to be a unital local tracial Ω-algebra if for any $\varepsilon > 0$ and any finite subset $F \subseteq A$, any nonzero positive element b, there exist a nonzero projection $p \in A$ and a C^*-algebra B in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : B \to A$ (not necessarily injective) and $\varphi(1_B) = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon \varphi(B)$ for all $x \in F$,
(3) $[1 - p] \leq [p]$.

Definition 2.4. Let Ω be a class of separable unital C^*-algebras. A unital C^*-algebra A is said to be a unital local tracial Ω-algebra if for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element b, and any integer $n > 0$, there exist a nonzero projection $p \in A$, and a C^*-algebra B in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : B \to A$ (not necessarily injective) and $\varphi(1_B) = p$ such that
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon \varphi(B)$ for all $x \in F$,
(3) $n[1 - p] \leq [p]$, $n[f_{\sigma_1}^{\sigma_2}((1 - p)b(1 - p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

We will prove that if A is a simple unital local tracial Ω-algebra, then Definition 2.3 is equivalent to Definition 2.4 given by Yang and Fang.

Definition 2.5. ([12]) A unital C^*-algebra A is said to belong to the class $TA\Omega$ if for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$ containing a nonzero positive element b, any integer $n > 0$, there exist a nonzero projection $p \in A$, and a C^*-subalgebra B of A with $B \in \Omega$ and $1_B = p$ such that
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon B$ for all $x \in F$,
(3) $n[1 - p] \leq [p]$, $n[f_{\sigma_1}^{\sigma_2}((1 - p)b(1 - p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

When a unital C^*-algebra A is a unital local tracial Ω-algebra and each $\varphi(B) \in \Omega$, then A belong to the class $TA\Omega$.

Let A be a C^*-algebra and α either a single automorphism of A or a group action on A. We shall say A is α-simple if A does not have any non-trivial α-invariant closed two-sided ideals.

In [16], J. Hua introduced a certain tracial Rokhlin property for non-simple C^*-algebras with an action of the group $\mathbb{Z}$. X. Yang and X. Fang in [27] given the analogous tracial Rokhlin property for non-simple C^*-algebras to that defined by Hua for an action of a finite group.

Definition 2.6. ([27]) Let A be a finite unital C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say α has the tracial Rokhlin property if for any finite set $F \subseteq A$, any $\varepsilon > 0$, any nonzero positive element $b \in A$, and any element $x \in A$, there exist $g_i,\ g_j \in G$ and mutually orthogonal projections $e_g \in A$ for $g \in G$ such that
(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g,\ h \in G$,
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,
(3) $\|e_{g_i} x e_{g_i}\| \geq \|x\| - \varepsilon$,
(4) with $e = \Sigma_{g\in G} e_g$, $[\alpha_{g_j}(1 - e)] \leq [b]$.

Theorem 2.7. ([16, 27]) *Let A be a unital C^*-algebra. Let $\alpha : G \to \mathrm{Aut(A)}$ be an action of a finite group G on A which has the tracial Rokhlin property.*

Suppose that A is α-simple. Then the crossed product C^*-algebra $C^*(G, A, \alpha)$ is a simple C^*-algebra.

Theorem 2.8. ([16,27]) *Let A be a unital C^*-algebra with the property SP and let $\alpha : G \to \mathrm{Aut(A)}$ be an action of a finite group G on A which has the tracial Rokhlin property. Then any non-zero hereditary C^*-subalgebra of the crossed product algebra $C^*(G, A, \alpha)$ has a nonzero projection which is equivalent to a projection in A.*

Theorem 2.9. ([18]) *For any $0 < \delta_1 < \delta_2 < \sigma_1 < \sigma_2 < 1$, there exists $\eta = \eta(\delta_1, \delta_2) > 0$ such that for any positive elements a and b with $\|a - b\| < \eta$ and $\|a\|, \|b\| \leq 1$ we have $[f_{\sigma_1}^{\sigma_2}(a)] \leq [f_{\delta_1}^{\delta_2}(b)] \leq [b]$.*

Theorem 2.10. ([24]) *Let $n \in \mathbb{N}$, and $(e_{i,j})_{1\leq j,\ k\leq n}$ be a system of matrix units for M_n. For every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever B is a unital C^*-algebra and $w_{j,k}$, for $1 \leq j,\ k \leq n$, are elements of B such that*

(1) $\|{w_{j,k}}^* - w_{k,j}\| < \delta$ for $1 \leq j,\ k \leq n$,

(2) $\|w_{j_1,k_1} w_{j_2,k_2} - \delta_{j_2,k_1} w_{j_1,k_2}\| < \delta$ for $1 \leq j_1,\ j_2,\ k_1,\ k_2 \leq n$,

(3) $w_{j,j}$ are orthogonal projections with $\Sigma_{j=1}^n w_{j,j} = 1$, there exists a unital homomorphism $\varphi : M_n \to B$ such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \leq j \leq n$ and $\|\varphi(e_{j,k}) - w_{j,k}\| < \varepsilon$ for $1 \leq j,\ k \leq n$.

The proof of the following theorem is similar to [12]. We do not give a proof.

Theorem 2.11. *Let Ω be a class of separable unital C^*-algebras. If A is a unital local tracial Ω-algebra, then pAp and $M_n(A)$ are unital tracial Ω-algebras for any projection $p \in A$ and positive integer n.*

3. **The main results**

The proof of the following theorem is similar to [11].

Theorem 3.1. *Let Ω be a class of unital C^*-algebras. The Definition 2,3 and Definition 2.4 are equivalent for any untial simple C^*-algebra A.*

Proof. Firstly we prove that Definition 2.3 implies Definition 2.4. We need to show that for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element b, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, and any integer $n > 0$, there exist a projection $p \in A$ and a C^*-algebra B in the finite saturation of Ω and unital $*$-homomorphism $\varphi : B \to A$ with $\varphi(1_B) = p$, such that

(1) $\|px - xp\| < \varepsilon$ for any $x \in F$,

(2) $pxp \in_\varepsilon \varphi(B)$ for any $x \in F$,

(3) $n[1 - p] \leq [p]$, $n[f_{\sigma_1}^{\sigma_2}((1 - p)b(1 - p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

Since A satisfies Definition 2.3, there exist a projection $p_1 \in A$ and a C^*-algebra A_1 in the finite saturation of Ω and unital $*$-homomorphism $\varphi : A_1 \to A$ with $\varphi(1_{A_1}) = p_1$ such that

(1′) $\|p_1 x - xp_1\| < \varepsilon$ for any $x \in F$, (2′) $p_1xp_1 \in_\varepsilon \varphi(A_1)$ for any $x \in F$,

(3′) $[1-p_1] \leq [b]$.

We may assume that A has the property SP. Since A has the property SP, there exist a nonzero projection $e' \in \mathrm{Her}(\mathrm{f}_{\sigma_1}^{\sigma_2}(\mathrm{p_1bp_1}))$ and there exist projections $e \in \mathrm{Her}(\mathrm{f}_{\sigma_1}^{\sigma_2}(\mathrm{p_1bp_1}))$, $\mathrm{f} \in (1-\mathrm{p_1})\mathrm{A}(1-\mathrm{p_1})$ such that $[e] \leq [e']$ and $[f] = [e]$.

By Theorem 2.11, $(1-p_1)A(1-p_1)$ is a local Ω-algebra, there exist a projection $p_2 \in (1-p_1)A(1-p_1)$ and a C^*-algebra A_2 in the finite saturation of Ω and unital $*$-homomorphism $\varphi' : A_2 \to (1-p_1)A(1-p_1)$ with $\varphi(1_{A_2}) = p_2$ such that

(1″) $\|p_2(1-p_1)x(1-p_1) - (1-p_1)x(1-p_1)p_2\| < \varepsilon$ for any $x \in F$,

(2″) $p_2(1-p_1)x(1-p_1)p_2 \in_\varepsilon \varphi(A_2)$ for any $x \in F$,

(3″) $[1-p_1-p_2] \leq [f]$.

Take $B = A_1 \bigoplus A_2$ and $p = p_1 + p_2$. Then B is in the finite saturation of Ω, we have a unital $*$-homomorphism $(\varphi+\varphi') : B \to A$ with $(\varphi+\varphi')(1_B) = p_1+p_2$ such that

(1) $\|x(p_1+p_2) - (p_1+p_2)x\| \leq \|xp_1 - p_1x + xp_2 - p_2x\| \leq \|p_1x - xp_1\| + \|p_2x - p_2xp_1 - xp_2 + p_1xp_2\| + \|p_2xp_1 - p_1xp_2\| \leq 4\varepsilon$ for any $x \in F$.

(2) $\|(p_1+p_2)x(p_1+p_2) - p_1xp_1 - p_2xp_2\| < 2\varepsilon$, $(p_1+p_2)x(p_1+p_2) \in_{2\varepsilon} B$ for any $x \in F$,

(3) $[f_{\sigma_1}^{\sigma_2}((1-p_1-p_2)b(1-p_1-p_2))] \leq [1-p_1-p_2] \leq [f] \leq [e] \leq [f_{\sigma_1}^{\sigma_2}(p_1bp_1)] \leq [f_{\sigma_1}^{\sigma_2}(p_1bp_1)] + [f_{\sigma_1}^{\sigma_2}(p_2bp_2)] \leq [f_{\sigma_3}^{\sigma_4}((p_1+p_2)b(p_1+p_2))]$.

Using the same method as in [12], we can prove that Definition 2.4 holds.

Secondly we prove that Definition 2.4 implies Definition 2.3. For any $\varepsilon > 0$, and finite subset $F \subseteq A$, any nonzero element $a \geq 0$, any integer $n > 0$, since A is a simple unital C^*-algebra, there are $x_i \in A$ $(i = 1,\ 2, \ldots,\ k)$ such that $\Sigma_{i=1}^k(x_iax_i^*) = 1$. Take $0 < d_1 < d_2 < 1$ such that $\|\Sigma_{i=1}^k x_ia^{1/2}f_{d_1}^{d_2}(a)a^{1/2}x_i^* - 1\| < 1$. Put $z = (\Sigma_{i=1}^k x_ia^{1/2}f_{d_1}^{d_2}(a)a^{1/2}x_i^*)^{-1}$, $y_i = z^{-1/2}x_ia^{1/2}$. Then we have $\Sigma_{i=1}^k y_if_{d_1}^{d_2}(a)y_i^* = 1$. We may assume that $\sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1$ and $\|a\| \leq 1$. Since A satisfies Definition 2.4, there exist a nonzero projection $r \in A$ and a C^*-algebra C in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : C \to A$ with $\varphi(1_C) = r$ such that

(1‴) $\|xr - rx\| < \varepsilon$ for all $x \in F'$,

(2‴) $rxr \in_\varepsilon \varphi(C)$ for all $x \in F'$ and

(3‴) $nk[f_{d_1}^{d_2}((1-r)a(1-r))] \leq [f_{d_3}^{d_4}(rar)]$.

where $F' = \{y_i, y_i^*,\ f_{d_1}^{d_2}(a),\ a,\ 1_A\} \cup F$.

By functional calculus, there are $z_i \in (1-r)A(1-r)$ such that $\Sigma_{i=1}^k z_if_{d_1}^{d_2}((1-r)a(1-r))z_i^* = 1-r$. We have

$$[1-r] \leq k[f_{d_1}^{d_2}((1-r)a(1-r))].$$

Therefore we have

$$\begin{aligned} n[1-r] &\le nk[f_{d_1}^{d_2}((1-r)a(1-r))] \le [f_{d_3}^{d_4}(rar)] + [f_{d_3}^{d_4}((1-r)a(1-r))] \\ &\le [f_{\sigma_3}^{\sigma_4}(a)] \le [a]. \end{aligned}$$

□

The method and technique of the following theorem is similar to [10].

Theorem 3.2. *Let Ω be a class of unital C^*-algebras. If A is a unital local tracial local tracial Ω-algebra, then A is a unital local tracial Ω-algebra.*

Proof. Since A is a unital local tracial local tracial Ω-algebra, for any $\delta > 0$ and any finite subset $G \subseteq A$, there exist a projection $p \in A$ and a C^*-algebra B in the finite saturation of local tracial Ω and a unital $*$-homomorphism $\varphi' : B \to A$ with $\varphi(1_B) = r$ such that

(1)$'$ $\|xp - px\| < \delta$ for all $x \in G$,

(2)$'$ $pxp \in_\delta \varphi'(B)$ for all $x \in G$,

(3)$'$ $2[1_A - p] \le [a]$.

Suppose that $[1_A - p] \ne 0$. By (3)$'$, there exist partial isometries $v_1,\ v_2 \in A$ such that $v_1^* v_1 = 1_A - p,\ v_2^* v_2 = 1_A - p,\ v_1 v_1^*,\ v_2 v_2^* \in \mathrm{Her(a)}$ and $(v_1 v_1^*)(v_2 v_2^*) = 0$. Set $a_1 = v_1 v_1^*,\ a_2 = v_2 v_2^*$. Then we have $a_1 \ne 0,\ a_2 \ne 0,\ a_1,\ a_2 \in \mathrm{Her(a)},\ \mathrm{a_1 a_2} = 0$.

For $H = F \cup \{a_1\}$ and $\varepsilon > 0$, there exist a projection $t \in A$ and a C^*-algebra C in the finite saturation of local tracial Ω and a unital $*$-homomorphism $\varphi'' : C \to A$ with $\varphi''(1_C) = t$ such that

(1)$''$ $\|xt - tx\| < \varepsilon$ for all $x \in H$,

(2)$''$ $txt \in_\varepsilon \varphi''(C)$ for all $x \in H$, and $\|ta_1 t\| \ge \|a_1\| - \varepsilon$,

(3)$''$ $[1_A - t] \le [a_2]$.

By (1)$''$ and (2)$''$, there exist $a_1' \in C$ and $a_1'' \in (1_A - t)A(1_A - t)$ such that $\|a_1 - a_1' - a_1''\| < 2\varepsilon$. We have $a_1' \ne 0$ and $[a_1'] \le [a_1]$.

For any $\varepsilon > 0$ and finite subset $H = \{tx_1 t,\ tx_2 t, \ldots, tx_n t,\ a_1'\}$, since C is in finite local finite saturation of Ω, there exist a projection $r \in A$ and a C^*-algebra in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : D \to A$ with $\varphi(1_D) = r$ such that

(1)$'''$ $\|txtr - rtxt\| < \varepsilon$ for all $x \in F$,

(2)$'''$ $rtxtr \in_\varepsilon D$ for all $x \in F$,

(3)$'''$ $[t - r] \le [a_1']$.

Therefore we have

(1) $\|xr - rx\| < 3\varepsilon$ for all $x \in F$,

(2) $rxr \in_{3\varepsilon} D$ for all $x \in F$,

(3) $[1_A - r] \le [1_A - t] + [t - r] \le [a_2] + [a_1'] \le [a_2] + [a_1] \le [a]$. □

Theorem 3.3. *Let Ω be a class of unital C^*-algebras. Let A be a local Ω-algebra and α-simple unital C^*-algebra with the property SP. Suppose that $\alpha :$*

$G \to \mathrm{Aut(A)}$ *is an action of a finite group* G *on* A *which has the tracial Rokhlin property. Then the crossed product algebra* $C^*(G, A, \alpha)$ *is a local* Ω*-algebra.*

Proof. $C^*(G, A, \alpha)$ is a simple C^*-algebra by Theorem 2.7. suppose F is a finite subset of the unit ball of A, and $G = \{g_1, g_2, \dots g_m\}$, and g_1 is the unit of G and $u_{g_i} \in C^*(G, A, \alpha)$ is the canonical unitary implementing of the automorphism α_{g_i}. For any finite subset G of the form $G = F \cup \{u_{g_i} : 1 \leq i \leq m\}$, any $\varepsilon > 0$, any nonzero positive element $b \in C^*(G, A, \alpha)$, we need to show that there exist a nonzero projection $e \in A$, and a C^*-algebra B in the finite saturation of Ω and a unital $*$-homomorphism $\varphi : B \to A$ with $\varphi(1_B) = e$ such that

(1) $\|ex - xe\| < \varepsilon$ for any $x \in G$,

(2) $exe \in_\varepsilon \varphi(B)$ for any $x \in G$,

(3) $[1_A - e] \leq [b]$.

By Theorem 2.8, there exists a nonzero projection $p \in A$ which is Murray-von Neumann equivalent to a projection in $\overline{bC^*(G, A, \alpha)b}$, i.e., $[p] \leq [b]$.

Set $\delta = \varepsilon/(16m)$. Choose $\eta > 0$ according to Theorem 2.10 for m given above and δ in place of ε. Moreover we may require $\eta < \varepsilon/[8m(m+1)]$. Applying Definition 2.6 to α with F given above, η in place with ε, and p in place of b, there are $g_k \in G$ and mutually orthogonal projections $e_{g_i} \in A$ for $1 \leq i \leq m$, such that

(1′) $\|\alpha_{g_i}(e_{g_j}) - e_{g_i g_j}\| < \eta$ for any $1 \leq i,\ j \leq m$,

(2′) $\|e_{g_i} a - a e_{g_i}\| < \eta$ for any $1 \leq i \leq m$ and any $a \in F$,

(3′) $[\alpha_{g_k}(1 - e)] \leq [p]$, with $e = \Sigma_{i=1}^m e_{g_i}$.

By (1′) and (2′), we have $\|ea - ae\| \leq \Sigma_{i=1}^m \|e_{g_i} a - a e_{g_i}\| < m\eta$.

Define $w_{g_i, g_j} = u_{g_i g_j{}^{-1}} e_{g_j}$ for every $1 \leq i,\ j \leq m$.

Using the same methods as in [24], we can prove that the elements $w_{g_i, g_j} \in eC^*(G, A, \alpha)e$ $(1 \leq i,\ j \leq m)$ satisfy the conditions in Theorem 2.10.

Let (f_{ij}) $(1 \leq i,\ j \leq m)$ be a system of matrix units for M_m. By Theorem 2.10, there exists a unital homomorphism $\varphi_0 : M_m \to eC^*(G, A, \alpha)e$ such that $\|\varphi_0(f_{ij}) - w_{g_i, g_j}\| < \delta$ for all $1 \leq i,\ j \leq m$, and $\psi_0(f_{ii}) = e_{g_i}$ for all $1 \leq i \leq m$. Now we define a unital homomorphism $\varphi : M_m \otimes e_{g_1} A e_{g_1} \to eC^*(G, A, \alpha)e$ by

$$\varphi(f_{ij} \otimes a) = \varphi_0(f_{i1}) a \varphi_0(f_{i1})$$

for all $1 \leq i,\ j \leq m$ and $a \in e_{g_1} A e_{g_1}$. Then

$$\varphi(f_{ij} \otimes e_{g_1}) = \varphi_0(f_{i1}) e_{g_1} \varphi_0(f_{1j}) = \varphi_0(f_{ij}) = e_{g_i} \varphi_0(f_{ij}) e_{g_j},$$

and $\varphi(1_{M_m} \otimes e_{g_1}) = e$.

Take $B = M_m \otimes e_{g_1} A e_{g_1}$. Then B is in the finite saturation of local tracial Ω. By Theorem 2.7, we have B is in the finite saturation of local tracial Ω.

Using the same method as in [14], we can prove that

(1) $\|ae - ea\| \leq m\eta < \varepsilon$, for any $a \in G$,

(2) $exe \in_\varepsilon \varphi(M_m \otimes e_{g_1} A e_{g_1})$ for any $x \in G$,

(3) $[1_A - e] = [\alpha_{g_k}(1 - e)] \leq [p] \leq [b]$. □

Acknowledgements

This article is supported by the National Natural Science Foundation of China (11501357).

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