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POSITIVE SOLUTIONS FOR ASYMPTOTICALLY PERIODIC KIRCHHOFF-TYPE EQUATIONS WITH CRITICAL GROWTH

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(Communicated by Asadollah Aghajani)

Abstract. In this paper, we consider the following Kirchhoff-type equations:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=\lambda f(x,u)+u^5, \quad \text{in } \mathbb{R}^3,\\ u(x)>0, \quad \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases}$$

where $a,b>0$ are constants and λ is a positive parameter. The aim of this paper is to study the existence of positive solutions for Kirchhoff-type equations with a nonlinearity in the critical growth under some suitable assumptions on $V(x)$ and $f(x,u)$. Recent results from the literature are improved and extended.

Keywords: Kirchhoff-type equations, critical growth, variational methods.
MSC(2010): Primary: 35J20; Secondary: 35J60.

1. Introduction and main results

In this paper, we consider the following Kirchhoff-type equations:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=\lambda f(x,u)+u^5, \quad \text{in } \mathbb{R}^3,\\ u(x)>0, \quad \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $a,b>0$ are constants and λ is a positive parameter. Moreover, $V(x)$ and $f(x,u)$ are continuous functions satisfying some conditions, which will be stated later on.

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Over the past decades, many papers have extensively considered the Kirchhoff equation

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Using variational methods, for various conditions of the nonlinearity $f(x,u)$, the existence and multiplicity of solutions for problem (1.2) have been extensively investigated in the literature, one can see [1, 4, 5, 7, 11, 15, 17] and the references therein.

It is well known that problem (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a+b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) \tag{1.3}$$

which was proposed by Kirchhoff in [11] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

There are also many works on the existence and multiplicity results for the following equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u + V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $f(x,u)$ satisfies certain conditions. More precisely, Wu [26] studied the existence of nontrivial solutions and infinitely many high energy solutions for problem (1.4) by using a symmetric mountain pass theorem. Liu and He [13] also studied the existence of infinitely many high energy solutions for superlinear Kirchhoff problem (1.4) by variant version of fountain theorem. Duan and Huang [6] dealt with problem (1.4) with sublinear case and the existence of infinitely many solutions for the problem is established by using the genus properties in critical point theory. For related topics, we refer the readers to [8–10, 14, 18, 24] and the references therein.

When $b \equiv 0$ and $a \equiv 1$ in (1.4), the equation itself turns to be a semilinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.5}$$

Many authors have studied the existence and multiplicity of solutions for problem (1.5) under various stipulations, one can see [12, 16, 19–23] and the references therein. More precisely, [16] and [21] studied the periodic case, Tang [23] and Lin [12] generalized the periodic case to asymptotically periodic case. Here we do not try to review the huge bibliography.

Motivated by the above facts, we want to look for the positive solutions for problem (1.1), where $V(x)$ and $f(x,u)$ in problem (1.1) are asymptotically periodic. Furthermore, we shall study the case that problem (1.1) has a nonlinearity in critical growth. We shall point out that the main difficulty of the present paper is the lack of compactness of the Sobolev embeddings, because we are working in whole $\mathbb{R}^3$, and the nonlinearity has a critical growth. These facts prevent from proving that the energy functional associated with (1.1) verifies the well known Palais-Smale condition, which is a key point to prove the existence of critical points for this functional in a lot of papers. Compared to the existing results, our result is different and extend the above results to some extent.

Related to the function $V(x)$, we assume that

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$, and there exists a function V_p, which is 1-periodic in each of x_i $(i = 1, 2, 3)$, such that

$$V_p(x) \geq V_0 > 0, \quad x \in \mathbb{R}^3,$$

where V_0 is a constant.

(V_2) There exists a function $W \in L^{\frac{3}{2}}(\mathbb{R}^3)$ with $W(x) \geq 0$ such that

$$V(x) = V_p(x) - W(x) \geq W_0, \quad x \in \mathbb{R}^3,$$

where W_0 is a positive constant and the inequality $W(x) > 0$ is strict on a subset of positive measure in $\mathbb{R}^3$.

On function f, since we intend to show the existence of positive solutions for problem (1.1), we assume that

(f_0) $f(x, u) = 0$, $\forall u \leq 0$.

We also suppose that $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(f_1) $|f(x, u)| \leq c(1+|u|^{q-1})$ for some $2 < q < 6$, where c is a positive constant.

(f_2) $f(x, u) = o(u^3)$ uniformly in x as $|u| \to 0$.

(f_3) $u \mapsto \frac{f(x,u)}{u^3}$ is nondecreasing on $(0, \infty)$.

(f_4) $\frac{F(x,u)}{|u|^4} \to +\infty$ uniformly in x as $u \to +\infty$, where $F(x, u) = \int_0^u f(x, s)\mathrm{d}s$.

Moreover, suppose f is asymptotically periodic in x in the following sense, there exists a function $f_p \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, which is 1-periodic in each of x_i $(i = 1, 2, 3)$, such that

(f_5) $|f(x, u)| \geq |f_p(x, u)|$, $\forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}$.

(f_6) $|f(x, u) - f_p(x, u)| \leq a(x)(|u| + |u|^{q-1})$, $\forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}$, where $a(x) \in L^\infty(\mathbb{R}^3)$ and for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^3 : |a(x)| \geq \varepsilon\}$ has finite Lebesgue measure.

(f_7) $u \mapsto \frac{f_p(x,u)}{u^3}$ is nondecreasing on $(0, \infty)$.

Theorem 1.1. *Suppose that* (V_1), (V_2) *and* $(f_0)-(f_7)$ *hold. Then, there exists* $\lambda^* > 0$ *such that problem* (1.1) *possesses a positive solution, for all* $\lambda > \lambda^*$.

The proof of Theorem 1.1 is mainly based on the method of the generalized Nehari manifold and the concentration-compactness principle. As in [3], we reduce the problem of looking for a positive solution into that of finding a minimizer on the Nehari manifold. Then we apply concentration-compactness principle to solve the minimizing problem.

The reminder of this paper is organized as follows. In Section 2, some preliminary results are presented. The proof of main results will be given in the last section.

2. **Variational setting and preliminaries**

In this section, we present some basic preliminary results and necessary lemmas, which will be used throughout this paper.

Let us first recall some notations. As usual, for $1 \leq s < \infty$, we denote

$$\|u\|_s = \left(\int_{\mathbb{R}^3} |u(x)|^s \mathrm{d}x\right)^{\frac{1}{s}}, \quad u \in L^s(\mathbb{R}^3)$$

and

$$\|u\|_\infty = \text{ess sup}_{x\in\mathbb{R}^3}|u(x)|, \quad u \in L^\infty(\mathbb{R}^3).$$

Let S be the sharp constant of the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, which is given by

$$S = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}.$$

Here, $\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$.

Define the function space

$$H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$$

with the usual norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

We consider the Sobolev space $H^1(\mathbb{R}^3)$ endowed with one of the following norms

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x, \quad \|u\|_{V_p}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V_p(x)u^2)\mathrm{d}x.$$

Under the assumptions (V_1) and (V_2), the norms $\|\cdot\|$ and $\|\cdot\|_{V_p}$ are equivalent to the standard norm $\|\cdot\|_{H^1}$.

The functional corresponding to the problem (1.1) is

$$\begin{aligned} I(u) = {} & \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x \\ & - \lambda\int_{\mathbb{R}^3} F(x,u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} u_+^6\mathrm{d}x. \end{aligned}$$

By our assumptions, I belongs to $C^1(H^1(\mathbb{R}^3),\mathbb{R})$ with

$$\begin{aligned}\langle I'(u),v\rangle = \left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u\nabla v\mathrm{d}x+\int_{\mathbb{R}^3}V(x)uv\mathrm{d}x\\ -\lambda\int_{\mathbb{R}^3}f(x,u)v\mathrm{d}x-\int_{\mathbb{R}^3}u_+^5v\mathrm{d}x,\end{aligned}$$

and its critical points are solutions of problem (1.1).

In the process of looking for the positive solutions of problem (1.1), its corresponding periodic equation is very important and defined by

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u+V_p(x)u=\lambda f_p(x,u)+u^5, & \text{in } \mathbb{R}^3,\\ u(x)>0, \quad \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{2.1}$$

and the associated functional is

$$\begin{aligned}I_p(u)=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2+\frac{1}{2}\int_{\mathbb{R}^3}V_p(x)u^2\mathrm{d}x\\ -\lambda\int_{\mathbb{R}^3}F_p(x,u)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}u_+^6\mathrm{d}x,\end{aligned} \tag{2.2}$$

where $F_p(x,u)=\int_0^u f_p(x,s)\mathrm{d}s$. It is well known that $I_p\in C^1(H^1(\mathbb{R}^3),\mathbb{R})$, with

$$\begin{aligned}\langle I_p'(u),v\rangle = \left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\int_{\mathbb{R}^3}\nabla u\nabla v\mathrm{d}x+\int_{\mathbb{R}^3}V_p(x)uv\mathrm{d}x\\ -\lambda\int_{\mathbb{R}^3}f_p(x,u)v\mathrm{d}x-\int_{\mathbb{R}^3}u_+^5v\mathrm{d}x\end{aligned} \tag{2.3}$$

for all $v\in H^1(\mathbb{R}^3)$. Hence, the critical points of I_p are positive solutions of problem (2.1).

Later, we give some lemmas that will be used in the proof of our theorems.

Lemma 2.1. *If (f_2) and (f_3) are satisfied, then*

$$0\le 4F(x,u)\le f(x,u)u,\quad \forall u\in\mathbb{R}.$$

If (f_2) and (f_5) are satisfied, then

$$f_p(x,u)=o(u^3) \text{ uniformly in } x \text{ as } u\to 0.$$

If (f_2), (f_5) and (f_7) are satisfied, then

$$0\le 4F_p(x,u)\le f_p(x,u)u,\quad \forall u\in\mathbb{R}. \tag{2.4}$$

In addition, if (f_3) is also satisfied, then

$$F(x,u)\ge F_p(x,u)\ge 0,\quad \forall u\in\mathbb{R}. \tag{2.5}$$

Proof. By (f_2) and (f_3), we have $f(x,u) \geq 0$, $\forall u > 0$. Thus, $F(x,u) \geq 0$, $\forall u > 0$. And we have

$$F(x,u) = \int_0^u \frac{f(x,s)}{s^3} s^3 ds \leq \int_0^u \frac{f(x,u)}{u^3} s^3 ds = \frac{1}{4} f(x,u)u.$$

The conclusion $f_p(x,u) = o(u^3)$ as $u \to 0$ is immediate from the assumptions (f_2) and (f_5). Similarly, we get (2.4) and $f_p(x,u) \geq 0$, $\forall u > 0$, then by (f_5), we obtain $f(x,u) \geq f_p(x,u) \geq 0$, $\forall u > 0$. So $F(x,u) \geq F_p(x,u) \geq 0$, $\forall u \in \mathbb{R}$. □

The next two lemmas show that the functional I_p verifies the mountain pass geometry.

Lemma 2.2. *Suppose that (V_1), (f_1), (f_2), (f_5) and (f_6) are satisfied, then there exist positive constants ρ and α such that*

$$I_p(u) \geq \alpha, \quad \forall u \in H^1(\mathbb{R}^3) : \|u\|_{V_p} = \rho.$$

Proof. By (f_1), (f_6) and Lemma 2.1, for all $\varepsilon > 0$, there exists C_ε such that

$$|f_p(x,u)| \leq \varepsilon |u| + C_\varepsilon |u|^{q-1}, \quad \forall u \in H^1(\mathbb{R}^3).$$

From the Sobolev embedding, we derive

$$\begin{aligned} I_p(u) &\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V_p(x) u^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} (\varepsilon |u|^2 + C_\varepsilon |u|^q) \mathrm{d}x \\ &\quad - \frac{1}{6} \int_{\mathbb{R}^3} u_+^6 \mathrm{d}x \\ &\geq \frac{1}{2} \min\{a,1\} \|u\|_{V_p}^2 - \lambda(\varepsilon \|u\|_{V_p}^2 + C\|u\|_{V_p}^q) - \frac{C}{6} \|u_+\|_{V_p}^6 \\ &\geq \frac{1}{2} (\min\{a,1\} - \lambda\varepsilon) \|u\|_{V_p}^2 - \lambda C \|u\|_{V_p}^q - C\|u\|_{V_p}^6 \end{aligned}$$

for some positive constant C. Since $2 < q < 6$, taking $\rho > 0$ sufficiently small, we conclude that there exists a constant $\alpha > 0$ such that

$$I_p(u) \geq \alpha, \quad \forall u \in H^1(\mathbb{R}^3) : \|u\|_{V_p} = \rho.$$

This completes the proof. □

Lemma 2.3. *Suppose that (V_1), (f_2), (f_3), (f_5) and (f_7) are satisfied. Thus, for all $\lambda > 0$, there exists $e \in H^1(\mathbb{R}^3)$ such that $I_p(e) < 0$ and $\|e\|_{V_p} > \rho$.*

Proof. Fix $v_0 \in H^1(\mathbb{R}^3)$ with $v_0 \geq 0$. By (2.2), we have

$$I_p(tv_0) \leq \frac{t^2}{2} \max\{a,1\} \|v_0\|_{V_p}^2 + \frac{bt^4}{4} \|v_0\|_{V_p}^4 - \frac{t^6}{6} \int_{\mathbb{R}^3} v_0^6 \mathrm{d}x.$$

Then obviously, $I_p(tv_0) \to -\infty$ as $t \to \infty$ and thus, there exists a constant $t_* > 0$ such that $\|t_* v_0\|_{V_p} > \rho$ and $I_p(t_* v_0) \leq 0$. The result directly follows by considering $e = t_* v_0$. □

Now we define

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \mid \gamma(0) = 0,\ I_p(\gamma(1)) \leq 0,\ \gamma(1) \neq 0\}$$

and

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_p(\gamma(t)).$$

Note that from Lemma 2.3, $\Gamma \neq \emptyset$. Moreover, from the argument in Lemma 2.2, clearly 0 is a local minimum of I_p. Consequently, using a version of Mountain Pass Theorem without (PS) condition founded in [25], we have the existence of sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying

$$I_p(u_n) \to c_\lambda, \quad \text{and} \quad I_p'(u_n) \to 0.$$

The above sequence is called a $(\mathrm{PS})_{c_\lambda}$ sequence for I_p.

Hereafter, $\mathcal{M}$ denotes the Nehari manifold associated with I_p, that is,

$$\mathcal{M} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I_p'(u), u\rangle = 0\}.$$

We prove the next lemma.

Lemma 2.4. *Suppose that* (V_1) *and* (f_7) *are satisfied, then for each* $u \in H^1(\mathbb{R}^3)$ *with* $u_+ \neq 0$*, there exists a unique* $t_0(u) > 0$ *such that* $t_0(u)u \in \mathcal{M}$ *and* $I_p(t_0u) = \max_{t\geq 0} I_p(tu)$*. Moreover* $c_\lambda = \overline{c} = \widetilde{c}$*, where* $\widetilde{c} = \inf_{\mathcal{M}} I_p$ *and* $\overline{c} = \inf_{u\in H^1(\mathbb{R}^3)\backslash\{0\}} \max_{t\geq 0} I_p(tu)$.

Proof. Let $u \in H^1(\mathbb{R}^3)$ with $u_+ \neq 0$ be fixed and define $g(t) := I_p(tu)$ on $[0,\infty)$. By Lemma 2.3, there exists $t_0 > 0$ such that

$$g(t_0) = \max_{t\geq 0} g(t) = \max_{t\geq 0} I_p(tu).$$

Clearly we have

$$\begin{aligned}
&g'(t_0) = 0 \Leftrightarrow t_0(u)u \in \mathcal{M} \\
&\Leftrightarrow at_0^2 \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + bt_0^4 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 + t_0^2 \int_{\mathbb{R}^3} V_p(x)u^2 \mathrm{d}x \\
&\quad = \lambda \int_{\mathbb{R}^3} f_p(x, t_0u)t_0u \mathrm{d}x + t_0^6 \int_{\mathbb{R}^3} u_+^6 \mathrm{d}x.
\end{aligned}$$

In the sequel, we will show that t_0 is unique. To this end, we suppose that there exists $t_1 > 0$ such that $t_1u \in \mathcal{M}$. This together with above equality implies that

$$\begin{aligned}
&\frac{a\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V_p(x)u^2 \mathrm{d}x}{t_0^2} + b\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 \\
&\quad = \lambda \int_{\mathbb{R}^3} \frac{f_p(x, t_0u)}{t_0^3u^3} u^4 \mathrm{d}x + t_0^2 \int_{\mathbb{R}^3} u_+^6 \mathrm{d}x
\end{aligned}$$

and

$$\frac{a\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^3}V_p(x)u^2\mathrm{d}x}{t_1^2}+b\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2$$
$$=\lambda\int_{\mathbb{R}^3}\frac{f_p(x,t_1u)}{t_1^3u^3}u^4\mathrm{d}x+t_1^2\int_{\mathbb{R}^3}u_+^6\mathrm{d}x.$$

Hence

$$\left(a\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^3}V_p(x)u^2\mathrm{d}x\right)\left(\frac{1}{t_0^2}-\frac{1}{t_1^2}\right)$$
$$=\lambda\int_{\mathbb{R}^3}\left(\frac{f_p(x,t_0u)}{t_0^3u^3}u^4-\frac{f_p(x,t_1u)}{t_1^3u^3}u^4\right)\mathrm{d}x+(t_0^2-t_1^2)\int_{\mathbb{R}^3}u_+^6\mathrm{d}x.$$

It follows from (f_7) that $t_0=t_1$. By using similar arguments in [25], it is easy to prove the rest of the proof. This concludes the proof. □

Now, we begin studying the behavior of mountain pass level c_λ related to the parameter λ.

Lemma 2.5. *Suppose that (V_1), (f_2), (f_3) and (f_7) are satisfied, then $\lim_{\lambda\to\infty}c_\lambda = 0$.*

Proof. If v_0 is the function given by Lemma 2.3, it follows that there exists $t_\lambda>0$ satisfying $I_p(t_\lambda v_0)=\max_{t\geq 0}I_p(tv_0)$. Then, we have

$$at_\lambda^2\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x+bt_\lambda^4\left(\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x\right)^2+t_\lambda^2\int_{\mathbb{R}^3}V_p(x)v_0^2\mathrm{d}x$$
$$=\lambda\int_{\mathbb{R}^3}f_p(x,t_\lambda v_0)t_\lambda v_0\mathrm{d}x+t_\lambda^6\int_{\mathbb{R}^3}v_0^6\mathrm{d}x.$$

By Lemma 2.1, we get

$$at_\lambda^2\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x+bt_\lambda^4\left(\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x\right)^2+t_\lambda^2\int_{\mathbb{R}^3}V_p(x)v_0^2\mathrm{d}x\geq t_\lambda^6\int_{\mathbb{R}^3}v_0^6\mathrm{d}x,$$

which implies that t_λ is bounded. Furthermore, we will show that $t_\lambda\to 0$ as $\lambda\to\infty$. If not, there exists a sequence $\lambda_n\to+\infty$ and a constant $t^*>0$ such that $t_{\lambda_n}\to t^*$ as $n\to\infty$. The boundedness of t_{λ_n} yields that there is $M>0$ such that

$$at_{\lambda_n}^2\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x+bt_{\lambda_n}^4\left(\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x\right)^2+t_{\lambda_n}^2\int_{\mathbb{R}^3}V_p(x)v_0^2\mathrm{d}x\leq M$$

and so

$$\lambda_n\int_{\mathbb{R}^3}f_p(x,t_{\lambda_n}v_0)t_{\lambda_n}v_0\mathrm{d}x+t_{\lambda_n}^6\int_{\mathbb{R}^3}v_0^6\mathrm{d}x\leq M.$$

If $t^*>0$, we have that

$$\lim_{n\to\infty}\left[\lambda_n\int_{\mathbb{R}^3}f_p(x,t_{\lambda_n}v_0)t_{\lambda_n}v_0\mathrm{d}x+t_{\lambda_n}^6\int_{\mathbb{R}^3}v_0^6\mathrm{d}x\right]\to\infty,$$

which is a contradiction. Thus $t_\lambda \to 0$ as $\lambda \to \infty$. Now, consider the path $\gamma^*(t) = te$ for $t \in [0,1]$, then $\gamma^* \in \Gamma$. Therefore we obtain

$$\begin{aligned} 0 < c_\lambda &\le \max_{t\in[0,1]} I_p(\gamma^*(t)) \le \max_{t\ge 0} I_p(tv_0) = I_p(t_\lambda v_0) \\ &\le a t_\lambda^2 \int_{\mathbb{R}^3} |\nabla v_0|^2 \mathrm{d}x + b t_\lambda^4 \left(\int_{\mathbb{R}^3} |\nabla v_0|^2 \mathrm{d}x\right)^2 + t_\lambda^2 \int_{\mathbb{R}^3} V_p(x) v_0^2 \mathrm{d}x \\ &\to 0. \end{aligned}$$

This completes the proof. □

Lemma 2.6. *Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence for I_p with $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then there is a $\lambda^* > 0$, when $\lambda > \lambda^*$, there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants R, $\eta > 0$ such that*

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \ge \eta > 0.$$

Proof. From Lemma 2.5, there exists a constant $\lambda^* > 0$ such that

$$c_\lambda < \frac{abS^3}{4} + \frac{S^6}{24}[(b^2 + 4aS^{-3})^{\frac{3}{2}} + b^3] \tag{2.6}$$

for all $\lambda > \lambda^*$. Now, we argue by contradiction. Suppose $\{u_n\}$ is vanishing, then from P.L. Lions Compactness Lemma [25] it directly follows that $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $2 < q < 6$. By (f_1), (f_2), (f_5) and (f_6), we have

$$\int_{\mathbb{R}^3} F_p(x, u_n) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} f_p(x, u_n) u_n \to 0.$$

Since $\{u_n\}$ is a $(PS)_{c_\lambda}$ sequence for I_p, we have

$$\begin{aligned} c_\lambda = &\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_p(x) u_n^2 \mathrm{d}x \\ &- \frac{1}{6}\int_{\mathbb{R}^3} u_{n+}^6 \mathrm{d}x + o_n(1) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &a \int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 + \int_{\mathbb{R}^3} V_p(x) u_n^2 \mathrm{d}x \\ &= \int_{\mathbb{R}^3} u_{n+}^6 \mathrm{d}x + o_n(1). \end{aligned} \tag{2.8}$$

We claim that $\int_{\mathbb{R}^3} |\nabla u_n|^2 \nrightarrow 0$. If not, $\int_{\mathbb{R}^3} |\nabla u_n|^2 \to 0$, then

$$\int_{\mathbb{R}^3} u_{n+}^6 \mathrm{d}x \le S^{-3}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^3 \to 0,$$

which implies that $c_\lambda = 0$ in (2.7). This is a contradiction with the fact that $c_\lambda > 0$.

By (2.8) we have

$$a + b\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \le S^{-3}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 + o_n(1).$$

Then we conclude

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \ge \frac{S^3}{2}(b + \sqrt{b^2 + 4aS^{-3}}) + o_n(1) \tag{2.9}$$

and

$$\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 \ge aS^3 + \frac{bS^6}{2}(b + \sqrt{b^2 + 4aS^{-3}}) + o_n(1) \tag{2.10}$$

It follows from (2.7) and (2.8) that

$$c_\lambda \ge \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 + o_n(1).$$

Combining with (2.9) and (2.10) we obtain

$$c_\lambda \ge \frac{abS^3}{4} + \frac{S^6}{24}[(b^2 + 4aS^{-3})^{\frac{3}{2}} + b^3],$$

which is a contradiction with (2.6). Hence $\{u_n\}$ is non-vanishing. This concludes the proof. □

The next result establishes the existence of solution for problem (2.1).

Theorem 2.7. *Suppose that (V_1), (V_2) and (f_0)–(f_7) hold. Then, there exists $\lambda^* > 0$ such that problem (2.1) possesses a positive solution, for all $\lambda > \lambda^*$.*

Proof. From Lemmas 2.2 and 2.3, there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying

$$I_p(u_n) \to c_\lambda, \quad \text{and} \quad I_p'(u_n) \to 0.$$

We first claim the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Indeed, noting that

$$\begin{aligned}
c_\lambda + \|u_n\|_{V_p} &\ge I_p(u_n) - \frac{1}{4}\langle I_p'(u_n), u_n\rangle \\
&\ge \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} V_p(x)u_n^2 \mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^3}\left[\frac{1}{4}f_p(x,u_n)u_n - F_p(x,u_n)\right]\mathrm{d}x \\
&\ge \frac{1}{4}\min\{a,1\}\|u_n\|_{V_p}^2,
\end{aligned}$$

where we used (2.4). Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

From boundedness of $\{u_n\}$, there is a subsequence of $\{u_n\}$, still denoted by itself, $\widetilde{u} \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup \widetilde{u} \quad \text{in } H^1(\mathbb{R}^3), \\ u_n &\to \widetilde{u} \quad \text{a.e. on } \mathbb{R}^3, \\ u_n &\to \widetilde{u} \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } s \in [2,6). \end{aligned}$$

Without loss of generality, we can assume that $\widetilde{u} \neq 0$. If not, $\widetilde{u} = 0$, then by Lemma 2.6, there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants R, $\eta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \eta > 0. \tag{2.11}$$

Since $u_n \to \widetilde{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ and $\widetilde{u} = 0$, we may suppose that $|y_n| \to \infty$ up to a subsequence. Denote $v_n(x) = u_n(x+y_n)$, once that V_p is periodic, we have that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Similarly, passing to a subsequence, we assume that

$$\begin{aligned} v_n &\rightharpoonup \widetilde{v} \quad \text{in } H^1(\mathbb{R}^3), \\ v_n &\to \widetilde{v} \quad \text{a.e. on } \mathbb{R}^3, \\ v_n &\to \widetilde{v} \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } s \in [2,6). \end{aligned}$$

By (2.11) one easily has that $\widetilde{v} \neq 0$. Furthermore, a standard computation leads to

$$I_p(v_n) \to c_\lambda \quad \text{and} \quad I'_p(v_n) \to 0.$$

Next we claim that $I'_p(\widetilde{u}) = 0$. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, passing to a subsequence, we may assume that there exists $m \geq 0$ such that $\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \to m$. Note that $I'_p(u_n) \to 0$, then $\widetilde{u}$ is a non-negative solution of the problem

$$-(a+bm)\Delta u + V_p(x)u = \lambda f_p(x,u) + u^5, \quad u \in H^1(\mathbb{R}^3).$$

To conclude our proof, we need to prove that $\int_{\mathbb{R}^3} |\nabla \widetilde{u}|^2 \mathrm{d}x = m$. From the weakly lower semi-continuous of the norm it follows that $m \geq \int_{\mathbb{R}^3} |\nabla \widetilde{u}|^2 \mathrm{d}x$. Then

$$\begin{aligned} &\left(a + b\int_{\mathbb{R}^3} |\nabla \widetilde{u}|^2 \mathrm{d}x\right) \int_{\mathbb{R}^3} |\nabla \widetilde{u}|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V_p(x)\widetilde{u}^2 \mathrm{d}x \\ &\quad \leq (a+bm) \int_{\mathbb{R}^3} |\nabla \widetilde{u}|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V_p(x)\widetilde{u}^2 \mathrm{d}x \\ &\quad = \lambda \int_{\mathbb{R}^3} f_p(x,\widetilde{u})\widetilde{u} \mathrm{d}x + \int_{\mathbb{R}^3} \widetilde{u}^6 \mathrm{d}x. \end{aligned} \tag{2.12}$$

This inequality implies that $\langle I'_p(\widetilde{u}), \widetilde{u}\rangle \leq 0$. By Lemma 2.4, there exists $\bar{t} \in (0,1]$ such that $\bar{t}\widetilde{u} \in \mathcal{M}$. Combining this results with the characterization of mountain pass level, we conclude

$$
\begin{aligned}
c_\lambda \le I_p(\bar{t}\widetilde{u}) &= I_p(\bar{t}\widetilde{u}) - \frac{1}{4}\langle I_p'(\bar{t}\widetilde{u}), \bar{t}\widetilde{u}\rangle \\
&= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla(\bar{t}\widetilde{u})|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} V_p(x)(\bar{t}\widetilde{u})^2 \mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |\bar{t}\widetilde{u}_+|^6 \mathrm{d}x \\
&\quad + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{4} f_p(x, \bar{t}\widetilde{u})\bar{t}\widetilde{u} - F_p(x, \bar{t}\widetilde{u})\right] \mathrm{d}x \\
&\le \frac{a}{4}\int_{\mathbb{R}^3} |\nabla\widetilde{u}|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} V_p(x)\widetilde{u}^2 \mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |\widetilde{u}_+|^6 \mathrm{d}x \\
&\quad + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{4} f_p(x, \widetilde{u})\widetilde{u} - F_p(x, \widetilde{u})\right] \mathrm{d}x \\
&\le \frac{a}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} V_p(x)u_n^2 \mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |u_{n+}|^6 \mathrm{d}x \\
&\quad + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{4} f_p(x, u_n)u_n - F_p(x, u_n)\right] \mathrm{d}x + o_n(1) \\
&= I_p(u_n) - \frac{1}{4} I_p'(u_n)u_n + o_n(1) = c_\lambda + o_n(1) \qquad (2.13)
\end{aligned}
$$

where we use Fatou's Lemma. So $\bar{t} = 1$, then $I_p'(\widetilde{u})\widetilde{u} = 0$, it follows from (2.12) that $\int_{\mathbb{R}^3} |\nabla\widetilde{u}|^2 \mathrm{d}x = m$. Then, $I_p'(\widetilde{u}) = 0$. Moreover, by (2.13), we have

$$
\begin{aligned}
I_p(\widetilde{u}) - \frac{1}{4}\langle I_p'(\widetilde{u}), \widetilde{u}\rangle &= \frac{a}{4}\int_{\mathbb{R}^3} |\nabla\widetilde{u}|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} V_p(x)\widetilde{u}^2 \mathrm{d}x \\
&\quad + \lambda \int_{\mathbb{R}^3} \left[\frac{1}{4} f_p(x, \widetilde{u})\widetilde{u} - F_p(x, \widetilde{u})\right] \mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |\widetilde{u}_+|^6 \mathrm{d}x \\
&= c_\lambda.
\end{aligned}
$$

From $I_p'(\widetilde{u}) = 0$ it directly follows that $I_p(\widetilde{u}) = c_\lambda$. This completes the proof. □

3. **Proof of main results**

We are now in a position to give the proof of Theorem 1.1. In this section, $\mathcal{N}$ denotes the Nehari manifold related to I, that is,

$$
\mathcal{N} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'(u), u\rangle = 0\}.
$$

Arguing as Lemmas 2.2 and 2.3, it is easy to prove that the functional I has the mountain pass geometry. Thus, there exists a $(\mathrm{PS})_{d_\lambda}$ sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying

$$
I(u_n) \to d_\lambda, \quad \text{and} \quad I'(u_n) \to 0,
$$

where $d_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$.

Arguing as Lemma 2.4, we obtain that

$$d_\lambda = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t\geq 0} I(tu) = \inf_{\mathcal{N}} I.$$

From the conditions in Theorem 1.1, we have that $F_p(x,u) \leq F(x,u)$. Combining with condition (V_2), it follows that $d_\lambda < c_\lambda$.

Now we give the proof of Theorem 1.1.

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_{d_\lambda}$ sequence for I. Arguing as in the proof of Theorem 2.7, we have that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Thus, there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

Now we will show that $u = 0$ cannot occur. Indeed, if $u = 0$, then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Since $W \in L^{\frac{3}{2}}(\mathbb{R}^3)$, we concludes that

$$\int_{\mathbb{R}^3} W u_n^2 \mathrm{d}x \to 0.$$

So

$$\begin{aligned}
|I_p(u_n) - I(u_n)| &= \left| \frac{1}{2}\int_{\mathbb{R}^3} W u_n^2 \mathrm{d}x + \int_{\mathbb{R}^3} (F(x,u_n) - F_p(x,u_n))\,\mathrm{d}x \right| \\
&\leq \frac{1}{2}\left|\int_{\mathbb{R}^3} W u_n^2 \mathrm{d}x\right| + \int_{\mathbb{R}^3} (|F_p(x,u_n)| + |F(x,u_n)|)\,\mathrm{d}x \\
&= o_n(1),
\end{aligned}$$

which implies that $I_p(u_n) \to d_\lambda$.

On the other hand, taking $\phi \in H^1(\mathbb{R}^3)$ with $\|\phi\| \leq 1$, we obtain that

$$\begin{aligned}
|\langle I_p'(u_n) - I'(u_n), \phi\rangle| &= \left| \int_{\mathbb{R}^3} W u_n \phi \mathrm{d}x + \int_{\mathbb{R}^3} (f(x,u_n) - f_p(x,u_n))\,\phi \mathrm{d}x \right| \\
&\leq C\left(\int_{\mathbb{R}^3} |W| u_n^2 \mathrm{d}x\right)^{\frac{1}{2}} + C(\|u_n\|_2 + \|u_n\|_q^{q-1}) \\
&= o_n(1).
\end{aligned}$$

Thus, $I_p'(u_n) = o_n(1)$. Let $t_n > 0$ such that $t_n u_n \in \mathcal{M}$. Using the same arguments in [2], it follows that $t_n \to 1$. Therefore,

$$c_\lambda \leq I_p(t_n u_n) = I_p(u_n) + o_n(1) = d_\lambda + o_n(1).$$

Letting $n \to \infty$, we get

$$c_\lambda \leq d_\lambda,$$

which is a contradiction with above fact $d_\lambda < c_\lambda$. Thus, $u \neq 0$, then arguing as in the proof of Theorem 2.7, u is a positive solution for problem (1.1). This completes the proof of Theorem 1.1. □

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