Title:
Some compact generalization of inequalities for polynomials with prescribed zeros

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SOME COMPACT GENERALIZATION OF INEQUALITIES FOR POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. Let \( p(z) = z^s h(z) \) where \( h(z) \) is a polynomial of degree at most \( n - s \) having all its zeros in \( |z| \geq k \) or in \( |z| \leq k \). In this paper we obtain some new results about the dependence of \( \max_{|z| = r} |p(Rz)| \) on \( |p(rz)| \) for \( r^2 \leq rR \leq k^2 \), \( k^2 \leq rR \leq R^2 \) and for \( R \leq r \leq k \). Our results refine and generalize certain well-known polynomial inequalities.

Keywords: Polynomial, inequality, zeros.


1. Introduction

Let \( P(z) \) be a polynomial of degree \( n \). It was shown by Govil \[4, Theorem 1\], that if \( P(z) \) has no zeros in \( |z| < 1 \), then for \( 0 \leq r \leq \rho \leq 1 \),

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{r+1}{\rho+1} \right)^n \max_{|z|=\rho} |P(z)|.
\]

Inequality (1.1) is best possible and equality holds for the polynomial \( P(z) = \left( \frac{1+z}{1+\rho} \right)^n \).

As an extension of (1.1), Aziz \[1\] proved that if \( P(z) \neq 0 \) in \( |z| < k \), where \( k \geq 1 \),

\[
\max_{|z|=r<1} |P(z)| \geq \left( \frac{r+k}{1+k} \right)^n \max_{|z|=1} |P(z)|.
\]

and in the case \( k \leq 1 \),

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{r+k}{1+k} \right)^n \max_{|z|=1} |P(z)|, \quad \text{for} \quad 0 \leq r \leq k^2.
\]

Aziz and Mohammad \[2\] obtained the upper bound for the \( \max_{|z|=R \geq 1} |P(z)| \) by proving the following result:
Theorem 1.1. If \( P(z) \) is a polynomial of degree \( n \) such that \( P(z) \neq 0 \) in \( |z| < k \), where \( k \geq 1 \), then

\[
\max_{|z|=R} |P(z)| \leq \left( \frac{R + k}{1 + k} \right)^n \max_{|z|=1} |P(z)|, \quad \text{for } 1 \leq R \leq k^2.
\]

Here equality holds if \( P(z) = (z + k)^n \).

As an extension of (1.2) Bidkham and Dewan [3] proved that:

Theorem 1.2. If \( P(z) \) is a polynomial of degree \( n \) such that \( P'(0) = 0 \) and \( P(z) \neq 0 \) in \( |z| < k \), where \( k \geq 1 \), then for \( 0 \leq r \leq \lambda \leq 1 \),

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{r + k}{\lambda + k} \right)^n \prod_{k=1}^{n} \left( 1 - \frac{(k - \lambda)(\lambda - r)n}{4k^3} \frac{k + r}{k + \lambda} \right)^{n-1} \max_{|z|=\lambda} |P(z)|.
\]

For the case of polynomials having all their zeros in \( |z| \leq k, k > 0 \), we have the following results due to Aziz [1].

Theorem 1.3. If \( P(z) \) is a polynomial of degree \( n \) which has all its zeros in the disk \( |z| \leq k \), where \( k \leq 1 \), then

\[
\max_{|z|=R} |P(z)| \geq \left( \frac{R + k}{1 + k} \right)^n \max_{|z|=1} |P(z)|.
\]

The result is sharp and equality holds for \( P(z) = (z + k)^n \).

Theorem 1.4. If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), where \( k \geq 1 \), then for every \( R \geq k^2 \),

\[
\max_{|z|=R} |P(z)| \geq \left( \frac{R + k}{1 + k} \right)^n \max_{|z|=1} |P(z)|.
\]

The result is sharp with equality for \( P(z) = (z + k)^n \).

Also Mir [5] proved the following theorem for polynomials with \( s \)-fold zeros at the origin.

Theorem 1.5. If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \leq 1 \) with \( s \)-fold zeros at the origin, then for \( R \leq k \leq 1 \),

\[
\max_{|z|=R} |P(z)| \leq R^s \left( \frac{R + k}{1 + k} \right)^n \max_{|z|=1} |P(z)|.
\]

The result is best possible for \( s = n-1 \) and equality holds for \( P(z) = z^{n-1}(z+k) \).
2. Main results

In this paper, we first extend inequalities (1.2), (1.3) and (1.4) to the class of polynomials of degree \( n \) with \( s \)-fold zeros at origin. In fact we prove:

**Theorem 2.1.** If \( P(z) \) is a polynomial of degree \( n \), with \( s \)-fold zeros at origin, \( 0 \leq s \leq n \) where the remaining \( n - s \) zeros in \( |z| \geq k \), then for every \( r^2 \leq Rr \leq k^2 
\begin{equation}
|P(rz)| \geq \left( \frac{r}{R} \right)^s \left( \frac{r + k}{R + k} \right)^{n-s} |P(Rz)|, \quad \text{for} \quad |z| = 1.
\end{equation}

If we use the Maximum Modulus Principle, the result is best possible and equality holds for \( P(z) = z^s(z + k)^{n-s} \).

**Remark 2.2.** If we take \( s = 0, R = 1 \), then Theorem 2.1 reduces to inequality (1.2). Also for \( s = 0, r = 1 \), inequality (2.1) reduces to (1.4). Finally, for \( s = 0, R = 1, k \leq 1 \), Theorem 2.1 reduces to inequality (1.3).

Next, we prove the following result which among other things includes Theorems 1.3 and 1.4 as special case.

**Theorem 2.3.** Let \( P(z) = z^s h(z) \) where \( h(z) \) is a polynomial of degree \( n - s \) having all its zeros in \( |z| \leq k \) and \( 0 \leq s \leq n \). Then for \( k^2 \leq r R \leq R^2 
\begin{equation}
|P(rz)| \leq \left( \frac{r}{R} \right)^s \left( \frac{r + k}{R + k} \right)^{n-s} |P(Rz)|, \quad \text{for} \quad |z| = 1.
\end{equation}

If we use the Maximum Modulus Principle, the result is best possible and equality holds for \( P(z) = z^s(z + k)^{n-s} \).

If we take \( R = 1 \) in Theorem 2.3, then we get the following result:

**Corollary 2.4.** Let \( P(z) = z^s h(z) \) where \( h(z) \) is a polynomial of degree \( n - s \) having all its zeros in \( |z| \leq k \), \( k \leq 1 \) and \( 0 \leq s \leq n \). Then for \( k^2 \leq r \leq 1 
\begin{equation}
\max_{|z|=r} |P(z)| \leq r^s \left( \frac{r + k}{1 + k} \right)^{n-s} \max_{|z|=1} |P(z)|.
\end{equation}

**Remark 2.5.** In general for \( k \leq 1 \), we can not compare Corollary 2.4 with Theorem 1.5 but, one can easily see that for \( k^2 \leq r \leq k \), the bound indicated in Corollary 2.4 is better than the bound obtained in Theorem 1.5.

If we take \( r = 1 \) in Theorem 2.3, then we get the following interesting result:

**Corollary 2.6.** Let \( P(z) = z^s h(z) \) where \( h(z) \) is a polynomial of degree \( n - s \) having all its zeros in \( |z| \leq k \) and \( 0 \leq s \leq n \). Then for \( R \geq \max\{1, k^2 \} 
\begin{equation}
\max_{|z|=R} |P(z)| \geq R^s \left( \frac{R + k}{1 + k} \right)^{n-s} \max_{|z|=1} |P(z)|.
\end{equation}
Remark 2.7. Corollary 2.6 not only includes Theorems 1.3 and 1.4 as special cases but also improves them. In fact:

(1) For $s = 0$, $k \leq 1$, Corollary 2.6 reduces to Theorem 1.3, so for $s \neq 0$, this result improves it.

(2) For $s = 0$, $k \geq 1$, Corollary 2.6 reduces to Theorem 1.4, so for $s \neq 0$, this result improves it also.

Finally, we give the following result which can be thought of as a generalization as well as an improvement of Theorem 1.2.

Theorem 2.8. Let $P(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^{\nu}$ be a polynomial of degree $n$ having all its zeros in $|z| \geq k$. Then for every $r \leq R \leq k$,

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{k + r}{k + R} \right)^n \left[ 1 - \frac{n(k^{n-1} - R^{n-1})(R - r)}{4k^n} \left( \frac{k + r}{k + R} \right)^{n-1} \right]^{-1} \times \left[ \max_{|z|=R} |P(z)| + \frac{n}{\mu} \left( \frac{R^n - r^n}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \right].
\]

By taking $\mu = 1$, we get the following improvement of result due to Bidkham and Dewan [3].

Corollary 2.9. Let $P(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \geq k$. Then for every $r \leq R \leq k$,

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{k + r}{k + R} \right)^n \left[ \max_{|z|=R} |P(z)| + n \left( \frac{R - r}{R + k} \right) \min_{|z|=k} |P(z)| \right].
\]

The result is best possible and equality holds for $P(z) = (z + k)^n$.

By taking $\mu = 2$, we get the following improvement of Theorem 1.2.

Corollary 2.10. Let $P(z) = a_0 + \sum_{\nu=2}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n$ having all its zeros in $|z| \geq k$. Then for every $r \leq R \leq k$,

\[
\max_{|z|=r} |P(z)| \geq \left( \frac{k + r}{k + R} \right)^n \left[ 1 - \frac{n(k-R)(R-r)}{4k^2} \left( \frac{k + r}{k + R} \right)^{n-1} \right]^{-1} \times \left[ \max_{|z|=R} |P(z)| + \frac{n}{2} \left( \frac{R^2 - r^2}{R^2 + k^2} \right) \min_{|z|=k} |P(z)| \right].
\]

If $P(z) = z^s h(z)$ where $h(z) = a_0 + \sum_{\nu=s}^{n} a_{\nu} z^{\nu}$ be a polynomial of degree $n - s$ having all its zeros in $|z| \geq k$, by using Theorem 2.8 for $h(z)$, we get the following interesting result.
Corollary 2.11. Let $P(z) = z^n h(z)$ where $h(z) = a_0 + \sum_{\nu=\mu}^{n-\mu} a_\nu z^\nu$ is a polynomial of degree $n - s$ having all its zeros in $|z| \geq k$. Then for every $r \leq R \leq k$,

$$\max_{|z|=r} |P(z)| \geq \left( \frac{r}{R} \right)^s \left( \frac{k + r}{k + R} \right)^{n-s} \times \left[ 1 - \frac{(n-s)(k^\mu - 1 - R^\mu - 1)(R - r)}{4k^\mu} \left( \frac{k + r}{k + R} \right)^{n-s-1} \right]^{-1} \times \left[ \max_{|z|=R} |P(z)| + \left( \frac{R}{k} \right)^s \frac{n-s}{\mu} \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \right].$$

(2.8)

For $\mu = 1$ in Corollary 2.11, we get the following result:

Corollary 2.12. Let $P(z) = z^n h(z)$ where $h(z)$ is a polynomial of degree $n - s$ having all its zeros in $|z| \geq k$. Then for every $r \leq R \leq k$,

$$\max_{|z|=r} |P(z)| \geq \left( \frac{r}{R} \right)^s \left( \frac{k + r}{k + R} \right)^{n-s} \times \left[ \max_{|z|=R} |P(z)| + (n-s) \left( \frac{R}{k} \right)^s \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \right].$$

(2.9)

Here equality holds for $P(z) = z^n(z + k)^{n-s}$.

3. Lemma

For the proof of Theorem 2.8, we need the following lemma.

Lemma 3.1. If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ is a polynomial of degree $n$, having no zeros in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + k^\mu} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\},$$

(3.1)

with equality for $P(z) = ((z^\mu + k^\mu)/(1 + k^\mu))^{\frac{\mu}{\mu}}$ where $n$ is a multiple of $\mu$.

This lemma is due to Pukhta [6].

4. Proofs of the theorems

Proof of Theorem 2.1. Since $P(z)$ has $s$-fold zeros at the origin and remaining $n - s$ zeros lie in $|z| \geq k$, we can write

$$P(z) = C z^n \prod_{j=1}^{n-s} (z - R_j e^{i\theta_j}),$$
where \( R_j \geq k, \) \( j = 1, 2, 3, \ldots, n - s. \) Therefore, for \( 0 \leq \theta < 2\pi, \) we have

\[
\left| \frac{P(Re^{i\theta})}{P(re^{i\theta})} \right| = \left( \frac{R}{r} \right)^s \prod_{j=1}^{n-s} \left| \frac{Re^{i\theta} - R_j e^{i\theta_j}}{re^{i\theta} - R_j e^{i\theta_j}} \right| \\
= \left( \frac{R}{r} \right)^s \prod_{j=1}^{n-s} \left| \frac{Re^{i(\theta - \theta_j)} - R_j}{re^{i(\theta - \theta_j)} - R_j} \right|.
\]

(4.1)

Now for \( r \geq R, \) \( Rr \geq R_j^2 \) \( (r \leq R, \ rR \leq R_j^2) \) and for each \( \theta, \ 0 \leq \theta < 2\pi, \) it can be easily seen that

\[
\left| \frac{Re^{i(\theta - \theta_j)} - R_j}{re^{i(\theta - \theta_j)} - R_j} \right|^2 = \frac{R^2 + R_j^2 - 2RR_j \cos(\theta - \theta_j)}{r^2 + R_j^2 - 2rR_j \cos(\theta - \theta_j)} \leq \left( \frac{R + R_j}{r + R_j} \right)^2.
\]

(4.2)

Since \( R_j \geq k, \) for all \( j = 1, 2, \ldots, n - s, \) it follows from (4.1) and (4.2) that if \( r^2 \leq rR \leq k^2, \) then

\[
\left| \frac{P(Re^{i\theta})}{P(re^{i\theta})} \right| \leq \left( \frac{R}{r} \right)^s \prod_{j=1}^{n-s} \left( \frac{R + R_j}{r + R_j} \right) \leq \left( \frac{R}{r} \right)^s \left( \frac{R + k}{r + k} \right)^{n-s}.
\]

(4.3)

Hence for \( r^2 \leq rR \leq k^2 \) and for each \( \theta, \ 0 \leq \theta < 2\pi, \) we have

\[
\left| P(Re^{i\theta}) \right| \leq \left( \frac{R}{r} \right)^s \left( \frac{R + k}{r + k} \right)^{n-s} \left| P(re^{i\theta}) \right|.
\]

This completes the proof of Theorem 2.1.  

**Proof of Theorem 2.3.** Similar to previous one for \( (r \leq R, \ Rr \geq R_j^2) \) or \( (r \geq R, \ rR \leq R_j^2) \) and for each \( \theta, \ 0 \leq \theta < 2\pi, \) it can be easily seen that

\[
\left| \frac{Re^{i(\theta - \theta_j)} - R_j}{re^{i(\theta - \theta_j)} - R_j} \right|^2 = \frac{R^2 + R_j^2 - 2RR_j \cos(\theta - \theta_j)}{r^2 + R_j^2 - 2rR_j \cos(\theta - \theta_j)} \geq \left( \frac{R + R_j}{r + R_j} \right)^2.
\]

(4.4)

Since \( R_j \leq k, \) for all \( j = 1, 2, \ldots, n - s, \) it follows from (4.1) that if \( k^2 \leq rR \leq R^2, \) then

\[
\left| \frac{P(Re^{i\theta})}{P(re^{i\theta})} \right| \geq \left( \frac{R}{r} \right)^s \prod_{j=1}^{n-s} \left( \frac{R + R_j}{r + R_j} \right) \geq \left( \frac{R}{r} \right)^s \left( \frac{R + k}{r + k} \right)^{n-s}.
\]

(4.5)

Hence for \( k^2 \leq rR \leq R^2 \) and for each \( \theta, \ 0 \leq \theta < 2\pi, \) we have

\[
\left| P(Re^{i\theta}) \right| \geq \left( \frac{R}{r} \right)^s \left( \frac{R + k}{r + k} \right)^{n-s} \left| P(re^{i\theta}) \right|.
\]

This completes the proof of Theorem 2.3.
Proof of Theorem 2.8. If \( P(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^{\nu} \) has no zeros in \( |z| < k \), and \( r \leq t \leq R \leq k \), then \( H(z) = P(tz) \) has no zeros in \( |z| < k/t \), where \( k/t \geq 1 \). Hence by Lemma 3.1,

\[
(4.6) \quad \max_{|z|=1} |tP'(tz)| \leq \frac{n}{1 + (k/t)^{\mu}} \left\{ \max_{|z|=1} |P(tz)| - \min_{|z|=\frac{k}{r}} |P(tz)| \right\},
\]

which gives

\[
(4.7) \quad \max_{|z|=\varepsilon} |P'(z)| \leq \frac{nt^{\mu-1}}{k^{\mu} + t^{\mu}} \left\{ \max_{|z|=\varepsilon} |P(z)| - \min_{|z|=\varepsilon} |P(z)| \right\},
\]

We have for \( r \leq t \leq R \leq k \), \( 0 \leq \theta < 2\pi \),

\[
|P(Re^{i\theta}) - P(re^{i\theta})| = \left| \int_{r}^{R} e^{i\theta} P'(te^{i\theta}) \, dt \right| \leq \int_{r}^{R} \left| P'(te^{i\theta}) \right| \, dt
\]

\[
\leq \int_{r}^{R} \frac{nt^{\mu-1}}{k^{\mu} + t^{\mu}} \left\{ \max_{|z|=r} |P(z)| - \min_{|z|=r} |P(z)| \right\} \, dt \quad \text{(by (4.7))}
\]

\[
\leq \int_{r}^{R} \frac{nt^{\mu-1}}{k^{\mu} + t^{\mu}} \left\{ \left( \frac{k + t}{k + r} \right)^{n} \max_{|z|=\varepsilon} |P(z)| - \min_{|z|=\varepsilon} |P(z)| \right\} \, dt
\]

\[
= \frac{n}{(k + r)^{n}} \max_{|z|=\varepsilon} |P(z)| \int_{r}^{R} \frac{t^{\mu-1}(k + t)^{n}}{k^{\mu} + t^{\mu}} \, dt
\]

which gives for \( r \leq R \leq k \),

\[
\max_{|z|=R} |P(z)| \leq \left\{ 1 + \frac{n}{(k + r)^{n}} \int_{r}^{R} \frac{t^{\mu-1}(k + t)^{n}}{k^{\mu} + t^{\mu}} \, dt \right\} \max_{|z|=\varepsilon} |P(z)|
\]

\[
- \min_{|z|=\varepsilon} |P(z)| \int_{r}^{R} \frac{nt^{\mu-1}}{k^{\mu} + t^{\mu}} \, dt
\]

\[
\leq \left\{ 1 + \frac{n}{(k + r)^{n}} \int_{r}^{R} \frac{R^{\mu-1}(k + R)^{n}}{k^{\mu} + R^{\mu}} \int_{r}^{R} (k + t)^{n-1} \, dt \right\} \max_{|z|=\varepsilon} |P(z)|
\]

\[
- \frac{n}{k^{\mu} + R^{\mu}} \min_{|z|=\varepsilon} |P(z)| \int_{r}^{R} t^{\mu-1} \, dt
\]

\[
= \left[ 1 + \frac{R^{\mu-1}(k + R)^{n}}{(k + r)^{n}} \left( 1 - \left( \frac{k + r}{k + R} \right)^{n} \right) \right] \max_{|z|=\varepsilon} |P(z)|
\]

\[
- \frac{n}{\mu} \left( \frac{R^{\mu} - r^{\mu}}{k^{\mu} + R^{\mu}} \right) \min_{|z|=\varepsilon} |P(z)|
\]

\[
= \left[ k^{\mu} - \frac{R^{\mu-1}k}{k^{\mu} + R^{\mu}} + \frac{R^{\mu-1}(k + R)}{k^{\mu} + R^{\mu}} \left( \frac{k + R}{k + r} \right)^{n} \right] \max_{|z|=\varepsilon} |P(z)|
\]
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\[ -\frac{n}{\mu} \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \]
\[ = \left( \frac{k + R}{k + r} \right)^n \left[ 1 - \frac{k^\mu - R^\mu - 1 k}{k^\mu + R^\mu} \left\{ 1 - \left( \frac{k + R}{k + r} \right)^n \right\} \right] \max_{|z|=r} |P(z)| \]
\[ -\frac{n}{\mu} \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \]
\[ = \left( \frac{k + R}{k + r} \right)^n \left[ 1 - \frac{n(k^\mu - R^\mu - 1 k)(R - r)}{(k^n + R^n)(k + R)} \left\{ 1 - \left( \frac{k + R}{k + r} \right)^n \right\} \right] \max_{|z|=r} |P(z)| \]
\[ -\frac{n}{\mu} \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \]
\[ \leq \left( \frac{k + R}{k + r} \right)^n \left[ 1 - \frac{n(k^\mu - R^\mu - 1 k)(R - r)}{4k^{\mu+1}} \left( \frac{k + r}{k + R} \right)^{n-1} \right] \max_{|z|=r} |P(z)| \]
\[ -\frac{n}{\mu} \left( \frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \]

This completes the proof of Theorem 2.8.

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