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SOME COMPACT GENERALIZATION OF INEQUALITIES FOR POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. Let $p(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree at most $n - s$ having all its zeros in $|z| \geq k$ or in $|z| \leq k$. In this paper we obtain some new results about the dependence of $|p(Rz)|$ on $|p(rz)|$ for $r^2 \leq rR \leq k^2$, $k^2 \leq rR \leq R^2$ and for $R \leq r \leq k$. Our results refine and generalize certain well-known polynomial inequalities.
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MSC(2010): Primary: 30A10; Secondary: 30C10, 30D15.

1. Introduction

Let $P(z)$ be a polynomial of degree n. It was shown by Govil [4, Theorem 1], that if $P(z)$ has no zeros in $|z| < 1$, then for $0 \leq r \leq \rho \leq 1$,

$$\max_{|z|=r} |P(z)| \geq \left(\frac{r+1}{\rho+1}\right)^n \max_{|z|=\rho} |P(z)|. \tag{1.1}$$

Inequality (1.1) is best possible and equality holds for the polynomial $P(z) = \left(\frac{1+z}{1+\rho}\right)^n$.
As an extension of (1.1), Aziz [1] proved that if $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$,

$$\max_{|z|=r<1} |P(z)| \geq \left(\frac{r+k}{1+k}\right)^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

and in the case $k \leq 1$,

$$\max_{|z|=r} |P(z)| \geq \left(\frac{r+k}{1+k}\right)^n \max_{|z|=1} |P(z)|, \qquad \text{for } 0 \leq r \leq k^2. \tag{1.3}$$

Aziz and Mohammad [2] obtained the upper bound for the $\max_{|z|=R\geq 1} |P(z)|$ by proving the following result:

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Theorem 1.1. *If $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then*

$$\max_{|z|=R} |P(z)| \leq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)|, \quad \text{for} \quad 1 \leq R \leq k^2. \tag{1.4}$$

Here equality holds if $P(z) = (z+k)^n$.

As an extension of (1.2) Bidkham and Dewan [3] proved that:

Theorem 1.2. *If $P(z)$ is a polynomial of degree n such that $P'(0) = 0$ and $P(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then for $0 \leq r \leq \lambda \leq 1$,*

$$\begin{aligned} \max_{|z|=r} |P(z)| \geq & \left(\frac{r+k}{\lambda+k}\right)^n \\ & \times \left[1 - \frac{(k-\lambda)(\lambda-r)n}{4k^3}\left(\frac{k+r}{k+\lambda}\right)^{n-1}\right]^{-1} \max_{|z|=\lambda} |P(z)|. \end{aligned} \tag{1.5}$$

For the case of polynomials having all their zeros in $|z| \leq k, k > 0$, we have the following results due to Aziz [1].

Theorem 1.3. *If $P(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq k$, where $k \leq 1$, then*

$$\max_{|z|=R>1} |P(z)| \geq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is sharp and equality holds for $P(z) = (z+k)^n$.

Theorem 1.4. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 1$, then for every $R \geq k^2$,*

$$\max_{|z|=R} |P(z)| \geq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is sharp with equality for $P(z) = (z+k)^n$.

Also Mir [5] proved the following theorem for polynomials with s-fold zeros at the origin.

Theorem 1.5. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin, then for $R \leq k \leq 1$,*

$$\max_{|z|=R} |P(z)| \leq R^s \left(\frac{R+k}{1+k}\right) \max_{|z|=1} |P(z)|. \tag{1.8}$$

The result is best possible for $s = n-1$ and equality holds for $P(z) = z^{n-1}(z+k)$.

2. Main results

In this paper, we first extend inequalities (1.2), (1.3) and (1.4) to the class of polynomials of degree n with s-fold zeros at origin. In fact we prove:

Theorem 2.1. *If $P(z)$ is a polynomial of degree n, with s-fold zeros at origin, $0 \le s \le n$ where the remaining $n-s$ zeros in $|z| \ge k$, then for every $r^2 \le Rr \le k^2$,*

$$|P(rz)| \ge \left(\frac{r}{R}\right)^s \left(\frac{r+k}{R+k}\right)^{n-s} |P(Rz)|, \qquad \text{for} \qquad |z| = 1. \tag{2.1}$$

If we use the Maximum Modulus Principle, the result is best possible and equality holds for $P(z) = z^s(z+k)^{n-s}$.

Remark 2.2. If we take $s = 0$, $R = 1$, then Theorem 2.1 reduces to inequality (1.2). Also for $s = 0$, $r = 1$, inequality (2.1) reduces to (1.4). Finally, for $s = 0$, $R = 1$, $k \le 1$, Theorem 2.1 reduces to inequality (1.3).

Next, we prove the following result which among other things includes Theorems 1.3 and 1.4 as special case.

Theorem 2.3. *Let $P(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \le k$ and $(0 \le s \le n)$. Then for $k^2 \le rR \le R^2$,*

$$|P(rz)| \le \left(\frac{r}{R}\right)^s \left(\frac{r+k}{R+k}\right)^{n-s} |P(Rz)|, \qquad \text{for} \qquad |z| = 1. \tag{2.2}$$

If we use the Maximum Modulus Principle, the result is best possible and equality holds for $P(z) = z^s(z+k)^{n-s}$.

If we take $R = 1$ in Theorem 2.3, then we get the following result:

Corollary 2.4. *Let $P(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \le k$, $k \le 1$ and $(0 \le s \le n)$. Then for $k^2 \le r \le 1$,*

$$\max_{|z|=r} |P(z)| \le r^s \left(\frac{r+k}{1+k}\right)^{n-s} \max_{|z|=1} |P(z)|. \tag{2.3}$$

Remark 2.5. In general for $k \le 1$, we can not compare Corollary 2.4 with Theorem 1.5 but, one can easily see that for $k^2 \le r \le k$, the bound indicated in Corollary 2.4 is better than the bound obtained in Theorem 1.5.

If we take $r = 1$ in Theorem 2.3, then we get the following interesting result:

Corollary 2.6. *Let $P(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \le k$ and $(0 \le s \le n)$. Then for $R \ge \max\{1,\ k^2\}$,*

$$\max_{|z|=R} |P(z)| \ge R^s \left(\frac{R+k}{1+k}\right)^{n-s} \max_{|z|=1} |P(z)|. \tag{2.4}$$

Remark 2.7. Corollary 2.6 not only includes Theorems 1.3 and 1.4 as special cases but also improves them. In fact:

(1) For $s = 0$, $k \leq 1$, Corollary 2.6 reduces to Theorem 1.3, so for $s \neq 0$, this result improves it.

(2) For $s = 0$, $k \geq 1$, Corollary 2.6 reduces to Theorem 1.4, so for $s \neq 0$, this result improves it also.

Finally, we give the following result which can be thought of as a generalization as well as an improvement of Theorem 1.2.

Theorem 2.8. *Let* $P(z) = a_0 + \Sigma_{\nu=\mu}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *having all its zeros in* $|z| \geq k$*. Then for every* $r \leq R \leq k$,

$$\max_{|z|=r}|P(z)| \geq \left(\frac{k+r}{k+R}\right)^n \times \left[1 - \frac{n(k^{\mu-1} - R^{\mu-1})(R-r)}{4k^\mu}\left(\frac{k+r}{k+R}\right)^{n-1}\right]^{-1} \times \left[\max_{|z|=R}|P(z)| + \frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|\right]. \tag{2.5}$$

By taking $\mu = 1$, we get the following improvement of result due to Bidkham and Dewan [3].

Corollary 2.9. *Let* $P(z)$ *be a polynomial of degree* n *having all its zeros in* $|z| \geq k$*. Then for every* $r \leq R \leq k$,

$$\max_{|z|=r}|P(z)| \geq \left(\frac{k+r}{k+R}\right)^n \left[\max_{|z|=R}|P(z)| + n\left(\frac{R-r}{R+k}\right)\min_{|z|=k}|P(z)|\right]. \tag{2.6}$$

The result is best possible and equality holds for $P(z) = (z+k)^n$.

By taking $\mu = 2$, we get the following improvement of Theorem 1.2.

Corollary 2.10. *Let* $P(z) = a_0 + \Sigma_{\nu=2}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \geq k$*. Then for every* $r \leq R \leq k$,

$$\max_{|z|=r}|P(z)| \geq \left(\frac{k+r}{k+R}\right)^n \left[1 - \frac{n(k-R)(R-r)}{4k^2}\left(\frac{k+r}{k+R}\right)^{n-1}\right]^{-1} \times \left[\max_{|z|=R}|P(z)| + \frac{n}{2}\left(\frac{R^2 - r^2}{R^2 + k^2}\right)\min_{|z|=k}|P(z)|\right]. \tag{2.7}$$

If $P(z) = z^s h(z)$ where $h(z) = a_0 + \Sigma_{\nu=\mu}^{n-s} a_\nu z^\nu$ be a polynomial of degree $n - s$ having all its zeros in $|z| \geq k$, by using Theorem 2.8 for $h(z)$, we get the following interesting result.

Corollary 2.11. *Let* $P(z) = z^s h(z)$ *where* $h(z) = a_0 + \Sigma_{\nu=\mu}^{n-s} a_\nu z^\nu$ *is a polynomial of degree* $n-s$ *having all its zeros in* $|z| \geq k$*. Then for every* $r \leq R \leq k$,

$$
\begin{aligned}
\max_{|z|=r} |P(z)| \geq & \left(\frac{r}{R}\right)^s \left(\frac{k+r}{k+R}\right)^{n-s} \\
& \times \left[1 - \frac{(n-s)(k^{\mu-1} - R^{\mu-1})(R-r)}{4k^\mu} \left(\frac{k+r}{k+R}\right)^{n-s-1}\right]^{-1} \\
& \times \left[\max_{|z|=R} |P(z)| + \left(\frac{R}{k}\right)^s \frac{n-s}{\mu} \left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right) \min_{|z|=k} |P(z)|\right].
\end{aligned}
\tag{2.8}
$$

For $\mu = 1$ in Corollary 2.11, we get the following result:

Corollary 2.12. *Let* $P(z) = z^s h(z)$ *where* $h(z)$ *is a polynomial of degree* $n-s$ *having all its zeros in* $|z| \geq k$*. Then for every* $r \leq R \leq k$,

$$
\begin{aligned}
\max_{|z|=r} |P(z)| \geq & \left(\frac{r}{R}\right)^s \left(\frac{k+r}{k+R}\right)^{n-s} \\
& \times \left[\max_{|z|=R} |P(z)| + (n-s) \left(\frac{R}{k}\right)^s \left(\frac{R-r}{R+k}\right) \min_{|z|=k} |P(z)|\right].
\end{aligned}
\tag{2.9}
$$

Here equality holds for $P(z) = z^s (z+k)^{n-s}$.

3. **Lemma**

For the proof of Theorem 2.8, we need the following lemma.

Lemma 3.1. *If* $P(z) = a_0 + \Sigma_{\upsilon=\mu}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n*, having no zeros in* $|z| < k,\ k \geq 1$*, then*

$$
\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^\mu} \left\{\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)|\right\},
\tag{3.1}
$$

with equality for $P(z) = ((z^\mu + k^\mu)/(1+k^\mu))^{\frac{n}{\mu}}$ *where* n *is a multiple of* μ .

This lemma is due to Pukhta [6].

4. **Proofs of the theorems**

Proof of Theorem 2.1. Since $P(z)$ has s-fold zeros at the origin and remaining $n-s$ zeros lie in $|z| \geq k$, we can write

$$
P(z) = C z^s \prod_{j=1}^{n-s} \left(z - R_j e^{i\theta_j}\right),
$$

where $R_j \geq k,\ j = 1, 2, 3, \ldots, n-s$. Therefore, for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \left(\frac{R}{r}\right)^s \prod_{j=1}^{n-s} \left|\frac{Re^{i\theta} - R_j e^{i\theta_j}}{re^{i\theta} - R_j e^{i\theta_j}}\right| \\ &= \left(\frac{R}{r}\right)^s \prod_{j=1}^{n-s} \left|\frac{Re^{i(\theta-\theta_j)} - R_j}{re^{i(\theta-\theta_j)} - R_j}\right|. \end{aligned} \tag{4.1}$$

Now for $r \geq R,\ Rr \geq R_j^2$ ($r \leq R,\ \ rR \leq R_j^2$) and for each $\theta,\ 0 \leq \theta < 2\pi$, it can be easily seen that

$$\left|\frac{Re^{i(\theta-\theta_j)} - R_j}{re^{i(\theta-\theta_j)} - R_j}\right|^2 = \frac{R^2 + R_j^2 - 2RR_j\cos(\theta-\theta_j)}{r^2 + R_j^2 - 2rR_j\cos(\theta-\theta_j)} \leq \left(\frac{R+R_j}{r+R_j}\right)^2. \tag{4.2}$$

Since $R_j \geq k$, for all $j = 1, 2, \ldots, n-s$, it follows from (4.1) and (4.2) that if $r^2 \leq rR \leq k^2$, then

$$\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| \leq \left(\frac{R}{r}\right)^s \prod_{j=1}^{n-s} \left(\frac{R+R_j}{r+R_j}\right) \leq \left(\frac{R}{r}\right)^s \left(\frac{R+k}{r+k}\right)^{n-s}. \tag{4.3}$$

Hence for $r^2 \leq rR \leq k^2$ and for each $\theta,\ 0 \leq \theta < 2\pi$, we have

$$|P(Re^{i\theta})| \leq \left(\frac{R}{r}\right)^s \left(\frac{R+k}{r+k}\right)^{n-s} |P(re^{i\theta})|.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.3. Similar to previous one for ($r \leq R,\ Rr \geq R_j^2$) or ($r \geq R,\ \ rR \leq R_j^2$) and for each $\theta,\ 0 \leq \theta < 2\pi$, it can be easily seen that

$$\left|\frac{Re^{i(\theta-\theta_j)} - R_j}{re^{i(\theta-\theta_j)} - R_j}\right|^2 = \frac{R^2 + R_j^2 - 2RR_j\cos(\theta-\theta_j)}{r^2 + R_j^2 - 2rR_j\cos(\theta-\theta_j)} \geq \left(\frac{R+R_j}{r+R_j}\right)^2. \tag{4.4}$$

Since $R_j \leq k$, for all $j = 1, 2, \ldots, n-s$, it follows from (4.1) that if $k^2 \leq rR \leq R^2$, then

$$\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| \geq \left(\frac{R}{r}\right)^s \prod_{j=1}^{n-s} \left(\frac{R+R_j}{r+R_j}\right) \geq \left(\frac{R}{r}\right)^s \left(\frac{R+k}{r+k}\right)^{n-s}. \tag{4.5}$$

Hence for $k^2 \leq rR \leq R^2$ and for each $\theta,\ 0 \leq \theta < 2\pi$, we have

$$|P(Re^{i\theta})| \geq \left(\frac{R}{r}\right)^s \left(\frac{R+k}{r+k}\right)^{n-s} |P(re^{i\theta})|.$$

This completes the proof of Theorem 2.3.

Proof of Theorem 2.8. If $P(z) = a_0 + \Sigma_{\nu=\mu}^{n} a_\nu z^\nu$ has no zeros in $|z| < k$, and $r \le t \le R \le k$, then $H(z) = P(tz)$ has no zeros in $|z| < k/t$, where $k/t \ge 1$. Hence by Lemma 3.1,

$$\max_{|z|=1} |tP'(tz)| \le \frac{n}{1+(k/t)^\mu} \left\{ \max_{|z|=1} |P(tz)| - \min_{|z|=\frac{k}{t}} |P(tz)| \right\}, \tag{4.6}$$

which gives

$$\max_{|z|=t} |P'(z)| \le \frac{nt^{\mu-1}}{k^\mu + t^\mu} \left\{ \max_{|z|=t} |P(z)| - \min_{|z|=k} |P(z)| \right\}. \tag{4.7}$$

We have for $r \le t \le R \le k,\ 0 \le \theta < 2\pi$,

$$\begin{aligned} |P(Re^{i\theta}) - P(re^{i\theta})| &= \left| \int_r^R e^{i\theta} P'(te^{i\theta}) dt \right| \le \int_r^R \left| P'(te^{i\theta}) \right| dt \\ &\le \int_r^R \frac{nt^{\mu-1}}{k^\mu + t^\mu} \left\{ \max_{|z|=t} |P(z)| - \min_{|z|=k} |P(z)| \right\} dt \qquad \text{(by (4.7))} \\ &\le \int_r^R \frac{nt^{\mu-1}}{k^\mu + t^\mu} \left\{ \left(\frac{k+t}{k+r} \right)^n \max_{|z|=r} |P(z)| - \min_{|z|=k} |P(z)| \right\} dt \\ &= \frac{n}{(k+r)^n} \max_{|z|=r} |P(z)| \int_r^R \frac{t^{\mu-1}(k+t)^n}{k^\mu + t^\mu} dt \\ &- \min_{|z|=k} |P(z)| \int_r^R \frac{nt^{\mu-1}}{k^\mu + t^\mu} dt, \end{aligned}$$

which gives for $r \le R \le k$,

$$\begin{aligned} &\max_{|z|=R} |P(z)| \le \left\{ 1 + \frac{n}{(k+r)^n} \int_r^R \frac{t^{\mu-1}(k+t)^n}{k^\mu + t^\mu} dt \right\} \max_{|z|=r} |P(z)| \\ &- \min_{|z|=k} |P(z)| \int_r^R \frac{nt^{\mu-1}}{k^\mu + t^\mu} dt \\ &\le \left\{ 1 + \frac{n}{(k+r)^n} \frac{R^{\mu-1}(k+R)}{k^\mu + R^\mu} \int_r^R (k+t)^{n-1} dt \right\} \max_{|z|=r} |P(z)| \\ &- \frac{n}{k^\mu + R^\mu} \min_{|z|=k} |P(z)| \int_r^R t^{\mu-1} dt \\ &= \left[1 + \frac{R^{\mu-1}(k+R)(k+R)^n}{(k+r)^n (k^\mu + R^\mu)} \left\{ 1 - \left(\frac{k+r}{k+R} \right)^n \right\} \right] \max_{|z|=r} |P(z)| \\ &- \frac{n}{\mu} \left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu} \right) \min_{|z|=k} |P(z)| \\ &= \left[\frac{k^\mu - R^{\mu-1}k}{k^\mu + R^\mu} + \frac{R^{\mu-1}(k+R)}{k^\mu + R^\mu} \left(\frac{k+R}{k+r} \right)^n \right] \max_{|z|=r} |P(z)| \end{aligned}$$

$$-\frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|$$
$$=\left(\frac{k+R}{k+r}\right)^n\left[1-\frac{k^\mu - R^{\mu-1}k}{k^\mu + R^\mu}\left\{1-\left(\frac{k+r}{k+R}\right)^n\right\}\right]\max_{|z|=r}|P(z)|$$
$$-\frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|$$
$$=\left(\frac{k+R}{k+r}\right)^n\left[1-\frac{(k^\mu - R^{\mu-1}k)(R-r)}{(k^\mu + R^\mu)(k+R)\left(1-\frac{k+r}{k+R}\right)}\left\{1-\left(\frac{k+r}{k+R}\right)^n\right\}\right]\max_{|z|=r}|P(z)|$$
$$-\frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|$$
$$\leq\left(\frac{k+R}{k+r}\right)^n\left[1-\frac{n(k^\mu - R^{\mu-1}k)(R-r)}{(k^\mu + R^\mu)(k+R)}\left(\frac{k+r}{k+R}\right)^{n-1}\right]\max_{|z|=r}|P(z)|$$
$$-\frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|$$
$$\leq\left(\frac{k+R}{k+r}\right)^n\left[1-\frac{n(k^\mu - R^{\mu-1}k)(R-r)}{4k^{\mu+1}}\left(\frac{k+r}{k+R}\right)^{n-1}\right]\max_{|z|=r}|P(z)|$$
$$-\frac{n}{\mu}\left(\frac{R^\mu - r^\mu}{R^\mu + k^\mu}\right)\min_{|z|=k}|P(z)|.$$

This completes the proof of Theorem 2.8.

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