ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 1, pp. 171–192

Title:

Finite p-groups and centralizers of non-cyclic abelian subgroups

Author(s):

J. Wang and X. Guo

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 1, pp. 171–192
Online ISSN: 1735-8515

FINITE p-GROUPS AND CENTRALIZERS OF NON-CYCLIC ABELIAN SUBGROUPS

J. WANG AND X. GUO*

(Communicated by Ali Reza Ashrafi)

Abstract. A p-group G is called a $\mathcal{CAC}$-p-group if $C_G(H)/H$ is cyclic for every non-cyclic abelian subgroup H in G with $H \nleq Z(G)$. In this paper, we give a complete classification of finite $\mathcal{CAC}$-p-groups.
Keywords: Finite p-group, centralizer, normal rank, cyclic group.
MSC(2010): Primary: 20D15.

1. Introduction

All groups considered in this paper are finite.

Let H be an abelian subgroup of a group G. Then

$$1 \leq H \leq C_G(H) \leq G$$

is always true. It is clear that G is abelian if and only if $|G : C_G(H)| = 1$ for every abelian subgroup H. So it is interesting to investigate the structure of a group G if $|G : C_G(H)|$ is small for every abelian subgroup H. In fact, K. Ishikawa in [4,5] investigates the structure of a p-group G with $|G : C_G(x)| = p$ and the structure of a p-group G with $|G : C_G(x)| = p^2$ for every $x \in G$ and gives the classifications for these kind of p-groups. On the other hand, it is also interesting to investigate the structure of a group G if $|C_G(H) : H|$ is small for every abelian subgroup H. In fact, Li and Zhang in [6] investigate the structure of a p-group G with $|C_G(x) : \langle x\rangle| \leq p^k$ for $k = 1$ or 2 and $p > 2$. Moreover, many authors investigated the structure of groups by using the some kind of index of subgroups, for example [10–12]. Now it is natural to ask the following question, which is proposed by Berkovich in [1]:

Article electronically published on February 22, 2017.
Received: 25 April 2015, Accepted: 2 November 2015.
*Corresponding author.

Question 1: Classify the p-groups G such that $C_G(H)/H$ is cyclic for every noncentral cyclic subgroup H in G.

Question 1 has been answered in [9]. We may also ask the following questions:

Question 2: How about the structure of a p-group G with $C_G(H)/H$ cyclic for every abelian subgroup H in G with $H \nleq Z(G)$?

Question 3: How about the structure of a p-group G with $C_G(H)/H$ cyclic for every non-cyclic abelian subgroup H in G with $H \nleq Z(G)$?

It is clear that Question 3 is more general than Question 2. Furthermore, we have the following proposition.

Proposition 1.1. *Let G be a non-abelian p-group. If $C_G(x)/\langle x\rangle$ is cyclic for every non-central element $x \in G$. Then, for every non-cyclic abelian subgroup H in G with $H \nleq Z(G)$, $C_G(H)/H$ is cyclic.*

In fact, let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$. Then there exists an element $x \in H$ with $x \notin Z(G)$. By the hypothesis, $C_G(x)/\langle x\rangle$ is cyclic. Noticing that $C_G(x)$ is abelian and $H \trianglelefteq C_G(x)$, we see $C_G(x)/H$ is cyclic. It follows from $C_G(H) \leq C_G(x)$ that $C_G(H)/H$ is cyclic.

Remark 1.2. Assume $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [b, a] = c, [c, a] = [c, b] = 1\rangle$. Then it is easy to see that $C_G(H)/H$ is cyclic for every non-cyclic abelian subgroup H in G with $H \nleq Z(G)$. However, $a \notin Z(G)$ and $C_G(a)/\langle a\rangle = \langle a, b^2, c\rangle/\langle a\rangle$ is not cyclic. So Question 3 is more general than Question 2.

In this paper we hope to investigate the structure of a p-group G in which $C_G(H)/H$ is cyclic for every non-cyclic abelian subgroup H in G with $H \nleq Z(G)$. For convenience, we call this kind of p-groups $\mathcal{CAC}$-p-groups.

It is clear that every abelian p-group must be a $\mathcal{CAC}$-p-group. So in the following $\mathcal{CAC}$-p-groups means non-abelian $\mathcal{CAC}$-p-groups.

2. **Preliminaries**

For convenience, we first introduce some notions and notations.

Let G be a p-group. Then $r(G) = \max\{\log_p|E| \mid E$ is an elementary abelian subgroup in G $\}$ and $r_n(G) = \max\{\log_p|E| \mid E$ is an elementary abelian normal subgroup in G $\}$ are called the rank and the normal rank of G respectively.

We use $\mathrm{M}_p(m, n)$ to denote the p-group

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}}\rangle, \text{ where } m \geq 2,$$

and $\mathrm{M}_p(m,n,1)$ to denote the p-group

$$\langle a,b,c \mid a^{p^m}=b^{p^n}=c^p=1,[a,b]=c,[c,a]=[c,b]=1\rangle,$$

where $m\geq n$, and $m+n\geq 3$ if $p=2$. We also use C_{p^m} and $C_{p^m}^n$ to denote the cyclic group of order p^m and the direct product of n cyclic groups of order p^m respectively. If H and K are groups, then $H*K$ denotes a central product of H and K. $M\lessdot G$ means M is a maximal subgroup of G. For other notation and terminology the reader is referred to [3].

Lemma 2.1. [8, Lemma 2.2] *Let G be a p-group. Then the following conditions are equivalent.*
(1) *G is a minimal non-abelian p-group;*
(2) *$d(G)=2$ and $|G'|=p$;*
(3) *$d(G)=2$ and $\Phi(G)=Z(G)$.*

Lemma 2.2. *Let G be a p-group and $c(G)=2$. Then G' is elementary abelian if and only if $G/Z(G)$ is elementary abelian.*

Proof. Since $c(G)=2$, G' is elementary abelian if and only if $[a^p,b]=[a,b]^p=1$ for all $a,b\in G$, and $[a^p,b]=[a,b]^p=1$ for all $a,b\in G$ if and only if $G/Z(G)$ is elementary abelian. Thus the lemma is true. □

Lemma 2.3. [1, Section 1, Lemma 1.1] *If a non-abelian p-group G has an abelian maximal subgroup, then $|G|=p|G'||Z(G)|$.*

Lemma 2.4. ([7]) *Let p be an odd prime and let G be a metacyclic p-group. Then there are non-negative integers r,s,t,u with $r\geq 1, u\leq r$ such that $G=\langle a,b \mid a^{p^{r+s+u}}=1, b^{p^{r+s+t}}=a^{p^{r+s}}, a^b=a^{1+p^r}\rangle$. Furthermore, different values of the parameters r,s,t and u with the above conditions give non-isomorphic metacyclic p-groups.*

Lemma 2.5. [2, Theorem 4.1] *Let G be a group of order p^n with $p>2$ and $n\geq 5$. If $r_n(G)=2$. Then G is one of the following groups:*
(1) *G is metacyclic;*
(2) *$G\cong \mathrm{M}_p(1,1,1)*C_{p^{n-2}}$;*
(3) *G is a 3-group of maximal class of order $\geq 3^5$;*
(4) *$G=\langle a,x,y \mid a^{p^{n-2}}=x^p=y^p=1,[a,x]=y,[x,y]=a^{ip^{n-3}},[y,a]=1\rangle$, $i=1$ or σ, where σ is a fixed square non-residue modulo p.*

Lemma 2.6. *Let G be a p-group. Then*
(1) [1, Section 7, Theorem 7.1] *If $K_{p-1}(G)$ is cyclic, then G is regular.*
(2) [1, Section 9, Exercise 10] *Let G be a 3-group of maximal class. Then $G_1=C_G(K_2(G)/K_4(G))$ is abelian or metacyclic minimal non-abelian.*
(3) [1, Section 9, Exercise 1(c)] *Let G be a maximal class p-group of order p^n. If $p>2$ and $n>3$. Then G has no cyclic normal subgroups of order p^2.*

(4) [1, Section 10, Corollary 10.2] *Suppose that N is a normal subgroup of G, and A is a maximal G-invariant abelian subgroup of N with* $\exp(A) = p^n, p^n > 2$. *Then* $\Omega_n(C_N(A)) = A$.

(5) [1, Section 41, Remarks.2] *G is metacyclic if and only if $\Omega_2(G)$ is metacyclic.*

Lemma 2.7. *Let G be a p-group of order p^n and $n \geq 4$. Then there exists a maximal subgroup M of G such that M is not of maximal class.*

Proof. Let $N \trianglelefteq G$ and $|N| = p^2$. Then $G/C_G(N) \lesssim Aut(N)$. Thus $|G : C_G(N)| \leq p$. Let $M \leq C_G(N)$ such that $N \leq M$ and $|G : M| = p$. Since $|Z(M)| \geq |N| = p^2$, M is not of maximal class. □

3. Some properties of $\mathcal{CAC}$-p-groups

In this section we discuss the properties of $\mathcal{CAC}$-p-groups which will be used later.

Lemma 3.1. *If G is a $\mathcal{CAC}$-p-group, then $r(G) \leq 3$.*

Proof. If not, then there exists $A \leq G$ and $A \cong C_p^4$. If $A \nleq Z(G)$, then there exist $a \in A \setminus Z(G)$ and $b \in A$ such that $\langle a, b\rangle$ is not cyclic. Since A is abelian, we see $A \leq C_G(\langle a, b\rangle)$ and $C_G(\langle a, b\rangle)/\langle a, b\rangle$ is not cyclic, in contradiction to the hypothesis. If $A \leq Z(G)$, then, for any $x \in G \setminus Z(G)$, there exists $a \in A$ such that $\langle a, x\rangle$ is not cyclic. Since $\langle A, x\rangle \leq C_G(\langle a, x\rangle)$ and $\langle A, x\rangle/\langle a, x\rangle$ is not cyclic, we see $C_G(\langle a, x\rangle)/\langle a, x\rangle$ is not cyclic, another contradiction. □

Lemma 3.2. *Let G be a $\mathcal{CAC}$-p-group with $r(G) = 3$ and $A \leq G$ with $A \cong C_p^3$. If $A \nleq Z(G)$. Then $C_G(A) = A$. If $A \leq Z(G)$, then $\Omega_1(G) = A$ and $\mho_1(G) \leq Z(G)$.*

Proof. Assume $A = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$. Then Lemma 3.1 implies $\Omega_1(C_G(A)) = A$.

If $A \nleq Z(G)$, then, we claim $C_G(A) = \Omega_1(C_G(A))$. Otherwise, there exists $x \in C_G(A) \setminus \Omega_1(C_G(A))$ with $o(x) = p^k$ and $k \geq 2$. Thus $x^{p^{k-1}} \in \Omega_1(C_G(A)) = A$. On the other hand, since $A \nleq Z(G)$, we may assume that $a \notin Z(G)$. If $\langle x^{p^{k-1}}\rangle \neq \langle a\rangle$, then $\langle a, x^{p^{k-1}}\rangle$ is not cyclic and $\langle x^{p^{k-1}}, a\rangle \nleq Z(G)$. Thus, by the hypotheses of the lemma, $\langle a, b, c, x\rangle/\langle a, x^{p^{k-1}}\rangle$ is a cyclic group. However, it is impossible. If $\langle x^{p^{k-1}}\rangle = \langle a\rangle$, then, by the hypotheses of the lemma, $\langle a, b, c, x\rangle/\langle a, b\rangle$ is cyclic. It is also impossible. So $C_G(A) = \Omega_1(C_G(A)) = A$.

If $A \leq Z(G)$, then $\Omega_1(G) = \Omega_1(C_G(A)) = A$. For any $x \in G$ with $o(x) = p^k$, if $x^p \notin Z(G)$, then $k \geq 3$. Furthermore, for any $y \in A \setminus \langle x^{p^{k-1}}\rangle$, $\langle x^p, y\rangle$ is not cyclic and $\langle x^p, y\rangle$ is abelian. By the hypotheses, $\langle a, b, c, x\rangle/\langle x^p, y\rangle$ is cyclic. However, it is impossible. Hence $x^p \in Z(G)$. So $\mho_1(G) \leq Z(G)$ and the lemma is proved. □

Lemma 3.3. *Suppose that G is a metacyclic p-group and $p > 2$. Then G is a $\mathcal{CAC}$-p-group if and only if G is a minimal non-abelian group.*

Proof. If G is a minimal non-abelian group, then $Z(G) = \Phi(G)$ by Lemma 2.1(3). Thus, for every non-cyclic abelian subgroup H in G with $H \not\leq Z(G)$, we have $H \not\leq \Phi(G)$ and so $H \not\leq \Phi(C_G(H))$. Since G is metacyclic, $C_G(H)$ is metacyclic. Thus $d(C_G(H)) \leq 2$. It follows that there exists $g \in G$ such that $C_G(H) = \langle H, g\rangle$. Hence $C_G(H)/H$ is cyclic. That is, G is a $\mathcal{CAC}$-p-group.

Conversely, let G be a $\mathcal{CAC}$-p-group. By Lemma 2.4, we may assume $G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r}\rangle$, where r, s, t, u are non-negative integers with $r \geq 1, u \leq r$. By calculation, we get $Z(G) = \langle a^{p^{s+u}}, b^{p^{s+u}}\rangle$. If $a^p \in Z(G)$, then $|G'| = p$. By Lemma 2.1, G is minimal non-abelian. Thus we may assume $a^p \notin Z(G)$. If $\langle a^p, b^{p^{s+u}}\rangle$ is cyclic, then $\langle b^{p^{s+u}}\rangle \leq \langle a\rangle \cap \langle b\rangle = \langle b^{p^{r+s+t}}\rangle$, which implies $t = 0$ and $r = u$. Let $b_1 = ba^{-1}$, then $b_1^{p^{r+s}} = 1$ and $\langle a^{p^{r+s}}, b_1^{p^{r+s-1}}\rangle \not\leq Z(G)$. Since G is a $\mathcal{CAC}$-p-group, $\langle a^p, b_1\rangle/\langle a^{p^{s+r}}, b_1^{p^{s+r-1}}\rangle$ is cyclic. Noticing that $\langle a^{p^{s+r}}, b_1^{p^{s+r-1}}\rangle \leq \Phi(\langle a^p, b_1\rangle)$, we see $\langle a^p, b_1\rangle$ is cyclic, a contradiction. If $\langle a^p, b^{p^{s+u}}\rangle$ is not cyclic, then $\langle a, b^{p^{s+u-1}}\rangle/\langle a^p, b^{p^{s+u}}\rangle$ is cyclic and therefore $\langle a, b^{p^{s+u-1}}\rangle$ is cyclic, another contradiction. □

It is easy to see that the arguments in the proof of Lemma 3.3 is true for ordinary metacyclic 2-groups. Thus we have the following lemma without proof.

Lemma 3.4. *Let G be an ordinary metacyclic 2-group. Then G is a $\mathcal{CAC}$-2-group if and only if G is a minimal non-abelian group.*

Lemma 3.5. *Let G be a $\mathcal{CAC}$-p-group of order p^n and $n \geq 6$. Then G has no abelian maximal subgroup M such that $r(M) = 3$.*

Proof. If not, assume $M \lessdot G$ and $M = \langle x\rangle \times \langle y\rangle \times \langle z\rangle$ with $o(x) = p^i, o(y) = p^j, o(z) = p^k$, where $i \geq 1, j \geq 1, k \geq 1$. Then Lemma 3.2 implies $\Omega_1(G) = \Omega_1(M) \leq Z(G)$ and $\mho_1(G) \leq Z(G)$. Since $M \not\leq Z(G)$, we may assume $x \notin Z(G)$. Thus, by the hypotheses of the lemma, $\langle x, y, z\rangle/\langle x, y^{p^{j-1}}\rangle$ is cyclic, which implies $j = 1$. Similarly, $k = 1$. Hence $\langle x^p\rangle \times \langle y\rangle \times \langle z\rangle = Z(G)$ and $i \geq 3$. If there exists $a \in G \setminus M$ such that $\langle a, x^{p^2}\rangle$ is not cyclic, then, by hypothesis, $\langle a, x^p, y, z\rangle/\langle a, x^{p^2}\rangle$ is cyclic. However, it is impossible. So for any $a \in G \setminus M$, $\langle a, x^{p^2}\rangle$ is cyclic. It follows from $a^p \in Z(G)$ that $o(a) \leq o(x)$ and $\langle a\rangle \not\leq \langle x^{p^2}\rangle$. Thus we may assume $x^{p^2} = a^p$ or $x^{p^2} = a^{p^2}$. If $x^{p^2} = a^p$, then $a^{-1}x^p \in \Omega_1(G) \leq M$ and therefore $a \in M$, a contradiction. If $x^{p^2} = a^{p^2}$, then, since $[a, x] \in Z(G)$, we see $o(ax^{-1}) = p^2$. Noticing that $ax^{-1} \notin M$, we have $x^{p^2} = (ax^{-1})^p$ by the above, a contradiction . □

Lemma 3.6. *Suppose that G is a $\mathcal{CAC}$-p-group of order p^n with $n \geq 6$ and $r(G) = 3$. If $p > 2$, then G has no abelian maximal subgroup. If $p = 2$ and G*

has an abelian maximal subgroup, then G is isomorphic to one of the following non-isomorphic groups:

(1) $D_{2^{n-1}} \times C_2$;

(2) $SD_{2^{n-1}} \times C_2$;

(3) $\langle a, b, c \mid a^4 = b^2 = c^{2^{n-3}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$.

Proof. If there exists a maximal subgroup M in G such that M is abelian, then $r(M) \leq 2$ according to Lemma 3.5. The hypotheses $r(G) = 3$ implies that $r(M) = 2$. Let $M = \langle x \rangle \times \langle y \rangle$ with $o(x) = p^i$ and $o(y) = p^j$ for $i \geq 1$ and $j \geq 1$, and let $A \leq G$ with $A \cong C_p^3$. Since $Z(G) \leq M$, we have $A \nleq Z(G)$. Thus $A = C_G(A)$ by Lemma 3.2. Hence $Z(G) = M \cap A = \Omega_1(M)$. Since $n \geq 6$, we may assume $i \geq 3$. In this case $\langle x^p, y^{p^{j-1}} \rangle \nleq Z(G)$. Thus, by the hypotheses, $\langle x, y \rangle / \langle x^p, y^{p^{j-1}} \rangle$ is cyclic. It follows that $o(x) = p^{n-2}$ and $o(y) = p$. For any $g \in A \setminus M$, $G = \langle x, y, g \rangle$. By Lemma 2.3, $|G'| = p^{n-3}$. Now assume $[x, g] = x^{ps} y^t$. It is easy to see that $G' = \langle x^{ps} y^t \rangle$ and so $(s, p) = 1$. If $p > 2$, then, by Lemma 2.6(1), G is regular. Thus $[x, g^p] = 1$ if and only if $[x, g]^p = 1$. However, $[x, g]^p = x^{sp^2} \neq 1$, a contradiction. If $p = 2$, then, according to $[x, g^2] = 1$, $[x, g] = x^{-2}$ or $x^{2^{n-3}-2}$ or $x^{-2}y$ or $x^{2^{n-3}-2}y$. If $[x, g] = x^{-2}$, then $G \cong D_{2^{n-1}} \times C_2$. If $[x, g] = x^{2^{n-3}-2}$, then $G \cong SD_{2^{n-1}} \times C_2$. If $[x, g] = x^{-2}y$, then $G = \langle x_1, g, y_1 \mid x_1^4 = g^2 = y_1^{2^{n-3}} = 1, [g, x_1] = y_1, [y_1, g] = [y_1, x_1] = y_1^{-2} \rangle$ when we set $x_1 = gx$ and $y_1 = x^2 y$. In this case, G is the type (3). If $[x, g] = x^{2^{n-3}-2}y$, then G is also the type (3). □

Lemma 3.7. *Let G be a $\mathcal{CAC}$-p-group, and H be a non-abelian subgroup of G. Then*

(1) *H is a $\mathcal{CAC}$-p-group.*

(2) *If $Z(H)$ is not cyclic, then $Z(H) \leq Z(G)$.*

Proof. (1) If K is a non-cyclic abelian subgroup of H and $K \nleq Z(H)$, then $K \nleq Z(G)$. By the hypotheses, $C_G(K)/K$ is cyclic and therefore $C_H(K)/K$ is cyclic. Hence H is a $\mathcal{CAC}$-p-group.

The proof of (2) comes immediately from the definition of $\mathcal{CAC}$-p-groups. □

Lemma 3.8. *Let G be a p-group. If $Z(G)$ is a cyclic subgroup of index p^2, then G is a $\mathcal{CAC}$-p-group.*

Proof. Let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$. Then $C_G(H) < G$ and $C_G(H) \geq HZ(G)$. Thus $C_G(H) = HZ(G)$ and therefore $C_G(H)/H \cong Z(G)/Z(G) \cap H$ is cyclic. So G is a $\mathcal{CAC}$-p-group. □

4. $\mathcal{CAC}$-p-groups of odd order

In this section we investigate the $\mathcal{CAC}$-p-groups for $p > 2$.

Lemma 4.1. *Let G be a p-group of order p^n and $r(G)=2$ with $p>2$ and $n\geq 3$. Then G is a $\mathcal{CAC}$-p-group if and only if G is one of the following pairwise non-isomorphic groups:*

(1) *metacyclic minimal non-abelian p-groups;*
(2) $\mathrm{M}_p(1,1,1)$*;*
(3) $\langle a,b,c \mid a^9=c^3=1, b^3=a^3, [a,b]=c, [c,a]=1, [c,b]=a^{-3}\rangle$*;*
(4) $\mathrm{M}_p(1,1,1)*C_{p^{n-2}}$*;*
(5) $\langle a,x,y \mid a^{p^{n-2}}=x^p=y^p=1, [a,x]=y, [x,y]=a^{ip^{n-3}}, [y,a]=1\rangle$*, $i=1$ or σ, where σ is a fixed square non-residue modulo p.*

Proof. If $|G|\leq p^4$, then, the conclusion holds by checking the list of groups of order p^3 and p^4. Assume $|G|\geq p^5$. Since $r(G)=2$, $r_n(G)\leq 2$. If $r_n(G)=1$, then G is cyclic, a contradiction. So $r_n(G)=2$. Thus G is one of the groups listed in Lemma 2.5. We discuss case by case.

If G is of the type (1) in Lemma 2.5, then, by Lemma 3.3, G is of the type (1).

If G is of the type (2) in Lemma 2.5, then $Z(G)$ is a cyclic subgroup of index p^2. By Lemma 3.8, G is a $\mathcal{CAC}$-p-group of the type (4).

If G is of the type (3) in Lemma 2.5, then $G_1=C_G(K_2(G)/K_4(G))$ is abelian or metacyclic minimal non-abelian by Lemma 2.6(2). Thus $\Phi(G_1)\leq Z(G_1)$ by Lemma 2.1. On the other hand, by [3, Section 14, Theorem 14.4], $G_1\lessdot G$. Thus $|G_1|\geq 3^4$ and $|\Phi(G_1)|\geq 3^2$. Noticing that $|Z(G)|=3$, we see $\Phi(G_1)\nleq Z(G)$. Furthermore, by Lemma 2.6(3), $\Phi(G_1)$ and $G_1/\Phi(G_1)$ are not cyclic, which means that G is not a $\mathcal{CAC}$-p-group.

If G is of the type (4) in Lemma 2.5, then, by [9, Theorem 4.1] and Proposition 1, we see G is a $\mathcal{CAC}$-p-group of the type (5).

Conversely, every group listed in the lemma is a $\mathcal{CAC}$-p-group and they are pairwise non-isomorphic. □

Lemma 4.2. *Let G be a $\mathcal{CAC}$-p-group of order p^n with $p>2$ and $n\geq 6$. If $r(G)=3$, then, for every maximal subgroup M of G, $r(M)=3$.*

Proof. Let $A\leq G$ with $A\cong C_p^3$. If there exists a $M\lessdot G$ such that $r(M)=2$, then, by Lemma 3.6, M is not abelian. Thus, according to Lemma 3.7, M is a $\mathcal{CAC}$-p-group of order p^{n-1}. So M is of type $(1),(4)$, or (5) listed in Lemma 4.1.

If M is of type $(4),(5)$ or type (1) with $\exp(M)=p^{n-2}$ in Lemma 4.1, then, by calculation, we see $Z(M)$ is cyclic and $|Z(M)|\geq p^2$. Let $Z(M)=\langle a\rangle$ with $o(a)=p^k$. Since $\langle a^{p^{k-1}}\rangle\trianglelefteq G$ and $|\langle a^{p^{k-1}}\rangle|=p$, $\langle a^{p^{k-1}}\rangle\leq Z(G)$. Furthermore, for any $b\in M\cap A\setminus\langle a^{p^{k-1}}\rangle$, we have $b\notin Z(G)$. Thus, by the hypotheses of the lemma, $\langle a,A\rangle/\langle a^{p^{k-1}},b\rangle$ is cyclic. However, it is impossible.

If M is of type (1) with $\exp(M)<p^{n-2}$ in Lemma 4.1, then assume $M=\langle a,b \mid a^{p^u}=b^{p^v}=1, [a,b]=a^{p^{u-1}}\rangle$, where $u\geq 2, v\geq 2$ and $u+v=n-1$. Thus

$Z(M)$ is not cyclic. By Lemma 3.7, $Z(M) \leq Z(G)$. It follows from Lemma 3.2 that $A \leq Z(G)$. Since $n \geq 6$, we may assume $u \geq 3$. Then, by the hypotheses, $\langle a^p, b, A\rangle/\langle a^{p^{u-1}}, b\rangle$ is cyclic. It is also impossible. $\square$

Lemma 4.3. *Let G be a $\mathcal{CAC}$-p-group of order p^6 and $p > 2$. If $r(G) = 3$, then, for every maximal subgroup M of G, $Z(M) = \Omega_1(G) = \mho_1(G) = Z(G) = G' = \Phi(G) \cong C_p^3$.*

Proof. Let M be a maximal subgroup of G. Then, by Lemma 3.6 and Lemma 4.2, M is not abelian and $r(M) = 3$. Let $A \leq G$ with $A \cong C_p^3$. We consider the following two cases:

Case 1. $A \leq Z(G)$.

In this case, it is clear that $A = Z(G)$. Then, by Lemma 3.2, $\mho_1(G) \leq Z(G) = \Omega_1(G)$, which implies $\exp(G) = p^2$. Since $r(M) = 3$, we have $\Omega_1(G) = Z(G) = Z(M) \leq \Phi(G)$. If $Z(G) < \Phi(G)$, then $d(G) = 2$. Assume $G = \langle g_1, g_2\rangle$ and $[g_1, g_2] = x$. If $o(x) = p$, then $x \in Z(G)$ and therefore $|G'| = p$. So G is minimal non-abelian by Lemma 2.1, a contradiction. If $o(x) = p^2$, then, by calculation, we get $[g_1, g_2^p] = x^p[x, g_2]^{\frac{p(p-1)}{2}} = x^p \neq 1$, in contradiction to $\mho_1(G) \leq Z(G)$. So $Z(G) = \Phi(G)$ and G is regular. By [1, Section 7, Theorem 7.2], $|G/\Omega_1(G)| = |\mho_1(G)|$ and therefore $\Omega_1(G) = \mho_1(G)$. If $|G'| < p^3$, then there exist x_1 and x_2 in G with $o(x_1) = o(x_2) = p^2$ such that $x_1 \in G\backslash\langle x_2, \Phi(G)\rangle$ and $[x_1, x_2] = 1$. If $\langle x_1\rangle \cap \langle x_2\rangle = 1$, then $|\langle x_1, x_2, A\rangle| = p^5$, in contradiction to that G has no abelian maximal subgroup. If $\langle x_1\rangle \cap \langle x_2\rangle \neq 1$, then $\langle x_1^p\rangle = \langle x_2^p\rangle$. Obviously, there exists an element $a \in A$ such that $\langle x_1, a\rangle$ is not cyclic. Then, by the hypothesis, $\langle x_1, x_2, A\rangle/\langle x_1, a\rangle$ is cyclic. However, it is impossible. Hence, for every $M \lessdot G$, $Z(M) = \Omega_1(G) = \mho_1(G) = Z(G) = G' = \Phi(G) \cong C_p^3$.

Case 2. $A \nleq Z(G)$.

By Lemma 3.2, $C_G(A) = A$ and so $Z(G) < A$ in this case. Since $r(M) = 3$, there exists a $B \cong C_p^3$ such that $B \leq M$ and $C_G(B) = B$. Let $N \leq M$ with $Z(G) < B < N < M$. If $Z(G) < Z(M)$, then $Z(M)$ is not cyclic. By Lemma 3.7, $Z(M) \leq Z(G)$, a contradiction. Thus $Z(M) = Z(G)$. Similarly, $Z(G) = Z(N)$ and therefore $Z(G) = Z(N) = Z(M)$. Now we consider the following two subcases:

Subcase 1. $|Z(N)| = p$

By [1, Section 1, Exercise 4], N is of maximal class. Then $N' \cong C_p \times C_p$ and $B = C_N(N')$ by the classification of maximal class p-groups of order p^4. Since $M/C_M(N') \lesssim Aut(N')$, we have $C_M(N') \lessdot M$. By the hypotheses of the lemma, $C_M(N')/N'$ is cyclic and so $C_M(N')$ is abelian. It follows that $C_M(N') \leq C_G(B)$, in contradiction to $C_G(B) = B$.

Subcase 2. $|Z(N)| = p^2$

Since $r(N) = 3$, by checking the list of groups of order p^4, we see $N \cong \mathrm{M}_p(1, 1, 1) \times C_p$ or $\mathrm{M}_p(2, 1, 1)$ or $\mathrm{M}_p(2, 1) \times C_p$.

If $N \cong \mathrm{M}_p(1,1,1) \times C_p$, then we may assume $N = \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1, [b,a] = c, [c,a] = [d,a] = [c,b] = [d,b] = [c,d] = 1\rangle$. In this case $Z(N) = Z(G) = \langle c,d\rangle$. Since $|M| = p^5$, we have $|K_3(M)| \le p^2$. Thus $|G/C_G(K_3(M))| \mid p$. So $K_3(M) \le Z(C_G(K_3(M))) \le Z(G)$. Take $x \in M\setminus N$. If $[a,x] \notin Z(G)$, then $[a,x,x] \in Z(G)$. Without loss of generality, we may assume $[a,x] \in Z(G)$. Noticing that $C_G(a) = C_G(\langle a,c,d\rangle) = \langle a,c,d\rangle$ and $[b,a] = c$, we see $[g,a] \notin \langle c\rangle$ for any $g \in G \setminus N$. Thus we may assume $[a,x] = c^i d$. For every integer j, since $C_G(a^j b) = \langle a^j b, c, d\rangle$, we see $[b,x] \notin Z(G)$. It follows that $M' = \langle a,c,d\rangle$ and so $\langle a,c,d\rangle \trianglelefteq G$. Take $y \in G \setminus M$. Since $[a,y] \notin \langle c\rangle$, we may assume $[a,y] = c^k d$. It follows that $[a,xy^{-1}] \in \langle c\rangle$ and so $xy^{-1} \in N$, a contradiction.

If $N \cong \mathrm{M}_p(2,1,1)$, then we may assume $N = \langle a,b,c \mid a^{p^2} = b^p = c^p = 1, [b,a] = c, [c,a] = [c,b] = 1\rangle$. Thus $Z(N) = Z(G) = \langle a^p, c\rangle$, $B = \Omega_1(N) = \langle a^p, b, c\rangle$. So $C_G(b) = B$. Since $|M/\Omega_1(N)| = p^2$, $M' \le \Omega_1(N)$. Take $x \in M \setminus N$. Then we may assume $[x,b] = a^p c^i$. Thus $x^p \in C_G(b)$, which implies $\exp(M) = p^2$. If $o(x) = p$, then $\langle a^p, b, c, x\rangle \cong \mathrm{M}_p(1,1,1) \times C_p$, a contradiction. So $o(x) = p^2$ and $\Omega_1(N) = \Omega_1(M)$. Take $y \in G \setminus M$ and assume $[y,b] = y_1, [y_1,b] = y_2$. If $o(y_1) = p^2$, then $[y,b^p] = y_1^p y_2^{\frac{p(p-1)}{2}} = y_1^p \ne 1$, a contradiction. If $o(y_1) = p$, then $[y,b] \in \Omega_1(M) = \Omega_1(N)$. So we may assume $[y,b] = a^p c^j$. Thus $[xy^{-1}, b] \in \langle c\rangle$ and therefore $xy^{-1} \in N$, a contradiction.

If $N \cong \mathrm{M}_p(2,1) \times C_p$, then, by the similar arguments as in the case $N \cong \mathrm{M}_p(2,1,1)$, we may also have a contradiction. □

Lemma 4.4. *Let G be a $\mathcal{CAC}$-p-group of order p^n with $p > 2$ and $n \ge 7$. Then $r(G) = 2$.*

Proof. Without loss of generality, we may assume $n = 7$ by Lemma 3.6, Lemma 3.7, and Lemma 4.2. If $r(G) \ne 2$, then $r(G) = 3$ by Lemma 3.1. Let M be a maximal subgroup of G. Then, according to Lemma 3.6, Lemma 3.7, and Lemma 4.2, M is not abelian, $r(M) = 3$ and G has no abelian subgroup of index p^2. Furthermore, by Lemma 4.3, $\Omega_1(M) = \mho_1(M) = Z(M) = M' \cong C_p^3$. Thus $\Omega_1(G) = Z(M) \le Z(G)$ and $\mho_1(G) \le Z(G)$ by Lemma 3.2. If $Z(M) < Z(G)$, then G has an abelian subgroup of index p^2, a contradiction. Hence $\mho_1(G) = \Omega_1(G)$. For any $a, b \in G$, if $[a,b] = x$ and $[x,b] = y$, then $y \in Z(G)$. By calculation, $[a,b^p] = x^p y^{\frac{p(p-1)}{2}} = x^p$. Thus $o(x) \le p$, and therefore $G' \le Z(G)$ and G is regular. According to [1, Section 7, Theorem 7.2], we see $|G/\Omega_1(G)| = |\mho_1(G)|$ and therefore $|G| = p^6$, in contradiction to the hypothesis. □

According to Lemma 4.1 and Lemma 4.4, we have the following result:

Theorem 4.5. *Let G be a p-group of order p^n with $p > 2$ and $n \ge 7$. Then G is a $\mathcal{CAC}$-p-group if and only if G is one of the following pairwise non-isomorphic groups:*

(1) *metacyclic minimal non-abelian p-groups;*
(2) $\mathrm{M}_p(1,1,1) * C_{p^{n-2}}$*;*
(3) $\langle a,x,y \mid a^{p^{n-2}} = x^p = y^p = 1, [a,x] = y, [x,y] = a^{ip^{n-3}}, [y,a] = 1\rangle$*,* $i = 1$ *or* σ*, where* σ *is a fixed square non-residue modulo* p*.*

5. $\mathcal{CAC}$-p-groups of even order

In this section we investigate the $\mathcal{CAC}$-2-groups.

Lemma 5.1. *Let* G *be a* $\mathcal{CAC}$*-*p*-group and* H *be a subgroup of* G*. If there exist* $a, b,$ *and* c *in* G *such that* $a \in H \setminus Z(H)$*,* $b \in Z(G) \cap H \setminus \langle a\rangle$*, and* $c \in C_G(a) \setminus H$*, then* $\langle C_H(a), c\rangle = \langle a, b, c\rangle$ *is abelian.*

Proof. By the hypotheses of the lemma, and $c \notin H$, we see $\langle C_H(a), c\rangle/\langle a, b\rangle = \langle \bar{c}\rangle$. So $\langle C_H(a), c\rangle = \langle a, b, c\rangle$ is abelian. □

Lemma 5.2. *Let* G *be a* $\mathcal{CAC}$*-2-group and* M *be a non-abelian maximal subgroup of* G*. If* $\exp(M) = 4$ *and* $Z(M) \cong C_2^3$*, then* $\Phi(G) \le Z(M)$ *and for any* $a \in M \setminus Z(M)$*,* $b \in G \setminus M$*, we have* $[a, b] \ne 1$ *and* $o(a) = o(b) = 4$*.*

Proof. By Lemma 3.7, $Z(M) \le Z(G)$. It follows from Lemma 3.2 that $\Phi(G) \le Z(G)$ and so $\Phi(G) \le Z(M)$. For any $x \in G \setminus Z(M)$, if $o(x) = 2$, then $Z(M)\langle x\rangle \cong C_2^4$, in contradiction to the Lemma 3.1. Thus $o(a) = o(b) = 4$. If $[a, b] = 1$ and $a^2 = b^2$, then $o(ab) = 2$, a contradiction. If $[a, b] = 1$ and $a^2 \ne b^2$, then $\langle a, b^2\rangle$ is not cyclic. By the hypotheses, $\langle a, b, Z(M)\rangle/\langle a, b^2\rangle$ is cyclic which is impossible. So $[a, b] \ne 1$. □

Lemma 5.3. *Let* G *be a* $\mathcal{CAC}$*-2-group of order* 2^n *with* $n \ge 6$*, and* M *be a maximal subgroup of* G*. If* M *is metacyclic minimal non-abelian. Then* G *is one of the following pairwise non-isomorphic groups:*
(1) $D_8 * C_{2^{n-2}}$;
(2) $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4, c^2 = a^{2^{n-3}}, [b, a] = c, [c, b] = c^2, [c, a] = 1\rangle$*;*
(3) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1\rangle$*;*
(4) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1\rangle$.

Proof. Let $M = \langle a, b \mid a^{2^u} = b^{2^v} = 1, [a, b] = a^{2^{u-1}}\rangle$, where $u \ge 2, v \ge 1$ and $u + v = n - 1$. We consider the following two cases: $v = 1$ and $v \ne 1$.

Case 1. $v = 1$

In this case, $M = \langle a, b \mid a^{2^{n-2}} = b^2 = 1, [a, b] = a^{2^{n-3}}\rangle$. Take $d \in G \setminus M$. Since $[b^2, d] = 1$, we have $[b, d] = 1$ or $a^{2^{n-3}}$. If $[b, d] = a^{2^{n-3}}$, then $[b, ad] = 1$. Without loss of generality, we may assume $[b, d] = 1$. Noticing that $Z(M) = \langle a^2\rangle$, we see $\langle a^{2^{n-3}}\rangle \le Z(G)$. By Lemma 5.1, $a^2 \in C_M(b) \le \langle a^{2^{n-3}}, b, d\rangle$. Since $d \notin M$, $a^2 \in \langle a^{2^{n-3}}, b, d^2\rangle$. Clearly, $\exp(G) = p^{n-2}$. Thus we may assume $d^2 = a^2$ or a^2b.

If $d^2 = a^2$, then $[a, d] = 1$ or $a^{2^{n-3}}$. If $[a, d] = 1$, then, by letting $a_1 = ad^{-1}$, $G = \langle a_1, b\rangle * \langle d\rangle \cong D_8 * C_{2^{n-2}}$. If $[a, d] = a^{2^{n-3}}$, then, by letting $d_1 = bd$, we see $d_1^2 = a^2$ and $[a, d_1] = [b, d_1] = 1$. So we may also have $G \cong D_8 * C_{2^{n-2}}$.

If $d^2 = a^2b$, then $[a^2, d] = [b, d] = 1$ and $[a, d^2] = a^{2^{n-3}}$. By calculation, $[a, d] = a^{\pm 2^{n-4}}b$. Then $G = \langle a_1, c, d \mid a_1^2 = c^4 = d^{2^{n-2}} = 1, c^2 = d^{2^{n-3}}, [a_1, d] = c, [c, a_1] = c^2, [c, d] = 1\rangle$ when we set $a_1 = a^{\pm 2^{n-5}-1}d$ and $c = a^{\pm 2^{n-4}}b$. Thus G is the type (2).

Case 2. $v \neq 1$

In this case, $Z(M) = \langle a^2, b^2\rangle$ is not cyclic. By Lemma 3.7, $Z(M) \leq Z(G)$. Take $d \in G \setminus M$. Since $[a^2, d] = 1$, $[a, d] = a^{2^{u-1}i}b^{2^{v-1}j}$, where i, j are integers. It follows that $[a, d^2] = 1$. Similarly, $[b, d^2] = 1$. Thus $d^2 \in Z(M) \leq Z(G)$. Noticing that $\Phi(M) = Z(M) \leq Z(G)$, we see $\Phi(G) \leq Z(G)$. So $G/Z(G)$ is elementary abelian. By Lemma 2.2, G' is elementary abelian. In particular, $G' \leq \Omega_1(M) = \langle a^{2^{u-1}}, b^{2^{v-1}}\rangle$. If there exists an element $g \in G \setminus M$ such that $o(g) = 2$, then $\Omega_1(M)\langle g\rangle \cong C_2^3$. It follows from Lemma 3.2 that $g \in Z(G)$. Since $n \geq 6$, we may assume $u \geq 3$. Then, by the hypotheses, $\langle a^2, b, g\rangle/\langle a^{2^{u-1}}, b\rangle$ is cyclic. However it is impossible. So there is not an involution in $G \setminus M$.

Now we consider the following three subcases:

Subcase 1. $u \geq 3$ and $v \geq 3$

Let $W = \langle a^{2^{u-2}}, b^{2^{v-2}}\rangle$. Then $W \cong C_4 \times C_4$ and $C_G(W) = G$. By Lemma 2.6(4), $\Omega_2(C_G(W)) = W$. Then Lemma 2.6(5) implies G is metacyclic. Thus $d(G) = 2$ and $|G'| = 2$. By Lemma 2.1, G is minimal non-abelian, a contradiction.

Subcase 2. $v = 2$

In this case, $M = \langle a, b \mid a^{2^{n-3}} = b^{2^2} = 1, [a, b] = a^{2^{n-4}}\rangle$. By the above, $G' \leq \langle a^{2^{n-4}}, b^2\rangle$. Take $d \in G \setminus M$. Then $d^2 \in Z(G) \cap M = \langle a^2, b^2\rangle$. If $o(d) < 2^{n-3}$, then, by letting $d_1 = ad$, we see $o(d_1) = 2^{n-3}$. Without loss of generality, we assume $o(d) = 2^{n-3}$. Thus we may assume $d^2 = a^2b^2$ or $d^2 = a^2$.

If $d^2 = a^2$, then $o(a^{-1}d) = 2$ if $[a, d] = 1$ and $o(da^{2^{n-5}-1}) = 2$ if $[a, d] = a^{2^{n-4}}$, which contradict that there is not an involution in $G\setminus M$. Thus $[a, d] = b^2$ or $a^{2^{n-4}}b^2$. Since $o(abd^{-1}) = 2$ if $[ab, d] = b^2a^{2^{n-4}}$ and $o(a^{1+2^{n-5}}bd^{-1}) = 2$ if $[ab, d] = b^2$, we see $[ab, d] = 1$ or $a^{2^{n-4}}$. It follows that $[b, d] = b^2$ or $a^{2^{n-4}}b^2$. If $[a, d] = b^2$ and $[b, d] = b^2$, then, by letting $a_1 = a^{1+2^{n-5}}b$, $G = \langle a_1, b, d \mid a_1^{2^{n-3}} = b^{2^2} = 1, [a_1, b] = a_1^{2^{n-4}}, [b, d] = b^2, [a_1, d] = 1, d^2 = a_1^2b^2\rangle$. By calculation, G is isomorphic to the group of type (4). If $[a, d] = b^2$ and $[b, d] = a^{2^{n-4}}b^2$, then $G = \langle a_1, b_1, d \mid a_1^{2^{n-3}} = b_1^{2^2} = 1, [a_1, b_1] = 1, [b_1, d] = b_1^2, [a_1, d] = a_1^{2^{n-4}}, d^2 = a_1^2b_1^2\rangle$ when we set $a_1 = a^{1+2^{n-5}}b$ and $b_1 = ad^{-1}$. Thus G is the type (4). If $[a, d] = a^{2^{n-4}}b^2$, then, by setting $d_1 = bd$ if $[b, d] = b^2$ and $d_1 = a^{2^{n-5}}bd$ if

$[b,d]=a^{2^{n-4}}b^2$, we see $d_1^2=a^2$ and $[a,d_1]=b^2$. So we may also have the group of type (4).

If $d^2=a^2b^2$, then, by letting $a_1=a^{1+2^{n-5}}b$, we see $d^2=a_1^2$, which is reduced to the case of $d^2=a^2$.

Subcase 3. $u=2$

In this case, $M=\langle a,b \mid a^{2^2}=b^{2^{n-3}}=1,[a,b]=a^2\rangle$ and $G'\le\langle a^2,b^{2^{n-4}}\rangle$. Take $d\in G\setminus M$. Without loss of generality, we may assume $d^2=a^2b^2$ or $d^2=b^2$.

If $d^2=b^2$, then $o(b^{-1}d)=2$ if $[b,d]=1$ and $o(db^{2^{n-5}-1})=2$ if $[b,d]=b^{2^{n-4}}$. So $[b,d]=a^2$ or $b^{2^{n-4}}a^2$. Since $(ab)^2=b^2$, we see $[ab,d]=a^2$ or $b^{2^{n-4}}a^2$. It follows that $[a,d]=1$ or $b^{2^{n-4}}$. If $[a,d]=1$ and $[b,d]=a^2$, then, by letting $d_1=ad$, $G=\langle a,b,d_1 \mid a^4=b^{2^{n-3}}=1,[a,b]=a^2,d_1^2=a^2b^2,[d_1,a]=[d_1,b]=1\rangle$. Thus G is the type (3). If $[a,d]=1$ and $[b,d]=b^{2^{n-4}}a^2$, then G is isomorphic to the group of type (4). If $[a,d]=b^{2^{n-4}}$ and $[b,d]=a^2$ or $b^{2^{n-4}}a^2$, then G is also the type (4).

If $d^2=a^2b^2$, then $o(b^{-1}d)=2$ if $[b,d]=a^2$ and $o(db^{2^{n-5}-1})=2$ if $[b,d]=a^2b^{2^{n-4}}$. So $[b,d]=1$ or $b^{2^{n-4}}$. Similarly, $[ab,d]=1$ or $b^{2^{n-4}}$. It follows that $[a,d]=1$ or $b^{2^{n-4}}$. Let $d_1=ad$ if $[a,d]=1$ and $d_1=b^{2^{n-5}}ad$ if $[a,d]=b^{2^{n-4}}$. In the two cases, we have $d_1^2=b^2$, which is reduced to the case of $d^2=b^2$. □

Lemma 5.4. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n\ge 6$. If there is a maximal subgroup M in G such that $M\cong D_8*C_{2^{n-3}}$, then G is one of the following pairwise non-isomorphic groups:*

(1) $D_8*C_{2^{n-2}}$;

(2) $\langle a,b,c \mid a^{2^{n-2}}=b^2=c^4=1,c^2=a^{2^{n-3}},[b,a]=c,[c,b]=c^2,[c,a]=1\rangle$.

Proof. Let $M=\langle a,b,c \mid a^{2^{n-3}}=b^2=c^2=1,[c,b]=a^{2^{n-4}},[b,a]=[c,a]=1\rangle$. Then $Z(M)=\langle a\rangle\trianglerighteq G$ and so $\langle a^{2^{n-4}}\rangle\le Z(G)$. Take $d\in G\setminus M$. Since $[b^2,d]=1$, by calculation, we have $[b,d]=1$ or $a^{2^{n-4}}$ or $a^{\pm 2^{n-5}}c$ or $a^{i2^{n-4}}bc$, where i is an integer. If $[b,d]=a^{\pm 2^{n-5}}c$, then, since $[b,d^2]\in\langle a^{2^{n-4}}\rangle$, we see $[c,d]=a^{2^{n-4}}$ or 1. If $[b,d]=a^{i2^{n-4}}bc$, then $[bc,d]=a^{2^{n-4}}$ or 1. Without loss of generality, we may assume $[b,d]=1$, or $a^{2^{n-4}}$. Now we consider $o(d)=2^{n-2}$ and $o(d)\le 2^{n-3}$.

If $o(d)=2^{n-2}$, then $\langle d^4\rangle=\langle a^2\rangle$. If $[b,d]=a^{2^{n-4}}$, then, by Lemma 2.1, $\langle b,d\rangle$ is metacyclic minimal non-abelian of order 2^{n-1}. If $[b,d]=1$, then $[b,cd]=a^{2^{n-4}}$ and so $\langle b,cd\rangle$ is metacyclic minimal non-abelian of order 2^{n-1}. Thus we may have the groups listed in lemma by Lemma 5.3.

If $o(d)\le 2^{n-3}$ and $[b,d]=1$, then, by Lemma 5.1, we see $a\in C_M(b)\le\langle a^{2^{n-4}},b,d\rangle$. Thus $a\in\langle a^{2^{n-4}},b,d^2\rangle$, in contradiction to $o(d)\le 2^{n-3}$. If $[b,d]=a^{2^{n-4}}$, then $[b,cd]=1$. We may also have a contradiction. □

Lemma 5.5. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. If there is a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^{2^{n-4}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1\rangle$, then G is one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1\rangle$;

(2) $\langle a, b, c \mid a^{2^{n-3}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1\rangle$.

Proof. By calculation, we see $Z(M) = \langle b^2, c\rangle$, $\Phi(M) = \langle a^2, b^2\rangle$, and $\Omega_1(M) = \langle c^{2^{n-5}}, b^2\rangle$. By Lemma 3.7, $Z(M) \leq Z(G)$. For any $d \in G \setminus M$, if $d^2 \notin Z(G)$, then there exists an element $x \in \Phi(M)$ such that $\langle x, d^2\rangle$ is not cyclic. It follows from Lemma 5.1 that $c \in C_M(d^2) \leq \langle x, d\rangle$ and so $d^2 \in Z(G)$, a contradiction. Thus $d^2 \in Z(G)$. So $\Phi(G) \leq Z(G)$ and $G/Z(G)$ is elementary abelian. By Lemma 2.2, G' is elementary abelian. In particular, $G' \leq \Omega_1(M)$. We consider $\exp(G) = 2^{n-4}$ and $\exp(G) = 2^{n-3}$.

If $\exp(G) = 2^{n-3}$, then $o(d) = 2^{n-3}$. Since $d^2 \in Z(M) = \langle b^2, c\rangle$, $\langle d^4\rangle = \langle c^2\rangle$. If $[b, d] = b^2$ or $c^{2^{n-5}}$, then $\langle b, d\rangle$ is metacyclic minimal non-abelian of order 2^{n-1}. If $[b, d] = 1$ or $b^2c^{2^{n-5}}$, then $[b, ad] = b^2$ or $c^{2^{n-5}}$ and so $\langle b, ad\rangle$ is metacyclic minimal non-abelian of order 2^{n-1}. Thus we may get the groups listed in lemma by Lemma 5.3.

If $\exp(G) = 2^{n-4}$, then $d^2 \in \langle b^2, c^2\rangle$. Since $[a, b] = b^2$, we may assume $[a, d] = 1$ or $a^{2^{n-5}}$. If $[a, d] = 1$, then, by Lemma 5.1, we see $c \in C_M(a) \leq \langle a, b^2, d\rangle$, a contradiction. So $[a, d] = a^{2^{n-5}}$. Similarly $[b, d] = a^{2^{n-5}}$. Then $[ab, d] = 1$. We may also have a contradiction. □

Lemma 5.6. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. Then*

(1) *If there is a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^{2^{n-4}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-5}}, [c, b] = 1\rangle$, then $n = 6$ and $G \cong \langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^2 = d^2, [a, b] = b^2, [a, c] = a^2, [a, d] = b^2, [b, c] = 1, [b, d] = a^2, [c, d] = c^2\rangle$.*

(2) *If $n = 6$, then there is not a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^4 = b^4 = c^4 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = c^2, [c, b] = 1\rangle$.*

Proof. Assume $M \lessdot G$ and M is isomorphic to the maximal subgroup listed in (1) or (2). Then $Z(M) = \Phi(M) = \langle b^2, c^2\rangle = \langle a^2, c^2\rangle \leq Z(G)$.

It is easy to see that $\langle b, c\rangle$ is the unique abelian maximal subgroup of M. Thus $\langle b, c\rangle$ *char* $M \trianglelefteq G$ and so $G' \leq \langle b, c\rangle$. For any $d \in G \setminus M$, it follows from $[b^2, d] = 1$ that $[b, d^2] = 1$. Thus $d^2 \in C_G(b) \cap M = C_M(b) = \langle b, c\rangle$. If $d^2 \notin Z(G)$, then there exists an element $x \in Z(M)$ such that $\langle x, d^2\rangle$ is not cyclic. By the hypotheses, $\langle b, c, d\rangle/\langle x, d^2\rangle$ is cyclic. Noticing that $\langle x, d^2\rangle \leq \Phi(\langle b, c, d\rangle)$, we see $\langle b, c, d\rangle$ is cyclic, a contradiction. Thus $d^2 \in Z(G)$ and so $\Phi(G) \leq Z(G)$. Thus $G/Z(G)$ is elementary abelian. By Lemma 2.2, G' is elementary abelian. In particular, $G' = \Omega_1(M) = M'$.

For any $d \in G \setminus M$, if $o(d) = 2$, then $\Omega_1(M)\langle d\rangle \cong C_2^3$, which implies $r(G) = 3$. If $d \in Z(G)$, then, by the hypotheses, $\langle b, c, d\rangle/\langle b, c^2\rangle$ is cyclic. However it is impossible. If $d \notin Z(G)$, then $C_G(d) = \Omega_1(M)\langle d\rangle$ by Lemma 3.2. It follows from $G' \cong C_2^2$ that there exists an element $x \in M \setminus \Phi(M)$ such that $[x, d] = 1$. Thus $x \in C_G(d) = \Omega_1(M)\langle d\rangle$. It is also impossible. So there is not an involution in $G \setminus M$.

Noticing that $[a, M] = G'$, we may take a suitable $d \in G \setminus M$ such that $[a, d] = 1$. If $[b, d] = 1$, then, by Lemma 5.1, we see $c \in C_M(b) \leq \langle b, d, c^2\rangle$, a contradiction. If $[b, d] = b^2$, then $[b, ad] = 1$. We may also have a contradiction. Thus $[b, d] \notin \langle b^2\rangle$.

If $M \cong \langle a, b, c \mid a^{2^{n-4}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-5}}, [c, b] = 1\rangle$, then, since $d^2 \in Z(M)$, we may assume $d^2 = a^{2i}b^{2j}$, where i, j are integers. Replacing d by da^{-i}, we have $d^2 = b^{2j}$ and so $d^2 = b^2$. Since $[b, d] \notin \langle b^2\rangle$, $[b, d] = a^{2^{n-5}}b^2$ or $a^{2^{n-5}}$. Similarly $[c, d] = b^2$ or $b^2a^{2^{n-5}}$. If $n \geq 7$, then $a^{2^{n-6}} \in Z(G)$. Since $(bda^{2^{n-6}})^2 = [b, d]a^{2^{n-5}} \neq 1$, we see $[b, d] = a^{2^{n-5}}b^2$. It follows that $(abc^{-1}d)^2 = b^2[c, d]$ and so $[c, d] = b^2a^{2^{n-5}}$. Thus $[bc, d] = 1$. By Lemma 5.1, $c \in C_M(bc) \leq \langle bc, d, b^2\rangle$, a contradiction. So $n = 6$. Since $(abd)^2 = a^2b^2[b, d]$, we see $[b, d] = a^2$. Thus $(bcd)^2 = b^2[c, d]$ and so $[c, d] = b^2a^2 = c^2$. Hence $G = \langle a, b, c, d\rangle$ is isomorphic to the group in lemma.

If $M \cong \langle a, b, c \mid a^4 = b^4 = c^4 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = c^2, [c, b] = 1\rangle$, then we may assume $d^2 = c^2$. Since $(bd)^2 = b^2c^2[b, d]$ and $(acd)^2 = a^2c^2[c, d]$, we see $[b, d] \neq a^2c^2$ and $[c, d] \neq a^2c^2$. Thus $[b, d] = c^2$ and $[c, d] = a^2$. It follows that $[bc, ad] = 1$. By Lemma 5.1, we see $c \in C_M(bc) \leq \langle bc, ad, b^2\rangle$, a contradiction. □

Lemma 5.7. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. If G has an abelian maximal subgroup and a maximal subgroup of maximal class, then G is one of the following pairwise non-isomorphic groups:*

(1) 2-*groups of maximal class;*
(2) $D_{2^{n-1}} \times C_2$*;*
(3) $SD_{2^{n-1}} \times C_2$*;*
(4) $Q_{2^{n-1}} \times C_2$*;*
(5) $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle \cong D_{2^{n-1}} * C_4 \cong Q_{2^{n-1}} * C_4 \cong SD_{2^{n-1}} * C_4$.

Proof. Let $M \lessdot G$ and M be of maximal class. Then $|Z(M)| = 2$ and $|M'| = 2^{n-3}$. Thus $2^{n-3} \leq |G'| \leq 2^{n-2}$. If $|G'| = 2^{n-2}$, then G is of maximal class. If $|G'| = 2^{n-3}$, then $|Z(G)| = 4$ by Lemma 2.3. So there exists an element $x \in Z(G)$ such that $x \notin M$. Then $x^2 \in M \cap Z(G) \leq Z(M)$. If $o(x) = 2$, then G is of the type (2), (3) or (4). If $o(x) = 4$, then G is of the type (5). □

Lemma 5.8. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. Then G has no maximal subgroup $M \cong SD_{2^{n-2}} \times C_2$ and if G has a maximal subgroup $M \cong D_{2^{n-2}} \times C_2$, then G is one of the groups listed in Lemma 3.6.*

Proof. Let $M \lessdot G$ and $M = \langle a, b, c \mid a^{2^{n-3}} = b^2 = c^2 = 1, [a, b] = a^{i2^{n-4}-2}, [c, a] = [c, b] = 1\rangle$, where $i = 0$ or 1. Then $r(M) = 3$. By Lemma 3.7, $Z(M) = \langle a^{2^{n-4}}, c\rangle \leq Z(G)$. Clearly, we may take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, d] = 1$. By Lemma 5.1, $\langle C_M(a^{2^{n-5}}), d\rangle = \langle a, c, d\rangle$ is an abelian maximal subgroup of G. So G is one of the groups listed in Lemma 3.6. Conversely, those groups listed in Lemma 3.6 have a maximal subgroup $M \cong D_{2^{n-2}} \times C_2$ and have no maximal subgroup $M \cong SD_{2^{n-2}} \times C_2$. □

Lemma 5.9. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. If there is a maximal subgroup M in G such that $M \cong Q_{2^{n-2}} \times C_2$, then G is one of the following pairwise non-isomorphic groups:*

(1) $Q_{2^{n-1}} \times C_2$;

(2) $SD_{2^{n-1}} \times C_2$;

(3) $\langle a, b, c \mid a^4 = b^4 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.

Proof. Let $M = \langle a, b, c \mid a^{2^{n-3}} = c^2 = 1, b^2 = a^{2^{n-4}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle$. Then $Z(M) = \langle a^{2^{n-4}}, c\rangle \leq Z(G)$. Since $\langle a, c\rangle$ is the unique abelian maximal subgroup of M, $G' \leq \langle a, c\rangle$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, d] = 1$. By Lemma 5.1, we see $a \in C_M(a^{2^{n-5}}) \leq \langle a^{2^{n-5}}, c, d\rangle$. Without loss of generality, we may assume $d^2 = a$. Then $[d^2, b] = [a, b] = a^{-2}$. It follows that $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$ or $a^{-1}c$ or $a^{2^{n-4}-1}c$. If $[d, b] = a^{-1}$, then $G = \langle b, c, d\rangle \cong Q_{2^{n-1}} \times C_2$. If $[d, b] = a^{2^{n-4}-1}$, then $G \cong SD_{2^{n-1}} \times C_2$. If $[d, b] = a^{-1}c$, then $G = \langle b, c_1, d_1 \mid b^4 = d_1^4 = c_1^{2^{n-3}} = 1, b^2 = c_1^{2^{n-4}}, [d_1, b] = c_1, [c_1, b] = [c_1, d_1] = c_1^{-2}\rangle$ when we set $d_1 = bd$ and $c_1 = a^{-1}c$. In this case, G is the type (3). If $[d, b] = a^{2^{n-4}-1}c$, then G is also the type (3). □

Lemma 5.10. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. Then G has no maximal subgroup $M \cong \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{2^{n-4}-2}\rangle$ and if G has a maximal subgroup $M \cong \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{-2}\rangle$, then G is one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{-2}\rangle$;

(2) $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{2^{n-3}-2}\rangle$;

(3) $\langle a, b \mid a^4 = b^2 = c^{2^{n-3}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$;

(4) $\langle a, b \mid a^4 = b^4 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.

Proof. Let $M \lessdot G$ and $M = \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{i2^{n-4}-2}\rangle$, where $i = 0$ or 1. It is easy to see $Z(M) = \langle a^2, b^{2^{n-4}}\rangle \leq Z(G)$ and $G' \leq \langle a^2, b\rangle$. Take a suitable $d \in G \setminus M$ such that $[b^{2^{n-5}}, d] = 1$. By Lemma 5.1, $\langle a^2, b, d\rangle$

is abelian and $b \in \langle a^2, b^{2^{n-5}}, d\rangle$. Without loss of generality, we may assume $d^2 = b$.

If $i = 1$, then $[d^2, a] = b^{2^{n-4}-2}$. Assume $[a, d] = a^{2j}b^k$. It follows from $[a^2, d] = 1$ that k is even and so $[a, d^2] \in \langle b^4\rangle$, a contradiction.

If $i = 0$, then $[d^2, a] = b^{-2}$. It follows that $[d, a] = b^{-1}$ or a^2b^{-1} or $a^2b^{2^{n-4}-1}$ or $b^{2^{n-4}-1}$. If $[d, a] = b^{-1}$, then G is the type (1). If $[d, a] = b^{2^{n-4}-1}$, then G is the type (2). If $[d, a] = a^2b^{-1}$, then $G = \langle a, b_1, c_1 \mid a^4 = b_1^2 = c_1^{2^{n-3}} = 1, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2}\rangle$ when we set $b_1 = ad$ and $c_1 = a^2b^{-1}$. In this case, G is the type (3). If $[d, a] = a^2b^{2^{n-4}-1}$, then, by letting $b_1 = ad$ and $c_1 = a^2b^{2^{n-4}-1}$, we see $G = \langle a, b_1, c_1 \mid a^4 = b_1^4 = c_1^{2^{n-3}} = 1, b_1^2 = c_1^{2^{n-4}}, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2}\rangle$. Thus G is the type (4). □

Lemma 5.11. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. If there is a maximal subgroup M in G such that $M \cong \langle a, b \mid a^8 = b^{2^{n-3}} = 1, a^4 = b^{2^{n-4}}, [b, a] = b^{-2}\rangle$, then G is one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b \mid a^8 = b^{2^{n-2}} = 1, a^4 = b^{2^{n-3}}, [b, a] = b^{-2}\rangle$;

(2) $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-3}} = 1, a^4 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.

Proof. Since $\langle a^2, b\rangle$ is the unique abelian maximal subgroup of M, $G' \leq \langle a^2, b\rangle$. Take $d \in G \setminus M$. Since $M' = \langle b^2\rangle$ and $Z(M) = \langle a^2\rangle$, we see $[b^{2^{n-5}}, d] = 1$ or $b^{2^{n-4}}$, and $[a^2, d] = 1$ or a^4. Thus $[a^2b^{2^{n-5}}, d] = 1$ or $b^{2^{n-4}}$. We may assume $[a^2b^{2^{n-5}}, d] = 1$. By Lemma 5.1, $b \in \langle a^4, a^2b^{2^{n-5}}, d\rangle$ and $\langle b, d, a^2b^{2^{n-5}}\rangle$ is abelian. Without loss of generality, we may assume $d^2 = b$ or ba^2. Then $[d^2, a] = b^{-2}$. By calculation, $[d, a] = b^{-1}$ or $b^{2^{n-4}-1}$. If $d^2 = b$ and $[d, a] = b^{-1}$ or $b^{2^{n-4}-1}$, then $G = \langle a, d\rangle$ is the type (1). Let $b_1 = a^3d$, $c_1 = b^{-1}$ if $d^2 = ba^2$, $[d, a] = b^{-1}$, and let $b_1 = ad$, $c_1 = b^{2^{n-4}-1}$ if $d^2 = ba^2$, $[d, a] = b^{2^{n-4}-1}$. In either case, we get $G = \langle a, b_1, c_1 \mid a^8 = b_1^2 = c_1^{2^{n-3}} = 1, a^4 = c_1^{2^{n-4}}, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2}\rangle$. Thus G is the type (2). □

Lemma 5.12. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. Then there is not a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^4 = b^2 = c^{2^{n-4}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.*

Proof. Otherwise, it is easy to see $G' \leq \langle a^2, ab\rangle$. If $(ab)^i a^{2j} \in G'$, where i is odd, then $|G'| = |\langle ab, a^2\rangle| = 2^{n-2}$. Thus G is of maximal class, a contradiction. It follows that $G' \leq \langle c, a^2\rangle$. Take a suitable $d \in G \setminus M$ such that $[c^{2^{n-6}}, d] = 1$. It is easy to see $[a^2, d] = 1$. Thus $a^2 \in Z(G)$. By Lemma 5.1, $ab \in \langle a^2, c^{2^{n-6}}, d\rangle$. It follows that $[a, d^2] = c^k$, where k is odd. On the other hand, we assume $[a, d] = a^{2s}c^t$ and so $[a, d^2] \in \langle c^2\rangle$, a contradiction. □

By similar arguments as in Lemma 5.12, we have the following result:

Lemma 5.13. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. Then there is not a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^4 = b^4 = c^{2^{n-4}} = 1, b^2 = c^{2^{n-5}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$ or $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-4}} = 1, a^4 = c^{2^{n-5}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.*

Lemma 5.14. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 6$. If there is a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^{2^{n-3}} = b^2 = c^4 = 1, c^2 = a^{2^{n-4}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle$, then G is one of the following pairwise non-isomorphic groups:*

(1) $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle$;
(2) $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-3}} = 1, a^4 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2}\rangle$.

Proof. It is easy to see $G' \leq \langle a, c\rangle$ and $\langle a^{2^{n-4}}\rangle = \langle c^2\rangle \leq Z(G)$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}c, d] = 1$. Then, by Lemma 5.1, $\langle a, c, d\rangle$ is abelian and $a \in \langle a^{2^{n-4}}, d, a^{2^{n-5}}c\rangle$. Without loss of generality, we may assume $d^2 = a$ or ac. Then $[d^2, b] = [a, b] = a^{-2}$. By calculation, $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$. If $d^2 = a$ and $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$, then $G = \langle b, c, d\rangle$ is isomorphic to the group of type (1). Let $d_1 = bd$, $c_1 = a^{-1}$ if $d^2 = ac$, $[d, b] = a^{-1}$, and let $d_1 = bd$, $c_1 = a^{2^{n-4}-1}$ if $d^2 = ac$, $[d, b] = a^{2^{n-4}-1}$. In either case, we have $G = \langle b, c_1, d_1 \mid b^2 = d_1^8 = c_1^{2^{n-3}} = 1, d_1^4 = c_1^{2^{n-4}}, [d_1, b] = c_1, [c_1, b] = [c_1, d_1] = c_1^{-2}\rangle$. Thus G is the type (2). □

Lemma 5.15. *Let G be a $\mathcal{CAC}$-2-group of order 2^n and $n \geq 7$. Then there is not a maximal subgroup M in G such that $M \cong \langle a, b, c \mid a^{2^{n-3}} = b^2 = c^4 = 1, c^2 = a^{2^{n-4}}, [b, a] = c, [c, b] = c^2, [c, a] = 1\rangle$.*

Proof. Otherwise, $G' \leq \langle a, c\rangle$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}c, d] = 1$. Then, by Lemma 5.1, $\langle a, c, d\rangle$ is abelian and $a \in \langle a^{2^{n-5}}c, a^{2^{n-4}}, d\rangle$. It follows that $[b, d^2] = c^i$, where i is odd. However it is impossible. □

By checking the list of groups of order 2^5, we get the following result:

Theorem 5.16. *Let G be a group of order 2^5. Then G is a $\mathcal{CAC}$-2-group if and only if G is one of the following pairwise non-isomorphic groups:*

(1) *metacyclic minimal non-abelian 2-groups;*
(2) *2-groups of maximal class;*
(3) $D_{2^4} \times C_2$;
(4) $SD_{2^4} \times C_2$;
(5) $Q_{2^4} \times C_2$;
(6) $\mathrm{M}_2(2, 2, 1)$;
(7) $\mathrm{M}_2(2, 2) \times \mathrm{C}_2$;
(8) $D_8 * C_{2^3}$;
(9) $\langle a, b \mid a^4 = b^8 = 1, [b, a] = b^{-2}\rangle$;

(10) $\langle a, b \mid a^4 = b^8 = 1, [b, a] = b^2 \rangle$;
(11) $\langle a, b \mid a^8 = b^8 = 1, a^4 = b^4, [b, a] = b^2 \rangle$;
(12) $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = [c, b] = 1 \rangle$;
(13) $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$;
(14) $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = c^2, [c, b] = 1 \rangle$;
(15) $\langle a, b, c \mid a^4 = b^2 = c^4 = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
(16) $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, b^2 = c^2, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
(17) $\langle a, b, c \mid a^8 = b^2 = c^4 = 1, a^4 = c^2, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$;
(18) $\langle a, b, c \mid a^2 = b^4 = c^4 = 1, [b, a] = c^2, [c, a] = b^2, [c, b] = 1 \rangle$;
(19) $\langle a, b, c \mid a^2 = b^4 = c^4 = 1, [b, a] = c^2, [c, a] = b^2c^2, [c, b] = 1 \rangle$;
(20) $\langle a, b, c \mid a^2 = b^4 = c^4 = 1, [b, a] = b^2, [c, a] = c^2, [c, b] = 1 \rangle$;
(21) $\langle a, b, c, d \mid a^4 = b^4 = c^2 = d^2 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = [d, a] = [c, b] = [d, b] = [d, c] = 1 \rangle$.

Theorem 5.17. *Let G be a group of order 2^n and $n \geq 6$. Then G is a $\mathcal{CAC}$-2-group if and only if G is one of the following pairwise non-isomorphic groups:*
(1) *metacyclic minimal non-abelian* 2*-groups;*
(2) 2*-groups of maximal class;*
(3) $D_{2^{n-1}} \times C_2$;
(4) $SD_{2^{n-1}} \times C_2$;
(5) $Q_{2^{n-1}} \times C_2$;
(6) $D_8 * C_{2^{n-2}}$;
(7) $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{-2} \rangle$;
(8) $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{2^{n-3}-2} \rangle$;
(9) $\langle a, b \mid a^8 = b^{2^{n-2}} = 1, a^4 = b^{2^{n-3}}, [b, a] = b^{-2} \rangle$;
(10) $\langle a, b, c \mid a^{2^{n-3}} = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = [c, b] = 1 \rangle$;
(11) $\langle a, b, c \mid a^{2^{n-3}} = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1 \rangle$;
(12) $\langle a, b, c \mid a^4 = b^2 = c^{2^{n-3}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
(13) $\langle a, b, c \mid a^4 = b^4 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
(14) $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-3}} = 1, a^4 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
(15) $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [b, a] = a^2, [c, a] = [c, b] = 1 \rangle$;
(16) $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [b, a] = c, [c, b] = c^2, [c, a] = 1 \rangle$;
(17) $\langle a, b, c, d, e \mid a^4 = d^2 = e^2 = 1, a^2 = b^2 = c^2, [b, a] = a^2, [c, a] = d, [c, b] = e, [d, a] = [e, a] = [d, b] = [e, b] = [c, d] = [c, e] = [d, e] = 1 \rangle$;
(18) $\langle a, b, c, d \mid a^4 = b^4 = d^2 = 1, b^2 = c^2, [b, a] = b^2, [c, a] = a^2, [c, b] = d, [d, a] = [d, b] = [c, d] = 1 \rangle$;
(19) $\langle a, b, c, d \mid a^4 = b^4 = d^2 = 1, a^2 = c^2, [b, a] = a^2, [c, a] = b^2c^2, [c, b] = d, [d, a] = [d, b] = [c, d] = 1 \rangle$;
(20) $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, [b, a] = c^2, [c, a] = b^2c^2, [c, b] = a^2b^2 \rangle$;

(21) $\langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^2 = d^2, [a, b] = b^2, [a, c] = a^2, [a, d] = b^2, [b, c] = 1, [b, d] = a^2, [c, d] = c^2\rangle$.

Proof. Assume each maximal subgroup of G is abelian. Then G is minimal non-abelian. If G is not metacyclic, then we may assume $G = \langle a, b, c \mid a^{2^u} = b^{2^v} = c^2 = 1, [b, a] = c, [c, a] = [c, b] = 1\rangle$, where $u \geq v \geq 1$. Since $n \geq 6$, $u \geq 3$. Noticing that $\langle a^2, b, c\rangle \leq C_G(\langle a^{2^2}, b\rangle)$ and $\langle a^2, b, c\rangle/\langle a^{2^2}, b\rangle$ is not cyclic, we see $C_G(\langle a^{2^2}, b\rangle)/\langle a^{2^2}, b\rangle$ is not cyclic, in contradiction to the hypothesis. Thus G is of the type (1).

If there exists a $M \lessdot G$ such that M is not abelian and M is of maximal class, then there exists a $M_1 \lessdot G$ such that M_1 is not of maximal class by Lemma 2.7. If M_1 is abelian, then G is of the type (2), (3), (4), (5), or (15) according to Lemma 5.7. Without loss of generality, we may assume that M is not abelian and M is not of maximal class. By Lemma 3.7, M is a $\mathcal{CAC}$-2-group.

If $n \geq 8$, then, by induction hypothesis, M is a group of types (1) and (3) − (16) with order 2^{n-1}. By Lemma 5.3−5.6 and Lemma 5.8−5.15, G is a group of types (3) − (16).

Now we consider $n = 6$ and $n = 7$.

Case 1. $n = 6$

In this case, M is one of the groups listed in Theorem 5.16 except the type (2). If M is a group of types (1), (3) − (5) and (8) − (17) listed in Theorem 5.16, then G is of the type (3) − (16) or (21) according to Lemma 5.3−5.6 and Lemma 5.8−5.14. Thus, we only need to consider that M is a group of the types (6), (7), (18), (19), (20), and (21) listed in Theorem 5.16.

If M is of the type (6) in Theorem 5.16, then we may assume $M = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = c, [c, a] = [c, b] = 1\rangle$. Then $Z(M) = \langle a^2, b^2, c\rangle \cong C_2^3$. By Lemma 5.2, $\Phi(G) = Z(M)$ and for any $g \in G \setminus M$, $x \in M \setminus Z(M)$, we have $[x, g] \neq 1$ and $o(g) = 4$. It follows from $M' = \langle c\rangle$ that $[x, g] \notin \langle c\rangle$. Thus $|G'| > 4$ and so $G' = Z(M)$. Without loss of generality, we may assume $g^2 = a^2$, c or a^2c.

If $g^2 = a^2$, then $o(bg) = 2$ if $[b, g] = a^2b^2$ and $o(abg) = 2$ if $[ab, g] = b^2c$. Thus $[b, g] \neq a^2b^2$ and $[ab, g] \neq b^2c$. Without loss of generality, we may assume $[a, g] = a^2$, b^2, a^2c or b^2c. If $[a, g] = a^2$, then $[b, g] = b^2$ or b^2c. If $[b, g] = b^2$, then $G = \langle a, b, c, g \mid a^4 = b^4 = c^2 = 1, a^2 = g^2, [a, b] = c, [a, g] = a^2, [b, g] = b^2, [a, c] = [b, c] = [c, g] = 1\rangle$. By a simple checking, G is the type (17). If $[b, g] = b^2c$, then G is also the type (17). If $[a, g] = b^2$, then $[b, g] = a^2$, a^2c or a^2b^2c. Then G is the type (18) if $[b, g] = a^2$ or a^2c, and G is the type (19) if $[b, g] = a^2b^2c$. If $[a, g] = a^2c$, then $[b, g] = b^2$, b^2c or a^2b^2c. It is easy to see that G is the type (17) if $[b, g] = b^2$, G is the type (18) if $[b, g] = b^2c$ and G is the type (19) if $[b, g] = a^2b^2c$. If $[a, g] = b^2c$, then $[b, g] = a^2$, a^2c or a^2b^2c. Thus G is the type (19) if $[b, g] = a^2c$ or a^2b^2c, and G is the type (18) if $[b, g] = a^2$.

If $g^2 = c$, then $o(ag) = 2$ if $[a, g] = a^2c$ and $(ag)^2 = (ab)^2$ if $[a, g] = b^2$. Thus, without loss of generality, we may assume $[a, g] = a^2$ or b^2c. Similarly, we may assume $[b, g] = b^2$, a^2b^2 or a^2c and $[ab, g] = a^2c$, b^2c or a^2b^2c. It follows that $[a, g] = b^2c$ and $[b, g] = a^2b^2$. Thus G is the type (20).

If $g^2 = a^2c$, then, without loss of generality, we may assume $[a, g] = a^2$ or b^2 and $[b, g] = a^2$ or b^2. It follows that $[ab, g] = a^2b^2$. Thus $(abg)^2 = a^2$, which is reduced to the case of $g^2 = a^2$.

If M is of the type (7) in Theorem 5.16, then, by using the similar arguments as that M is of the type (6), we have that G is of the type (17), (18) or (19).

If M is of the type (21) in Theorem 5.16, then, by using the similar arguments as that M is of the type (6), we have that G is of the type (17).

If M is of the type (18), (19) or (20) in Theorem 5.16, then, by the same arguments as in Lemma 5.6, $Z(M) = \langle b^2, c^2\rangle \leq Z(G)$ and $G' = M'$. Since $\langle a, b^2, c^2\rangle \cong C_2^3$ and $a \notin Z(G)$, we see $C_G(a) = \langle a, b^2, c^2\rangle$ by Lemma 3.2. Noticing that $[a, M] = G'$, we may take a suitable $d \in G \setminus M$ such that $[a, d] = 1$. Then $d \in C_G(a)$, a contradiction.

Case 2. $n = 7$

We only need to consider M is a group of types (17), (18), (19), (20) and (21) listed in theorem.

If M is of the type (17), then $M' = Z(M) = \langle a^2, d, e\rangle \cong C_2^3$. By Lemma 5.2, $G' = M'$ and for any $g \in G \setminus M$, $x \in M \setminus Z(M)$, we have $[x, g] \neq 1$. It follows from $[a, M] = \langle a^2, d\rangle$ that $[a, g] \notin \langle a^2, d\rangle$. Similarly, $[b, g] \notin \langle a^2, e\rangle$ and $[c, g] \notin \langle d, e\rangle$. We may take a suitable $h \in G \setminus M$ such that $[a, h] = e$. Then $[b, h] = d$, da^2, de or da^2e and $[c, h] = a^2$, a^2d, a^2e or a^2de. It follows that $[ac, bh] = 1$ if $[c, h] = a^2$ and $[ac, cbh] = 1$ if $[c, h] = a^2d$. Thus $[c, h] = a^2e$ or a^2de. If $[c, h] = a^2e$, then $[ab, ch] = 1$ if $[b, h] = d$, $[ab, ach] = 1$ if $[b, h] = da^2$, $[bc, ah] = 1$ if $[b, h] = de$ and $[abc, ch] = 1$ if $[b, h] = da^2e$. If $[c, h] = a^2de$, then, by letting $h_1 = ah$, we see $[a, h_1] = e$ and $[c, h_1] = a^2e$. So we may also have a contradiction.

If G has a maximal subgroup which is isomorphic to type (18), (19) or (20), then, by using the similar arguments as that M is of the type (17), we may have a contradiction.

If M is of the type (21), then $\Omega_1(M) = Z(M) = M' = \Phi(M) = \langle a^2, b^2\rangle \leq Z(G)$. We claim $\exp(G) = 4$. Otherwise, there exists an element $g \in G \setminus M$ such that $o(g) = 8$. Assume $g^2 = x_1$. It is clear that there exist $x_2 \in M \setminus \langle a^2, b^2, x_1\rangle$ and $x_3 \in \langle a^2, b^2\rangle$ such that $[x_1, x_2] = 1$ and $\langle x_1, x_3\rangle$ is not cyclic. By Lemma 5.1, we see $x_2 \in C_M(x_1) \leq \langle x_3, g\rangle$ and so $x_2 \in \langle x_1, x_3\rangle$, a contradiction. Thus the claim holds. Hence for any $x \in G \setminus M$, $x^2 \in \Omega_1(M) \leq Z(G)$ and therefore $\Phi(G) \leq Z(G)$. So $G' = M'$. Noticing that $[c, M] = G'$, we may take a suitable $x \in G \setminus M$ such that $[c, x] = 1$. By Lemma 5.1, we see $b \in C_M(c) \leq \langle c, x, a^2\rangle$. However it is impossible. So we may not have a $\mathcal{CAC}$-2-group.

It is easy to see that those groups in theorem are pairwise non-isomorphic. In following we prove those groups in theorem are $\mathcal{CAC}$-2-groups.

If G is of the type (1), then G is a $\mathcal{CAC}$-2-group by Lemma 3.4.

If G is of the type (2), then G is metacyclic and $\Phi(G)$ is cyclic. Let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$, then $H \nleq \Phi(G)$ and so $H \nleq \Phi(C_G(H))$. Since G is metacyclic, $C_G(H)$ is metacyclic. Thus $d(C_G(H)) \leq 2$. It follows that there exists an element $g \in G$ such that $C_G(H) = \langle H, g\rangle$. Hence $C_G(H)/H = \langle \bar{g}\rangle$ is cyclic. So G is a $\mathcal{CAC}$-2-group.

If G is of the type (3), then assume $G = \langle a, b, c \mid a^{2^{n-2}} = b^2 = c^2 = 1, [a, b] = a^{-2}, [c, a] = [c, b] = 1\rangle$. Let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$, then there exists an element $x \in H$ with $x \notin Z(G)$. Assume $x = a^i b^j c^k$ with $j = 1$ or 2. If $j = 2$, then $H \leq C_G(H) \leq C_G(a^i c^k) = \langle a, c\rangle \cong C_{2^{n-2}} \times C_2$. Thus $C_G(H)/H$ is cyclic. If $j = 1$, then $C_G(H) \leq C_G(a^i b c^k) = \langle a^i b, c, a^{2^{n-3}}\rangle$. Thus $|C_G(H)| \leq 8$. Since $|H| \geq 4$, $C_G(H)/H$ is cyclic. So G is a $\mathcal{CAC}$-2-group.

Similarly, if G is a group of types (4), (5), (7) – (9), and (12) – (15), then G is a $\mathcal{CAC}$-2-group.

If G is of the type (6), then $Z(G)$ is a cyclic subgroup of index 4. So G is a $\mathcal{CAC}$-2-group by Lemma 3.8.

If G is a group of types (10), (11) and (16), then $|Z(G)| \geq 2^{n-3}$, $\Phi(G) \leq Z(G)$ or $\Phi(G)$ is cyclic. Let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$. Noticing that $HZ(G) \leq Z(C_G(H))$ and $|C_G(H)/HZ(G)| \leq 2$, we see $C_G(H)$ is abelian. It is easy to check $r(G) = 2$. Then $d(C_G(H)) \leq 2$. Since $\Phi(G) \leq Z(G)$ or $\Phi(G)$ is cyclic, we have $H \nleq \Phi(G)$ and so $H \nleq \Phi(C_G(H))$. Thus there exists an element $g \in C_G(H)$ such that $C_G(H) = \langle H, g\rangle$. Hence $C_G(H)/H = \langle \bar{g}\rangle$. So G is a $\mathcal{CAC}$-2-group.

If G is a group of types (17) – (21), then $\Omega_1(G) = Z(G)$ and G has no abelian maximal subgroup. Let H be a non-cyclic abelian subgroup of G and $H \nleq Z(G)$. Then there exists an element $x \in H$ such that $o(x) = 4$. Thus $|H| \geq 8$. If G is a group of types (17) – (20), then, since $|Z(G)| = 8$, we see $|Z(G)H| \geq 16$. It follows that $|C_G(H)| = 16$. If G is the type (21), then $|Z(G)| = 4$. It is easy to check $Z(M) = Z(G)$ for all subgroups M of order 32. It follows that $|C_G(H)| \leq 16$. Thus $|C_G(H)/H| \leq 2$ and therefore G is a $\mathcal{CAC}$-2-group. □

Acknowledgements

The authors would like to thank the referee for valuable suggestions and comments that contributed to the version of this paper.

The research of the work was partially supported by the National Natural Science Foundation of China(11371237) and a grant of "The First-Class Discipline of Universities in Shanghai".

References

[1] Y. Berkovich, Groups of Prime Power Order, Vol. I, Walter de Gruyter, Berlin, New York, 2008.

[2] N. Blackburn, Generalizations of certain elementary theorems on p-groups, *Proc. Lond. Math. Soc.* **11** (1961) 1–22.
[3] B. Huppert, Endliche Gruppen I, Springer, New York, Berlin, 1967.
[4] K. Ishikawa, Finite p-groups up to isoclinism, which have only two conjugacy lengths, *J. Algebra* **220** (1999) 333–354.
[5] K. Ishikawa, On finite p-groups which have only two conjugacy lengths, *Israel J. Math.* **129** (2002) 119–123.
[6] X. Li and J. Zhang, Finite p-groups and centralizers of non-central elements, *Comm. Algebra* **41** (2013), no. 9, 3267–3276.
[7] M. F. Newman and M. Xu, Metacyclic groups of prime-power order (Research announcement), *Adv. Math.* **17** (1988) 106–107.
[8] M. Xu, L. An and Q. Zhang, Finite p-groups all of whose non-abelian proper subgroups are generated by two elements, *J. Algebra* **319** (2008), no. 9, 3603–3620.
[9] L. Zhang, J. Wang and H. Qu, Finite p-groups whose non-central cyclic subgroups have cyclic quotient groups in their centralizers, *Bull. Korean Math. Soc.* **52** (2015), no. 2, 367–376.
[10] X. Zhang and X. Guo, Finite p-groups whose non-normal cyclic subgroups have small index in their normalizers, *J. Group theory* **15** (2012), no. 5, 641–659.
[11] X. Zhang and X. Guo, On the Wielandt subgroup in a p-group of maximal class, *Chin. Ann. Math. Ser. B* **33** (2012), no. 1, 83–90.
[12] L. Zhao and X. Guo, Finite p-groups in which the normal closures of the non-normal cyclic subgroups have small index, *J. Algebra Appl.* **13** (2014), no. 2, Article ID 1350087, 7 pages.

(Jiao Wang) Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China.
E-mail address: wangjiaotiedan@163.com

(Xiuyun Guo) Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China.
E-mail address: xyguo@staff.shu.edu.cn