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Finite $p$-groups and centralizers of non-cyclic abelian subgroups

Author(s):
J. Wang and X. Guo
FINITE \( p \)-GROUPS AND CENTRALIZERS OF NON-CYCLIC ABELIAN SUBGROUPS

J. WANG AND X. GUO

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ABSTRACT. A \( p \)-group \( G \) is called a \( \text{CAC} \)-\( p \)-group if \( C_G(H)/H \) is cyclic for every non-cyclic abelian subgroup \( H \) in \( G \) with \( H \not\subseteq Z(G) \). In this paper, we give a complete classification of finite \( \text{CAC} \)-\( p \)-groups.

Keywords: Finite \( p \)-group, centralizer, normal rank, cyclic group.


1. Introduction

All groups considered in this paper are finite. Let \( H \) be an abelian subgroup of a group \( G \). Then

\[ 1 \leq H \leq C_G(H) \leq G \]

is always true. It is clear that \( G \) is abelian if and only if \( |G : C_G(H)| = 1 \) for every abelian subgroup \( H \). So it is interesting to investigate the structure of a group \( G \) if \( |G : C_G(H)| \) is small for every abelian subgroup \( H \). In fact, K. Ishikawa in [4, 5] investigates the structure of a \( p \)-group \( G \) with \( |G : C_G(x)| = p \) and the structure of a \( p \)-group \( G \) with \( |G : C_G(x)| = p^2 \) for every \( x \in G \) and gives the classifications for these kind of \( p \)-groups. On the other hand, it is also interesting to investigate the structure of a group \( G \) if \( |C_G(H) : H| \) is small for every abelian subgroup \( H \). In fact, Li and Zhang in [6] investigate the structure of a \( p \)-group \( G \) with \( |C_G(x) : \langle x \rangle| \leq p^k \) for \( k = 1 \) or 2 and \( p > 2 \). Moreover, many authors investigated the structure of groups by using the some kind of index of subgroups, for example [10–12]. Now it is natural to ask the following question, which is proposed by Berkovich in [1]:

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*Corresponding author.
Question 1: Classify the $p$-groups $G$ such that $C_G(H)/H$ is cyclic for every noncentral cyclic subgroup $H$ in $G$.

Question 1 has been answered in [9]. We may also ask the following questions:

Question 2: How about the structure of a $p$-group $G$ with $C_G(H)/H$ cyclic for every abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$?

Question 3: How about the structure of a $p$-group $G$ with $C_G(H)/H$ cyclic for every non-cyclic abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$?

It is clear that Question 3 is more general than Question 2. Furthermore, we have the following proposition.

Proposition 1.1. Let $G$ be a non-abelian $p$-group. If $C_G(x)/\langle x \rangle$ is cyclic for every non-central element $x \in G$. Then, for every non-cyclic abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$, $C_G(H)/H$ is cyclic.

In fact, let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\subseteq Z(G)$. Then there exists an element $x \in H$ with $x \not\in Z(G)$. By the hypothesis, $C_G(x)/\langle x \rangle$ is cyclic. Noticing that $C_G(x)$ is abelian and $H \subseteq C_G(x)$, we see $C_G(x)/H$ is cyclic. It follows from $C_G(H) \leq C_G(x)$ that $C_G(H)/H$ is cyclic.

Remark 1.2. Assume $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [b, a] = c, [c, a] = [c, b] = 1 \rangle$. Then it is easy to see that $C_G(H)/H$ is cyclic for every non-cyclic abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$. However, $a \not\in Z(G)$ and $C_G(a)/\langle a \rangle = \langle a, b^2, c \rangle/\langle a \rangle$ is not cyclic. So Question 3 is more general than Question 2.

In this paper we hope to investigate the structure of a $p$-group $G$ in which $C_G(H)/H$ is cyclic for every non-cyclic abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$. For convenience, we call this kind of $p$-groups CAC-$p$-groups.

It is clear that every abelian $p$-group must be a CAC-$p$-group. So in the following CAC-$p$-groups means non-abelian CAC-$p$-groups.

2. Preliminaries

For convenience, we first introduce some notions and notations.

Let $G$ be a $p$-group. Then $r(G) = \max \{ \log_p |E| \mid E$ is an elementary abelian subgroup in $G \}$ and $r_n(G) = \max \{ \log_p |E| \mid E$ is an elementary abelian normal subgroup in $G \}$ are called the rank and the normal rank of $G$ respectively.

We use $M_p(m, n)$ to denote the $p$-group

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle,$$ where $m \geq 2$,.
and \( M_p(m, n, 1) \) to denote the \( p \)-group
\[
\langle a, b, c \mid a^{p^n} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,
\]
where \( m \geq n \), and \( m + n \geq 3 \) if \( p = 2 \). We also use \( C_{p^n} \) and \( C_{p^n}^m \) to denote the cyclic group of order \( p^n \) and the direct product of \( n \) cyclic groups of order \( p^n \) respectively. If \( H \) and \( K \) are groups, then \( H \ast K \) denotes a central product of \( H \) and \( K \). \( M < G \) means \( M \) is a maximal subgroup of \( G \). For other notation and terminology the reader is referred to [3].

**Lemma 2.1.** [[8, Lemma 2.2]] Let \( G \) be a \( p \)-group. Then the following conditions are equivalent.
1. \( G \) is a minimal non-abelian \( p \)-group;
2. \( d(G) = 2 \) and \( |G'| = p \);
3. \( d(G) = 2 \) and \( \Phi(G) = Z(G) \).

**Lemma 2.2.** Let \( G \) be a \( p \)-group and \( c(G) = 2 \). Then \( G' \) is elementary abelian if and only if \( G/Z(G) \) is elementary abelian.

**Proof.** Since \( c(G) = 2 \), \( G' \) is elementary abelian if and only if \( [a^{p^r}, b] = [a, b]^p = 1 \) for all \( a, b \in G \), and \( [a^{p^r}, b] = [a, b]^p = 1 \) for all \( a, b \in G \) if and only if \( G/Z(G) \) is elementary abelian. Thus the lemma is true. \( \square \)

**Lemma 2.3.** [[1, Section 1, Lemma 1.1]] If a non-abelian \( p \)-group \( G \) has an abelian maximal subgroup, then \( |G| = p|G'||Z(G)| \).

**Lemma 2.4.** \(((7))\) Let \( p \) be an odd prime and let \( G \) be a metacyclic \( p \)-group. Then there are non-negative integers \( r, s, t, u \) with \( r \geq 1, u \leq r \) such that \( G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+u}} = a^{p^{r+s}}, a^b = a^{1+p^r} \rangle \). Furthermore, different values of the parameters \( r, s, t \) and \( u \) with the above conditions give non-isomorphic metacyclic \( p \)-groups.

**Lemma 2.5.** [[2, Theorem 4.1]] Let \( G \) be a group of order \( p^n \) with \( p > 2 \) and \( n \geq 5 \). If \( r_n(G) = 2 \). Then \( G \) is one of the following groups:
1. \( G \) is metacyclic;
2. \( G \cong M_p(1, 1, 1) \ast C_{p^{n-2}} \);
3. \( G \) is a 3-group of maximal class of order \( \geq 3^5 \);
4. \( G = \langle a, x, y \mid a^{p^{n-2}} = x^p = y^p = 1, [a, x] = y, [x, y] = a^{p^{n-3}}, [y, a] = 1 \rangle, i = 1 \) or \( \sigma \), where \( \sigma \) is a fixed square non-residue modulo \( p \).

**Lemma 2.6.** Let \( G \) be a \( p \)-group. Then
1. \([1, \text{Section 7, Theorem 7.1}] \) If \( K_{p-1}(G) \) is cyclic, then \( G \) is regular.
2. \([1, \text{Section 9, Exercise 10}] \) Let \( G \) be a 3-group of maximal class. Then \( G_1 = C_G(K_2(G)/K_4(G)) \) is abelian or metacyclic minimal non-abelian.
3. \([1, \text{Section 9, Exercise 1(c)}] \) Let \( G \) be a maximal class \( p \)-group of order \( p^n \). If \( p > 2 \) and \( n > 3 \). Then \( G \) has no cyclic normal subgroups of order \( p^2 \).
(4) [1, Section 10, Corollary 10.2] Suppose that $N$ is a normal subgroup of $G$, and $A$ is a maximal $G$-invariant abelian subgroup of $N$ with $\exp(A) = p^n, p^n > 2$. Then $\Omega_n(C_N(A)) = A$.

(5) [1, Section 41, Remarks 2] $G$ is metacyclic if and only if $\Omega_2(G)$ is metacyclic.

**Lemma 2.7.** Let $G$ be a $p$-group of order $p^n$ and $n \geq 4$. Then there exists a maximal subgroup $M$ of $G$ such that $M$ is not of maximal class.

**Proof.** Let $N \cong G$ and $|N| = p^2$. Then $G/C_G(N) \cong \text{Aut}(N)$. Thus $|G : C_G(N)| \leq p$. Let $M \leq C_G(N)$ such that $N \leq M$ and $|G : M| = p$. Since $|Z(M)| \geq |N| = p^2$, $M$ is not of maximal class. ∎

3. Some properties of $\mathcal{C}AC$-$p$-groups

In this section we discuss the properties of $\mathcal{C}AC$-$p$-groups which will be used later.

**Lemma 3.1.** If $G$ is a $\mathcal{C}AC$-$p$-group, then $r(G) \leq 3$.

**Proof.** If not, then there exists $A \leq G$ and $A \cong C_p^4$. If $A \not\cong Z(G)$, then there exist $a \in A \setminus Z(G)$ and $b \in A$ such that $\langle a, b \rangle$ is not cyclic. Since $A$ is abelian, we see $A \leq C_G((a, b))$ and $C_G((a, b))/(a, b)$ is not cyclic, in contradiction to the hypothesis. If $A \leq Z(G)$, then for any $x \in G \setminus Z(G)$, there exists $a \in A$ such that $\langle a, x \rangle$ is not cyclic. Since $\langle A, x \rangle \leq C_G((a, x))$ and $\langle A, x \rangle/(a, x)$ is not cyclic, we see $C_G((a, x))/(a, x)$ is not cyclic, another contradiction. ∎

**Lemma 3.2.** Let $G$ be a $\mathcal{C}AC$-$p$-group with $r(G) = 3$ and $A \leq G$ with $A \cong C_p^3$. If $A \not\cong Z(G)$. Then $C_G(A) = A$. If $A \leq Z(G)$, then $\Omega_1(G) = A$ and $\Omega_1(G) \leq Z(G)$.

**Proof.** Assume $A = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$. Then Lemma 3.1 implies $\Omega_1(C_G(A)) = A$.

If $A \not\cong Z(G)$, then we claim $C_G(A) = \Omega_1(C_G(A))$. Otherwise, there exists $x \in C_G(A) \setminus \Omega_1(C_G(A))$ with $o(x) = p^k$ and $k \geq 2$. Thus $xp^{k-1} \in \Omega_1(C_G(A)) = A$. On the other hand, since $A \not\cong Z(G)$, we may assume that $a \not\in Z(G)$. If $\langle xp^{k-1} \rangle \neq \langle a \rangle$, then $\langle a, xp^{k-1} \rangle$ is not cyclic and $\langle xp^{k-1}, a \rangle \not\cong Z(G)$. Thus, by the hypotheses of the lemma, $\langle a, b, c, x \rangle/(a, xp^{k-1})$ is a cyclic group. However, it is impossible. If $\langle xp^{k-1} \rangle = \langle a \rangle$, then, by the hypotheses of the lemma, $\langle a, b, c, x \rangle/(a, b)$ is cyclic. It is also impossible. So $C_G(A) = \Omega_1(C_G(A)) = A$.

If $A \leq Z(G)$, then $\Omega_1(G) = \Omega_1(C_G(A)) = A$. For any $x \in G$ with $o(x) = p^k$, if $xp \not\in Z(G)$, then $k \geq 3$. Furthermore, for any $y \in A \setminus \langle xp^{k-1} \rangle$, $\langle xp, y \rangle$ is not cyclic and $\langle xp, y \rangle$ is abelian. By the hypotheses, $\langle a, b, c, x \rangle/(xp, y)$ is cyclic. However, it is impossible. Hence $xp \in Z(G)$. So $\Omega_1(G) \leq Z(G)$ and the lemma is proved. ∎
Lemma 3.3. Suppose that $G$ is a metacyclic $p$-group and $p > 2$. Then $G$ is a $C\text{AC}$-$p$-group if and only if $G$ is a minimal non-abelian group.

Proof. If $G$ is a minimal non-abelian group, then $Z(G) = \Phi(G)$ by Lemma 2.1(3). Thus, for every non-cyclic abelian subgroup $H$ in $G$ with $H \not\subseteq Z(G)$, we have $H \not\subseteq \Phi(G)$ and so $H \not\subseteq \Phi(C_G(H))$. Since $G$ is metacyclic, $C_G(H)$ is metacyclic. Thus $d(C_G(H)) \leq 2$. It follows that there exists $g \in G$ such that $C_G(H) = \langle H, g \rangle$. Hence $C_G(H)/H$ is cyclic. That is, $G$ is a $C\text{AC}$-$p$-group.

Conversely, let $G$ be a $C\text{AC}$-$p$-group. By Lemma 2.4, we may assume $G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+1}} = a^{p^{r+s}}, a^b = a^{1+p^t} \rangle$, where $r, s, t, u$ are non-negative integers with $r \geq 1, u \leq r$. By calculation, we get $Z(G) = \langle a^{p^{r+s}}, b^{p^{r+s+1}} \rangle$. If $a^p \in Z(G)$, then $|G'| = p$. By Lemma 2.1, $G$ is minimal non-abelian. Thus we may assume $a^p \not\in Z(G)$. If $\langle a^p, b^{p^{r+s+1}} \rangle$ is cyclic, then $\langle b^{p^{r+s+1}} \rangle \leq \langle a \rangle \cap \langle b \rangle = \langle b^{p^{r+s+1}} \rangle$, which implies $t = 0$ and $r = u$. Let $b_1 = ba^{-1}$, then $b_1^{p^{r+s+1}} = 1$ and $\langle a^{p^{r+s}}, b_1^{p^{r+s+1}} \rangle \subseteq Z(G)$. Since $G$ is a $C\text{AC}$-$p$-group, $\langle a^p, b_1 \rangle/\langle a^{p^{r+s}}, b_1^{p^{r+s+1}} \rangle$ is cyclic. Noticing that $\langle a^{p^{r+s}}, b_1^{p^{r+s+1}} \rangle \subseteq \Phi(\langle a^p, b_1 \rangle)$, we see $\langle a^p, b_1 \rangle$ is cyclic, a contradiction. If $\langle a^p, b_1^{p^{r+s+1}} \rangle$ is not cyclic, then $\langle a, b_1^{p^{r+s+1}} \rangle/\langle a^p, b_1^{p^{r+s+1}} \rangle$ is cyclic and therefore $\langle a, b_1^{p^{r+s+1}} \rangle$ is cyclic, another contradiction. \hfill $\Box$

It is easy to see that the arguments in the proof of Lemma 3.3 is true for ordinary metacyclic 2-groups. Thus we have the following lemma without proof.

Lemma 3.4. Let $G$ be an ordinary metacyclic 2-group. Then $G$ is a $C\text{AC}$-2-group if and only if $G$ is a minimal non-abelian group.

Lemma 3.5. Let $G$ be a $C\text{AC}$-$p$-group of order $p^n$ and $n \geq 6$. Then $G$ has no abelian maximal subgroup $M$ such that $r(M) = 3$.

Proof. If not, assume $M \triangleleft G$ and $M = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ with $o(x) = p^i, o(y) = p^j, o(z) = p^k$, where $i \geq 1, j \geq 1, k \geq 1$. Then Lemma 3.2 implies $\Omega_1(G) = \Omega_1(M) \leq Z(G)$ and $\Omega_1(G) \leq Z(G)$. Since $M \not\subseteq Z(G)$, we may assume $x \not\in Z(G)$. Thus, by the hypotheses of the lemma, $\langle x, y, z \rangle/\langle x, y^{p^{i-1}} \rangle$ is cyclic, which implies $j = 1$. Similarly, $k = 1$. Hence $\langle x^p \rangle \times \langle y \rangle \times \langle z \rangle = Z(G)$ and $i \geq 3$. If there exists $a \in G \setminus M$ such that $\langle a, x^p \rangle$ is not cyclic, then, by hypothesis, $\langle a, x^p, y, z \rangle/\langle a, x^p \rangle$ is cyclic. However, it is impossible. So for any $a \in G \setminus M$, $\langle a, x^p \rangle$ is cyclic. It follows from $a^p \in Z(G)$ that $o(a) \leq o(x)$ and $\langle a \rangle \not\leq \langle x^p \rangle$. Thus we may assume $x^p = a^p$ or $x^p = a^p$. If $x^p = a^p$, then $a^{-1}x^p \in \Omega_1(G) \leq M$ and therefore $a \in M$, a contradiction. If $x^p = a^p$, then, since $[a, x] \in Z(G)$, we see $o(ax^{-1}) = p^2$. Noticing that $ax^{-1} \not\in M$, we have $x^p = (ax^{-1})^p$ by the above, a contradiction. \hfill $\Box$

Lemma 3.6. Suppose that $G$ is a $C\text{AC}$-$p$-group of order $p^n$ with $n \geq 6$ and $r(G) = 3$. If $p > 2$, then $G$ has no abelian maximal subgroup. If $p = 2$ and $G$
has an abelian maximal subgroup, then $G$ is isomorphic to one of the following non-isomorphic groups:

1. $D_{2n-1} \times C_2$;
2. $SD_{2n-1} \times C_2$;
3. $\langle a, b, c \mid a^4 = b^2 = c^{2n-3} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$.

Proof. If there exists a maximal subgroup $M$ in $G$ such that $M$ is abelian, then $r(M) \leq 2$ according to Lemma 3.5. The hypotheses $r(G) = 3$ implies that $r(M) = 2$. Let $M = \langle x \rangle \times \langle y \rangle$ with $o(x) = p^i$ and $o(y) = p^j$ for $i \geq 1$ and $j \geq 1$, and let $A \leq G$ with $A \cong C_p^3$. Since $Z(G) \leq M$, we have $A \not\leq Z(G)$. Thus $A = C_G(A)$ by Lemma 3.2. Hence $Z(G) = M \cap A = \Omega_1(M)$. Since $n \geq 6$, we may assume $i \geq 3$. In this case $\langle x^p, y^{p^i-1} \rangle \not\leq Z(G)$. Thus, by the hypotheses, $\langle x, y \rangle / \langle x^p, y^{p^i-1} \rangle$ is cyclic. It follows that $o(x) = p^{n-2}$ and $o(y) = p$. For any $g \in A \setminus M$, $G = \langle x, y, g \rangle$. By Lemma 2.3, $|G'| = p^{n-3}$. Now assume $[x, g] = x^{p^i}y^t$. It is easy to see that $G' = \langle x^{p^i}y^t \rangle$ and so $(s, p) = 1$. If $p > 2$, then, by Lemma 2.6(1), $G$ is regular. Thus $[x, g]^p = 1$ if and only if $[x, g]^p = 1$. However, $[x, g]^p = x^{p^i}y^t \neq 1$, a contradiction. If $p = 2$, then, according to $[x, g^2] = 1, [x, g] = x^{-2}$ or $x^{2n-3-2}$ or $x^{-2}y$ or $x^{2n-3-2}y$. If $[x, g] = x^{-2}$, then $G \cong D_{2n-1} \times C_2$. If $[x, g] = x^{2n-3-2}$, then $G \cong SD_{2n-1} \times C_2$. If $[x, g] = x^{-2}y$, then $G = \langle x_1, g, y_1 \mid x_1^4 = g^2 = y_1^{2n-3} = 1, [g, x_1] = y_1, [y_1, g] = [y_1, x_1] = y_1^{-2} \rangle$ when we set $x_1 = gx$ and $y_1 = x^2y$. In this case, $G$ is the type (3). If $[x, g] = x^{2n-3-2}y$, then $G$ is also the type (3).

Lemma 3.7. Let $G$ be a $\mathcal{CAC}$-p-group, and $H$ be a non-abelian subgroup of $G$. Then

1. $H$ is a $\mathcal{CAC}$-p-group.
2. If $Z(H)$ is not cyclic, then $Z(H) \leq Z(G)$.

Proof. (1) If $K$ is a non-cyclic abelian subgroup of $H$ and $K \not\leq Z(H)$, then $K \not\leq Z(G)$. By the hypotheses, $C_G(K)/K$ is cyclic and therefore $C_H(K)/K$ is cyclic. Hence $H$ is a $\mathcal{CAC}$-p-group.

The proof of (2) comes immediately from the definition of $\mathcal{CAC}$-p-groups.

Lemma 3.8. Let $G$ be a p-group. If $Z(G)$ is a cyclic subgroup of index $p^2$, then $G$ is a $\mathcal{CAC}$-p-group.

Proof. Let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\leq Z(G)$. Then $C_G(H) < G$ and $C_G(H) \geq HZ(G)$. Thus $C_G(H) = HZ(G)$ and therefore $C_G(H)/H \cong Z(G)/Z(G) \cap H$ is cyclic. So $G$ is a $\mathcal{CAC}$-p-group.

4. $\mathcal{CAC}$-p-groups of odd order

In this section we investigate the $\mathcal{CAC}$-p-groups for $p > 2$. 

Lemma 4.1. Let $G$ be a $p$-group of order $p^n$ and $r(G) = 2$ with $p > 2$ and $n \geq 3$. Then $G$ is a $\mathcal{CAC}$-group if and only if $G$ is one of the following pairwise non-isomorphic groups:

1. metacyclic minimal non-abelian $p$-groups;
2. $M_p(1, 1, 1)$;
3. $\langle a, b, c \mid a^2 = c^3 = 1, b^3 = a^3, [a, b] = c, [c, a] = 1, [c, b] = a^{-3} \rangle$;
4. $M_p(1, 1, 1) \ast C_{p^{n-2}}$;
5. $\langle a, x, y \mid a^b = x^p = y^p = 1, [a, x] = y, [x, y] = a^i y^{-3}, [y, a] = 1, i = 1 \rangle$ or $\sigma$, where $\sigma$ is a fixed square non-residue modulo $p$.

Proof. If $|G| \leq p^4$, then, the conclusion holds by checking the list of groups of order $p^3$ and $p^4$. Assume $|G| \geq p^5$. Since $r(G) = 2$, $r_n(G) \leq 2$. If $r_n(G) = 1$, then $G$ is cyclic, a contradiction. So $r_n(G) = 2$. Thus $G$ is one of the groups listed in Lemma 2.5. We discuss case by case.

If $G$ is of the type (1) in Lemma 2.5, then, by Lemma 3.3, $G$ is of the type (1).

If $G$ is of the type (2) in Lemma 2.5, then $Z(G)$ is a cyclic subgroup of index $p^2$. By Lemma 3.8, $G$ is a $\mathcal{CAC}$-group of the type (4).

If $G$ is of the type (3) in Lemma 2.5, then $G_1 = C_G(K_2(G)/K_4(G))$ is abelian or metacyclic minimal non-abelian by Lemma 2.6(2). Thus $\Phi(G_1) \leq Z(G_1)$ by Lemma 2.1. On the other hand, by [3, Section 14, Theorem 14.4], $G_1 < G$. Thus $|G_1| \geq 3^4$ and $|\Phi(G_1)| \geq 3^2$. Noticing that $|Z(G)| = 3$, we see $\Phi(G_1) \nsubseteq Z(G)$. Furthermore, by Lemma 2.6(3), $\Phi(G_1)$ and $G_1/\Phi(G_1)$ are not cyclic, which means that $G$ is not a $\mathcal{CAC}$-group.

If $G$ is of the type (4) in Lemma 2.5, then, by [9, Theorem 4.1] and Proposition 1, we see $G$ is a $\mathcal{CAC}$-group of the type (5).

Conversely, every group listed in the lemma is a $\mathcal{CAC}$-group and they are pairwise non-isomorphic. \hfill \Box

Lemma 4.2. Let $G$ be a $\mathcal{CAC}$-group of order $p^n$ with $p > 2$ and $n \geq 6$. If $r(G) = 3$, then, for every maximal subgroup $M$ of $G$, $r(M) = 3$.

Proof. Let $A \leq G$ with $A \cong C_p^3$. If there exists a $M \leq G$ such that $r(M) = 2$, then, by Lemma 3.6, $M$ is not abelian. Thus, according to Lemma 3.7, $M$ is a $\mathcal{CAC}$-group of order $p^{n-1}$. So $M$ is of type (1), (4), or (5) listed in Lemma 4.1.

If $M$ is of type (4), (5) or type (1) with $\exp(M) = p^{n-2}$ in Lemma 4.1, then, by calculation, we see $Z(M)$ is cyclic and $|Z(M)| \geq p^2$. Let $Z(M) = \langle a \rangle$ with $o(a) = p^k$. Since $\langle a^b \rangle \leq G$ and $|\langle a^b \rangle| = p$, $\langle a^b \rangle \leq Z(G)$. Furthermore, for any $b \in M \cap A \setminus \langle a^b \rangle$, we have $b \notin Z(G)$. Thus, by the hypotheses of the lemma, $\langle a, A \rangle/\langle a^b \rangle$ is cyclic. However, it is impossible.

If $M$ is of type (1) with $\exp(M) < p^{n-2}$ in Lemma 4.1, then assume $M = \langle a, b \mid a^v = b^u = 1, [a, b] = a^{u-1} \rangle$, where $u \geq 2, v \geq 2$ and $u + v = n - 1$. Thus
Z(M) is not cyclic. By Lemma 3.7, Z(M) ≤ Z(G). It follows from Lemma 3.2 that A ≤ Z(G). Since n ≥ 6, we may assume u ≥ 3. Then, by the hypotheses, ⟨a^{p^n}, b, A⟩/⟨a^{p^n-1}, b⟩ is cyclic. It is also impossible.  

**Lemma 4.3.** Let G be a CAC-p-group of order p^6 and p > 2. If r(G) = 3, then, for every maximal subgroup M of G, Z(M) = Ω₁(G) = Ω₂(G) = Z(G) = G′ = Φ(G) ∼= C_p^3.

**Proof.** Let M be a maximal subgroup of G. Then, by Lemma 3.6 and Lemma 4.2, M is not abelian and r(M) = 3. Let A ≤ G with A ∼= C_p^3. We consider the following two cases:

**Case 1. A ≤ Z(G).**

In this case, it is clear that A = Z(G). Then, by Lemma 3.2, Ω₁(G) ≤ Z(G) = Ω₁(G), which implies exp(G) = p^2. Since r(M) = 3, we have Ω₁(G) = Z(G) = Z(M) ≤ Φ(G). If Z(G) < Φ(G), then d(G) = 2. Assume G = ⟨g_1, g_2⟩ and [g_1, g_2] = x. If o(x) = p, then x ∈ Z(G) and therefore G′ = p. So G is minimal non-abelian by Lemma 2.1, a contradiction. If o(x) = p^2, then, by calculation, we get [g_1, g_2]^p = x^p[x, g_2]^{p^2-1} = x^p ≠ 1, in contradiction to Ω₁(G) ≤ Z(G). So Z(G) = Φ(G) and G is regular. By [1, Section 7, Theorem 7.2], [G/Ω₁(G)] = [Ω₁(G)] and therefore Ω₁(G) = Ω₁(G). If |G'| < p^3, then there exist x_1 and x_2 in G with o(x_1) = o(x_2) = p^2 such that x_1 ∈ G \ (x_2, Φ(G)) and [x_1, x_2] = 1. If ⟨x_1⟩ ∩ ⟨x_2⟩ = 1, then [⟨x_1, x_2, A⟩] = p^3, in contradiction to that G has no abelian maximal subgroup. If ⟨x_1⟩ ∩ ⟨x_2⟩ ≠ 1, then ⟨x_1^p⟩ = ⟨x_2^p⟩. Obviously, there exists an element a ∈ A such that ⟨x_1, a⟩ is not cyclic. Then, by the hypothesis, ⟨x_1, x_2, A⟩/⟨x_1, a⟩ is cyclic. However, it is impossible. Hence, for every M < G, Z(M) = Ω₁(G) = Ω₁(G) = Z(G) = G′ = Φ(G) ∼= C_p^3.

**Case 2. A ∉ Z(G).**

By Lemma 3.2, C_G(A) = A and so Z(G) < A in this case. Since r(M) = 3, there exists a B ∼= C_p^3 such that B ≤ M and C_G(B) = B. Let N ≤ M with Z(G) < B < N ≤ M. If Z(G) < Z(M), then Z(M) is not cyclic. By Lemma 3.7, Z(M) ≤ Z(G), a contradiction. Thus Z(M) = Z(G). Similarly, Z(G) = Z(N) and therefore Z(G) = Z(N) = Z(M). Now we consider the following two subcases:

**Subcase 1. |Z(N)| = p**

By [1, Section 1, Exercise 4], N is of maximal class. Then N′ ∼= C_p × C_p and B = C_N(N′) by the classification of maximal class p-groups of order p^4. Since M/C_M(N′) ≤ Aut(N′), we have C_M(N′) ⊆ M. By the hypotheses of the lemma, C_M(N′)/N′ is cyclic and so C_M(N′) is abelian. It follows that C_M(N′) ≤ C_G(B), in contradiction to C_G(B) = B.

**Subcase 2. |Z(N)| = p^2**

Since r(N) = 3, by checking the list of groups of order p^4, we see N ∼= M_p(1,1,1) × C_p or M_p(2,1,1) or M_p(2,1) × C_p.
If \( N \cong M_p(1,1,1) \times C_p \), then we may assume \( N = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, [b, a] = c, [c, a] = [d, a] = [c, b] = [d, b] = [c, d] = 1 \rangle \). In this case \( Z(N) = Z(G) = \langle c, d \rangle \). Since \( |M| = p^5 \), we have \( |K_3(M)| \leq p^2 \). Thus \( |G/C_G(K_3(M))| \mid p \). So \( K_3(M) \leq Z(C_G(K_3(M))) \leq Z(G) \). Take \( x \in M \setminus N \). If \( [a, x] \notin Z(G) \), then \( [a, x, x] \in Z(G) \). Without loss of generality, we may assume \( [a, x] \in Z(G) \). Noticing that \( C_G(a) = C_G(\langle a, c, d \rangle) = \langle a, c, d \rangle \) and \([b, a] = c\), we see \([g, a] \notin \langle c \rangle \) for any \( g \in G \setminus N \). Thus we may assume \([a, x] = c^d \). For every integer \( j \), since \( C_G(a^j b) = \langle a^j b, c, d \rangle \), we see \([b, x] \notin Z(G) \). It follows that \( M' = \langle a, c, d \rangle \) and \( \langle a, c, d \rangle \leq G \). Take \( y \in G \setminus M \). Since \([a, y] \notin \langle c \rangle \), we may assume \([a, y] = c^d \). It follows that \([a, xy^{-1}] \in \langle c \rangle \) and \( xy^{-1} \in N \), a contradiction.

If \( N \cong M_p(2,1,1) \), then we may assume \( N = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, [b, a] = c, [c, a] = [c, b] = 1 \rangle \). Thus \( Z(N) = Z(G) = \langle a^{p^2}, c \rangle, B = \Omega_1(N) = \langle a^{p^2}, b, c \rangle \). So \( C_G(b) = B \). Since \( |M/\Omega_1(N)| = p^2 \), \( M' \leq \Omega_1(N) \). Take \( x \in M \setminus N \). Then we may assume \([x, b] = a^{p^2} c \). Thus \( x^p \in C_G(b) \), which implies \( \exp(M) = p^2 \). If \( o(x) = p \), then \( \langle a^p, b, c, x \rangle \cong M_p(1,1,1) \times C_p \), a contradiction. So \( o(x) = p^2 \) and \( \Omega_1(N) = \Omega_1(M) \). Take \( y \in G \setminus M \) and assume \([y, b] = y_1, [y, y_1, b] = y_2 \). If \( o(y_1) = p^2 \), then \([y, b^p] = y_1^j y_2^k = y_1^p \neq 1 \), a contradiction. If \( o(y_1) = p \), then \([y, b] \in \Omega_1(M) = \Omega_1(N) \). So we may assume \([y, b] = a^{p^2} c \). Thus \([x y^{-1}, b] \in \langle c \rangle \) and therefore \( x y^{-1} \in N \), a contradiction.

If \( N \cong M_p(2,1) \times C_p \), then, by the similar arguments as in the case \( N \cong M_p(2,1,1) \), we may also have a contradiction. \( \square \)

**Lemma 4.4.** Let \( G \) be a \( CAC-p \)-group of order \( p^n \) with \( p > 2 \) and \( n \geq 7 \). Then \( r(G) = 2 \).

*Proof.* Without loss of generality, we may assume \( n = 7 \) by Lemma 3.6, Lemma 3.7, and Lemma 4.2. If \( r(G) \neq 2 \), then \( r(G) = 3 \) by Lemma 3.1. Let \( M \) be a maximal subgroup of \( G \). Then, according to Lemma 3.6, Lemma 3.7, and Lemma 4.2, \( M \) is not abelian, \( r(M) = 3 \) and \( G \) has no abelian subgroup of index \( p^2 \). Furthermore, by Lemma 4.3, \( \Omega_1(M) = \Omega_1(M) = Z(M) = M' \cong C_p^3 \). Thus \( \Omega_1(G) = Z(M) \leq Z(G) \) and \( \Omega_1(G) \leq Z(G) \) by Lemma 3.2. If \( Z(M) < Z(G) \), then \( G \) has an abelian subgroup of index \( p^2 \), a contradiction. Hence \( \Omega_1(G) = \Omega_1(G) \). For any \( a, b \in G \), if \([a, b] = x \) and \([x, b] = y \), then \( y \in Z(G) \). By calculation, \([a, b^p] = x^p y^{p(p-1) \over 2} = x^p \). Thus \( o(x) \leq p \), and therefore \( G' \leq Z(G) \) and \( G \) is regular. According to [1, Section 7, Theorem 7.2], we see \([G/\Omega_1(G)] = [\Omega_1(G)] \) and therefore \([G] = p^3 \), in contradiction to the hypothesis. \( \square \)

According to Lemma 4.1 and Lemma 4.4, we have the following result:

**Theorem 4.5.** Let \( G \) be a \( p \)-group of order \( p^n \) with \( p > 2 \) and \( n \geq 7 \). Then \( G \) is a \( CAC-p \)-group if and only if \( G \) is one of the following pairwise non-isomorphic groups:
(1) metacyclic minimal non-abelian $p$-groups;
(2) $M_p(1, 1, 1) \ast C_{p^{n-2}}$;
(3) $\langle a, x, y \mid a^{p^n-2} = x^p = y^p = 1, [a, x] = y, [x, y] = a^i a^{p^n-3}, [y, a] = 1, i = 1$ or $\sigma$, where $\sigma$ is a fixed square non-residue modulo $p$. 

5. CAC-$p$-groups of even order

In this section we investigate the CAC-2-groups.

**Lemma 5.1.** Let $G$ be a CAC-$p$-group and $H$ be a subgroup of $G$. If there exist $a, b,$ and $c$ in $G$ such that $a \in H \setminus Z(H)$, $b \in Z(G) \cap H \setminus \langle a \rangle$, and $c \in C_G(a) \setminus H$, then $\langle C_H(a), c \rangle = \langle a, b, c \rangle$ is abelian.

**Proof.** By the hypotheses of the lemma, and $c \notin H$, we see $\langle C_H(a), c \rangle / \langle a, b \rangle = \langle \langle \cdot \rangle \rangle$. So $\langle C_H(a), c \rangle = \langle a, b, c \rangle$ is abelian. □

**Lemma 5.2.** Let $G$ be a CAC-$2$-group and $M$ be a non-abelian maximal subgroup of $G$. If $\exp(M) = 4$ and $Z(M) \cong C_2^3$, then $\Phi(G) \leq Z(M)$ and for any $a \in M \setminus Z(M)$, $b \in G \setminus M$, we have $[a, b] \neq 1$ and $o(a) = o(b) = 4$.

**Proof.** By Lemma 3.7, $Z(M) \leq Z(G)$. It follows from Lemma 3.2 that $\Phi(G) \leq Z(G)$ and so $\Phi(G) \leq Z(M)$. For any $x \in G \setminus Z(M)$, if $o(x) = 2$, then $Z(M) \langle x \rangle \cong C_2^4$, in contradiction to the Lemma 3.1. Thus $o(a) = o(b) = 4$. If $[a, b] = 1$ and $a^2 = b^2$, then $o(ab) = 2$, a contradiction. If $[a, b] = 1$ and $a^2 \neq b^2$, then $\langle a, b \rangle$ is not cyclic. By the hypotheses, $\langle a, b, Z(M) \rangle / \langle a, b \rangle$ is cyclic which is impossible. So $[a, b] \neq 1$. □

**Lemma 5.3.** Let $G$ be a CAC-$2$-group of order $2^n$ with $n \geq 6$, and $M$ be a maximal subgroup of $G$. If $M$ is metacyclic minimal non-abelian. Then $G$ is one of the following pairwise non-isomorphic groups:

1. $D_8 \ast C_{2^{n-2}}$;
2. $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4, c^2 = a^{2^{n-3}}, [b, a] = c, [c, b] = c^2, [c, a] = 1 \rangle$;
3. $\langle a, b, c \mid a^{2^{n-3}} = b^2 = 1, c^2 = a^2 b^2, [a, b] = b^2, [c, a] = [c, b] = 1 \rangle$;
4. $\langle a, b, c \mid a^{2^{n-3}} = b^2 = 1, c^2 = a^2 b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1 \rangle$.

**Proof.** Let $M = \langle a, b \mid a^{2^n} = b^2 = 1, [a, b] = a^{2^{n-1}} \rangle$, where $u \geq 2, v \geq 1$ and $u + v = n - 1$. We consider the following two cases: $v = 1$ and $v \neq 1$.

**Case 1.** $v = 1$

In this case, $M = \langle a, b \mid a^{2^{n-2}} = b^2 = 1, [a, b] = a^{2^{n-3}} \rangle$. Take $d \in G \setminus M$. Since $[b^2, d] = 1$, we have $[b, d] = 1$ or $a^{2^{n-3}}$. If $[b, d] = a^{2^{n-3}}$, then $[b, ad] = 1$. Without loss of generality, we may assume $[b, d] = 1$. Noticing that $Z(M) = \langle a^2 \rangle$, we see $\langle a^{2^{n-3}} \rangle \leq Z(G)$. By Lemma 5.1, $a^2 \in C_M(b) \leq \langle a^{2^{n-3}}, b, d \rangle$. Since $d \notin M$, $a^2 \in \langle a^{2^{n-3}}, b, a^2 \rangle$. Clearly, $\exp(G) = 2^{n-2}$. Thus we may assume $a^2 = a^2$ or $a^2$. 

If \( d^2 = a^2 \), then \([a, d] = 1\) or \(a^{2n-3}\). If \([a, d] = 1\), then, by letting \(a_1 = ad^{-1}\), \(G = \langle a_1, b \rangle \cong D_8 \ast C_{2^{n-2}}\). If \([a, d] = a^{2n-3}\), then, by letting \(d_1 = bd\), we see \(d_1^2 = a^2\) and \([a, d_1] = [b, d_1] = 1\). So we may also have \(G \cong D_8 \ast C_{2^{n-2}}\).

If \( d^2 = a^2b \), then \([a^2, d] = [b, d] = 1\) and \([a, d^2] = a^{2n-3}\). By calculation, \([a, d] = a^{\pm 2n-4}b\). Then \(G = \langle a_1, c, d \mid a_1^2 = c^4 = d^{2n-2} = 1, c^2 = d^{2n-3}, [a, d] = c, [c, a_1] = c^2, [c, d] = 1\) when we set \(a_1 = a^{\pm 2n-5}d\) and \(c = a^{\pm 2n-4}b\). Thus \(G\) is the type \((2)\).

**Case 2.** \(v \neq 1\)

In this case, \(Z(M) = \langle a^2, b^2 \rangle\) is not cyclic. By Lemma 3.7, \(Z(M) \leq Z(G)\).

Take \(d \in G \setminus M\). Since \([a^2, d] = 1\), \([a, d] = a^{2n-1}b^{2n-1}\), where \(i, j\) are integers. It follows that \([a, d^2] = 1\). Similarly, \([b, d^2] = 1\). Thus \(d^2 \in Z(M) \leq Z(G)\). Noticing that \(\Phi(M) = Z(M) \leq Z(G)\), we see \(\Phi(G) \leq Z(G)\). So \(G/Z(G)\) is elementary abelian. By Lemma 2.2, \(G'\) is elementary abelian. In particular, \(G' \leq \Omega_1(M) = \langle a^{2n-1}, b^{2n-1} \rangle\). If there exists an element \(g \in G \setminus M\) such that \(o(g) = 2\), then \(\Omega_1(M)\langle g \rangle \cong C_2^3\). It follows from Lemma 3.2 that \(g \in Z(G)\). Since \(n \geq 6\), we may assume \(u \geq 3\). Then, by the hypotheses, \(\langle a^2, b, g \rangle / \langle a^{2n-1}, b \rangle\) is cyclic. However it is impossible. So there is not an involution in \(G \setminus M\).

Now we consider the following three subcases:

**Subcase 1.** \(u \geq 3\) and \(v \geq 3\)

Let \(W = \langle a^{2n-2}, b^{2n-2} \rangle\). Then \(W \cong C_4 \times C_4\) and \(C_G(W) = G\). By Lemma 2.6(4), \(\Omega_2(C_G(W)) = W\). Then Lemma 2.6(5) implies \(G\) is metacyclic. Thus \(d(G) = 2\) and \(|G'| = 2\). By Lemma 2.1, \(G\) is minimal non-abelian, a contradiction.

**Subcase 2.** \(v = 2\)

In this case, \(M = \langle a, b \mid a^{2n-3} = b^{2^{n-1}} = 1, [a, b] = a^{2n-4} \rangle\). By the above, \(G' \leq \langle a^{2n-4}, b^2 \rangle\). Take \(d \in G \setminus M\). Then \(d^2 \in Z(G) \cap M = \langle a^2, b^2 \rangle\). If \(o(d) < 2n-3\), then, by letting \(d_1 = ad\), we see \(o(d_1) = 2n-3\). Without loss of generality, we assume \(o(d) = 2n-3\). Thus we may assume \(d^2 = a^2b^2\) or \(d^2 = a^2\).

If \(d^2 = a^2\), then \(o(a^2d^{-1}) = 2\) if \([a, d] = 1\) and \(o(da^{2n-5}d^{-1}) = 2\) if \([a, d] = a^{2n-4}\), which contradict that there is not an involution in \(G \setminus M\). Thus \([a, d] = b^2\) or \(a^{2n-4}b^2\). Since \(o(abd^{-1}) = 2\) if \([ab, d] = b^2a^{2n-4}\) and \(o(a^{1+2n-3}bd^{-1}) = 2\) if \([ab, d] = b^2\), we see \([ab, d] = 1\) or \(a^{2n-4}\). It follows that \([b, d] = b^2\) or \(a^{2n-4}b^2\). If \([a, d] = b^2\) and \([b, d] = b^2\), then, by letting \(a_1 = a^{1+2n-5}, b, G = \langle a_1, b, d \mid a_1^{2^{n-1}} = b^2 = 1, [a_1, b] = a_1^{2n-4}, [a_1, d] = b^2, [a_1, d] = 1, d^2 = a_1^3b^2 \rangle\). By calculation, \(G\) is isomorphic to the group of type \((4)\). If \([a, d] = b^2\) and \([b, d] = a^{2n-4}b^2\), then \(G = \langle a_1, b_1, d \mid a_1^{2^{n-3}} = b_1^{2^{n-2}} = 1, [a_1, b_1] = 1, [b_1, d] = b_1^2, [a_1, d] = a_1^{2n-4}, d^2 = a_1^3b_1^2 \rangle\) when we set \(a_1 = a^{1+2n-5}b\) and \(b_1 = ad^{-1}\). Thus \(G\) is the type \((4)\). If \([a, d] = a^{2n-4}b^2\), then, by setting \(d_1 = bd\) if \([b, d] = b^2\) and \(d_1 = a^{2n-5}bd\) if
\[ [b, d] = a^{2n-4}b^2, \] we see \( d_1^2 = a^2 \) and \( [a, d_1] = b^2 \). So we may also have the group of type (4).

If \( d^2 = a^2b^2 \), then, by letting \( a_1 = a_1^{1+2n-5}b \), we see \( d^2 = a_1^2 \), which is reduced to the case of \( d^2 = a^2 \).

**Subcase 3.** \( u = 2 \)

In this case, \( M = \langle a, b \mid a^{2} = b^{2n-3} = 1, [a, b] = a^2 \rangle \) and \( G' \leq \langle a^2, b^{2n-4} \rangle \). Take \( d \in G \setminus M \).

Without loss of generality, we may assume \( d = a^2b^2 \) or \( d^2 = b^2 \).

If \( d^2 = b^2 \), then \( o(b^{-1}d) = 2 \) if \( [b, d] = 1 \) and \( o(db^{2n-5}) = 2 \) if \( [b, d] = b^{2n-4} \). So \( [b, d] = a^2 \) or \( b^{2n-4} a^2 \). Since \( (ab)^2 = b^2 \), we see \( [ab, d] = a^2 \) or \( b^{2n-4} a^2 \). It follows that \([a, d] = 1 \) or \( b^{2n-4} \).

If \([a, d] = 1 \) and \([b, d] = a^2 \), then, by letting \( d_1 = ad \), \( G = \langle a, b, d_1 \mid a^4 = b^{2n-3} = 1, [a, b] = a^2, d_1^2 = a^2b^2, [d_1, a] = [d_1, b] = 1 \rangle \). Thus \( G \) is the type (3). If \([a, d] = 1 \) and \([b, d] = b^{2n-4} a^2 \), then \( G \) is isomorphic to the group of type (4). If \([a, d] = b^{2n-4} \) and \([b, d] = a^2 \) or \( b^{2n-4} a^2 \), then \( G \) is also the type (4). In the two cases, we have \( d_1^2 = b^2 \), which is reduced to the case of \( d^2 = b^2 \).

**Lemma 5.4.** Let \( G \) be a CAC-2-group of order \( 2^n \) and \( n \geq 6 \). If there is a maximal subgroup \( M \) in \( G \) such that \( M \cong D_8 \rtimes C_{2n-4} \), then \( G \) is one of the following pairwise non-isomorphic groups:

(1) \( D_8 \rtimes C_{2n-2} \);

(2) \( \langle a, b, c \mid a^{2n-3} = b^2 = c^4 = 1, [a, b] = c, [c, b] = b^2, [c, a] = 1 \rangle \).

**Proof.** Let \( M = \langle a, b, c \mid a^{2n-3} = b^2 = c^4 = 1, [c, b] = a^{2n-4}, [b, a] = [c, a] = 1 \rangle \). Then \( Z(M) = \langle a \rangle \supseteq G \) and so \( \langle a^{2n-4} \rangle \leq Z(G) \). Take \( d \in G \setminus M \). Since \( [b^2, d] = 1 \), by calculation, we have \( [b, d] = 1 \) or \( a^{2n-4} \) or \( a^4b^{2n-5} c \) or \( a^{2n-4} bc \), where \( i \) is an integer. If \( [b, d] = a^{k+2n-5}c \), then, since \( [b, d^2] \in \langle a^{2n-4} \rangle \), we see \( [c, d] = a^{2n-4} \) or \( 1 \). If \( [b, d] = a^{2n-4} bc \), then \( [bc, d] = a^{2n-4} \) or \( 1 \). Without loss of generality, we may assume \( [b, d] = 1 \), or \( a^{2n-4} \). Now we consider \( o(d) = 2^{n-2} \) and \( o(d) \leq 2^{n-3} \).

If \( o(d) = 2^{n-2} \), then \( \langle d^4 \rangle = \langle a^2 \rangle \). If \( [b, d] = a^{2n-4} \), then, by Lemma 2.1, \( \langle b, d \rangle \) is metacyclic minimal non-abelian of order \( 2^{n-1} \). If \( [b, d] = 1 \), then \( [b, cd] = a^{2n-4} \) and so \( [b, cd] \) is metacyclic minimal non-abelian of order \( 2^{n-1} \). Thus we may have the groups listed in lemma by Lemma 5.3.

If \( o(d) \leq 2^{n-3} \) and \([b, d] = 1 \), then, by Lemma 5.1, we see \( a \in C_M(b) \leq \langle a^{2n-4}, b, d \rangle \). Thus \( a \in \langle a^{2n-4}, b, d^2 \rangle \), in contradiction to \( o(d) \leq 2^{n-3} \). If \([b, d] = a^{2n-4} \), then \( [b, cd] = 1 \). We may also have a contradiction. \( \square \)
Lemma 5.5. Let $G$ be a $\mathcal{CAC}$-2-group of order $2^n$ and $n \geq 6$. If there is a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^{2^{n-4}} = b^2 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1 \rangle$, then $G$ is one of the following pairwise non-isomorphic groups:

1. $\langle a, b, c \mid a^{2^{n-3}} = b^2 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1 \rangle$;
2. $\langle a, b, c \mid a^{2^{n-3}} = b^2 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1 \rangle$.

Proof. By calculation, we see $Z(M) = \langle b^2, c \rangle$, $\Phi(M) = \langle a^2, b^2 \rangle$, and $\Omega_1(M) = \langle c^{2^{n-5}}, b^2 \rangle$. By Lemma 3.7, $Z(M) \leq Z(G)$. For any $d \in G \setminus M$, if $d^2 \notin Z(G)$, then there exists an element $x \in \Phi(M)$ such that $\langle x, d^2 \rangle$ is not cyclic. It follows from Lemma 5.1 that $c \in C_M(d^2) \leq \langle x, d \rangle$ and so $d^2 \in Z(G)$, a contradiction. Thus $d^2 \in Z(G)$. So $\Phi(G) \leq Z(G)$ and $G/Z(G)$ is elementary abelian. By Lemma 2.2, $G'$ is elementary abelian. In particular, $G' \leq \Omega_1(M)$. We consider $\exp(G) = 2^{n-4}$ and $\exp(G) = 2^{n-3}$.

If $\exp(G) = 2^{n-3}$, then $o(d) = 2^{n-3}$. Since $d^2 \in Z(M) = \langle b^2, c \rangle$, $\langle d^2 \rangle = \langle c^2 \rangle$. If $[b, d] = b^2$ or $c^{2^{n-5}}$, then $[b, d]$ is metacyclic minimal non-abelian of order $2^{n-1}$. If $[b, d] = 1$ or $b^2c^{2^{n-5}}$, then $[b, ad] = b^2$ or $c^{2^{n-5}}$ and so $\langle b, ad \rangle$ is metacyclic minimal non-abelian of order $2^{n-1}$. Thus we may get the groups listed in lemma by Lemma 5.3.

If $\exp(G) = 2^{n-4}$, then $d^2 \in \langle b^2, c^2 \rangle$. Since $[a, b] = b^2$, we may assume $[a, d] = 1$ or $a^{2^{n-5}}$. If $[a, d] = 1$, then, by Lemma 5.1, we see $c \in C_M(a) \leq \langle a, b^2, d \rangle$, a contradiction. So $[a, d] = a^{2^{n-5}}$. Similarly $[b, d] = a^{2^{n-5}}$. Then $[ab, d] = 1$. We may also have a contradiction. □

Lemma 5.6. Let $G$ be a $\mathcal{CAC}$-2-group of order $2^n$ and $n \geq 6$. Then

1. If there is a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^{2^{n-4}} = b^2 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = [c, b] = 1 \rangle$, then $n = 6$ and $G \cong \langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^2 = d^2, [a, b] = b^2, [c, a] = a^2, [c, d] = c^2 \rangle$;
2. If $n = 6$, then there is not a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^4 = b^4 = c^2 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = c^2, [c, b] = 1 \rangle$.

Proof. Assume $M \leq G$ and $M$ is isomorphic to the maximal subgroup listed in (1) or (2). Then $Z(M) = \Phi(M) = \langle b^2, c^2 \rangle = \langle a^2, c^2 \rangle \leq Z(G)$.

It is easy to see that $\langle b, c \rangle$ is the unique abelian maximal subgroup of $M$. Thus $\langle x, d \rangle$ is cyclic. For any $d \in G \setminus M$, it follows from $[b^2, d] = 1$ that $[b, d^2] = 1$. Thus $d^2 \in C_G(b) \cap M = C_M(b) = \langle b, c \rangle$. If $d^2 \notin Z(G)$, then there exists an element $x \in Z(M)$ such that $\langle x, d^2 \rangle$ is not cyclic. By the hypotheses, $\langle b, c, d \rangle/\langle x, d^2 \rangle$ is cyclic. Noticing that $\langle x, d^2 \rangle \leq \Phi(\langle b, c, d \rangle)$, we see $\langle b, c, d \rangle$ is cyclic, a contradiction. Thus $d^2 \in Z(G)$ and so $\Phi(G) \leq Z(G)$. Thus $G/Z(G)$ is elementary abelian. By Lemma 2.2, $G'$ is elementary abelian. In particular, $G' = \Omega_1(M) = M'$. 


For any \(d \in G \setminus M\), if \(o(d) = 2\), then \(\Omega_1(M) \langle d \rangle \cong C_2^2\), which implies \(r(G) = 3\). If \(d \in Z(G)\), then, by the hypotheses, \(\langle b, c, d \rangle / \langle b, c^2 \rangle\) is cyclic. However it is impossible. If \(d \notin Z(G)\), then \(C_G(d) = \Omega_1(M) \langle d \rangle\) by Lemma 3.2. It follows from \(G' \cong C_2^2\) that there exists an element \(x \in M \setminus \Phi(M)\) such that \([x, d] = 1\). Thus \(x \in C_G(d) = \Omega_1(M) \langle d \rangle\). It is also impossible. So there is not an involution in \(G' \setminus M\).

Noticing that \([a, M] = G'\), we may take a suitable \(d \in G \setminus M\) such that \([a, d] = 1\). If \([b, d] = 1\), then, by Lemma 5.1, we see \(c \in C_M(b) \leq \langle b, d, c^2 \rangle\), a contradiction. If \([b, d] = b^2\), then \([b, ad] = 1\). We may also have a contradiction. Thus \([b, d] \neq (b^2)\).

If \(M \cong \langle a, b, c \mid a^{2^{n-4}} = b^{2^2} = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^{2^{n-5}}, [c, b] = 1\rangle\), then, since \(d^2 \in Z(M)\), we may assume \(d^2 = a^ib^j\), where \(i, j\) are integers. Replacing \(d\) by \(da^{-i}\), we have \(d^2 = b^{2j}\) and so \(d^2 = b^2\). Since \([b, d] \notin (b^2)\), \([b, d] = a^{2^{n-5}}b^2\) or \(a^{2^{n-5}}\). Similarly \([c, d] = b^2\) or \(b^2a^{2^{n-5}}\). If \(n \geq 7\), then \(a^{2^{n-6}} \in Z(G)\). Since \((bda^{-1})^2 = [b, d]a^{2^{n-6}} \neq 1\), we see \([b, d] = a^{2^{n-5}}b^2\). It follows that \((abc^{-1})^2 = b^2[c, d]\) and so \([c, d] = b^2a^{2^{n-5}}\). Thus \([bc, d] = 1\). By Lemma 5.1, \(c \in C_M(bc) \leq \langle bc, d, b^2 \rangle\), a contradiction. So \(n = 6\). Since \((abd)^2 = a^2b^2\), we see \([b, d] = a^2\). Thus \((bcd)^2 = b^2[c, d]\) and so \([c, d] = b^2a^2 = c^2\). Hence \(G = \langle a, b, c, d \rangle\) is isomorphic to the group in lemma.

If \(M \cong \langle a, b, c \mid a^4 = b^4 = c^4 = 1, a^2 = b^2, [b, a] = a^2, [c, a] = c^2, [c, b] = 1\rangle\), then we may assume \(d^2 = c^2\). Since \((bd)^2 = b^2c^2[b, d]\) and \((acd)^2 = a^2c^2[c, d]\), we see \([b, d] \neq a^2c^2\) and \([c, d] \neq a^2c^2\). Thus \([b, d] = c^2\) and \([c, d] = a^2\). It follows that \([bc, ad] = 1\). By Lemma 5.1, we see \(c \in C_M(bc) \leq \langle bc, ad, b^2 \rangle\), a contradiction.

**Lemma 5.7.** Let \(G\) be a CAC-2-group of order \(2^n\) and \(n \geq 6\). If \(G\) has an abelian maximal subgroup and a maximal subgroup of maximal class, then \(G\) is one of the following pairwise non-isomorphic groups:

1. 2-groups of maximal class;
2. \(D_{2^{n-1}} \times C_2^2\);
3. \(SD_{2^{n-1}} \times C_2^2\);
4. \(Q_{2^{n-1}} \times C_2^2\);
5. \(\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle \cong D_{2^{n-1}} \ast C_4 \cong Q_{2^{n-1}} \ast C_4 \cong SD_{2^{n-1}} \ast C_4\).

**Proof.** Let \(M < G\) and \(M\) be of maximal class. Then \(|Z(M)| = 2\) and \(|M'| = 2^{n-3}\). Thus \(2^{n-3} \leq |G'| \leq 2^{n-2}\). If \(|G'| = 2^{n-2}\), then \(G\) is of maximal class. If \(|G'| = 2^{n-3}\), then \(|Z(G)| = 4\) by Lemma 2.3. So there exists an element \(x \in Z(G)\) such that \(x \notin M\). Then \(x^2 \in M \cap Z(G) \leq Z(M)\). If \(o(x) = 2\), then \(G\) is of the type (2), (3) or (4). If \(o(x) = 4\), then \(G\) is of the type (5). \(\square\)
Lemma 5.8. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 6$. Then $G$ has no maximal subgroup $M \cong SD_{2^{n-2}} \times C_2$ and if $G$ has a maximal subgroup $M \cong D_{2^{n-2}} \times C_2$, then $G$ is one of the groups listed in Lemma 3.6.

Proof. Let $M \leq G$ and $M = \langle a, b, c \mid a^{2^{n-3}} = b^2 = c^2 = 1, [a, b] = a^{2^{n-4}-2}, [c, a] = [c, b] = 1 \rangle$, where $i = 0$ or 1. Then $r(M) = 3$. By Lemma 3.7, $Z(M) = \langle a^{2^{n-4}}, c \rangle \leq Z(G)$. Clearly, we may take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, d] = 1$. By Lemma 5.1, $\langle C_M(a^{2^{n-5}}), d \rangle = \langle a, c, d \rangle$ is an abelian maximal subgroup of $G$. So $G$ is one of the groups listed in Lemma 3.6. Conversely, those groups listed in Lemma 3.6 have a maximal subgroup $M \cong D_{2^{n-2}} \times C_2$ and have no maximal subgroup $M \cong SD_{2^{n-2}} \times C_2$. □

Lemma 5.9. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 6$. If there is a maximal subgroup $M$ in $G$ such that $M \cong Q_{2^{n-2}} \times C_2$, then $G$ is one of the following pairwise non-isomorphic groups:

1. $Q_{2^{n-1}} \times C_2$;
2. $SD_{2^{n-1}} \times C_2$;
3. $\langle a, b, c \mid a^{2^{n-3}} = b^4 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$.

Proof. Let $M = \langle a, b, c \mid a^{2^{n-3}} = c^2 = 1, b^2 = a^{2^{n-4}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$. Then $Z(M) = \langle a^{2^{n-4}}, c \rangle \leq Z(G)$. Since $\langle a, c \rangle$ is the unique abelian maximal subgroup of $M$, $G' \leq \langle a, c \rangle$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, d] = 1$. By Lemma 5.1, we see $a \in C_M(a^{2^{n-5}}) \leq \langle a^{2^{n-5}}, c, d \rangle$. Without loss of generality, we may assume $d^2 = a$. Then $[d^2, b] = [a, b] = a^{-2}$. It follows that $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$ or $a^{-1}c$ or $a^{2^{n-4}-1}c$. If $[d, b] = a^{-1}$, then $G = \langle b, c, d \rangle \cong Q_{2^{n-1}} \times C_2$. If $[d, b] = a^{2^{n-4}-1}$, then $G \cong SD_{2^{n-1}} \times C_2$. If $[d, b] = a^{-1}c$, then $G = \langle b, c_1, d_1 \mid b^4 = d_1^4 = c_1^{2^{n-3}} = 1, b^2 = c_1^{2^{n-4}}, [d_1, b] = c_1, [c_1, b] = [c_1, d_1] = c_1^{-1} \rangle$ when we set $d_1 = bd$ and $c_1 = a^{-1}c$. In this case, $G$ is the type (3). If $[d, b] = a^{2^{n-4}-1}c$, then $G$ is also the type (3). □

Lemma 5.10. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 6$. Then $G$ has no maximal subgroup $M \cong \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{2^{n-4}-2} \rangle$ and if $G$ has a maximal subgroup $M \cong \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{-2} \rangle$, then $G$ is one of the following pairwise non-isomorphic groups:

1. $\langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{-2} \rangle$;
2. $\langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{2^{n-4}-2} \rangle$;
3. $\langle a, b \mid a^4 = b^2 = c^{2^{n-3}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
4. $\langle a, b \mid a^4 = b^2 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [b, c] = c^{-2} \rangle$.

Proof. Let $M < G$ and $M = \langle a, b \mid a^4 = b^{2^{n-3}} = 1, [b, a] = b^{2^{n-4}-2}, \rangle$, where $i = 0$ or 1. It is easy to see $Z(M) = \langle a^2, b^{2^{n-4}} \rangle \leq Z(G)$ and $G' \leq \langle a^2, b \rangle$. Take a suitable $d \in G \setminus M$ such that $[b^{2^{n-5}}, d] = 1$. By Lemma 5.1, $\langle a^2, b, d \rangle$
is abelian and \( b \in \langle a^2, b^{2n-5}, d \rangle \). Without loss of generality, we may assume \( d^2 = b \).

If \( i = 1 \), then \( [d^2, a] = b^{2n-1-2} \). Assume \([a, d] = a^2b^k \). It follows from \([a^2, d] = 1 \) that \( k \) is even and so \([a, d^2] \in \langle b^4 \rangle \), a contradiction.

If \( i = 0 \), then \([d^2, a] = b^{-2} \). It follows that \([d, a] = b^{-1} \) or \( a^2b^{-1} \) or \( a^2b^{2n-1} \) or \( b^{2n-1} \). If \([d, a] = b^{-1} \), then \( G \) is the type (1). If \([d, a] = b^{2n-1} \), then \( G \) is the type (2). If \([d, a] = a^2b^{-1} \), then \( G = \langle a, b, c_1 | a^4 = b_1^3 = c_1^{-1} = 1, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2} \rangle \) when we set \( b_1 = ad \) and \( c_1 = a^2b^{-1} \). In this case, \( G \) is the type (3). If \([d, a] = a^2b^{2n-4} \), then, by letting \( b_1 = ad \) and \( c_1 = a^2b^{2n-4} \), we see \( G = \langle a, b, c_1 | a^4 = b_1 = c_1^{-1} \rangle = 1, b_1 = c_1^{-1}, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2} \rangle \). Thus \( G \) is the type (4).

\( \square \)

**Lemma 5.11.** Let \( G \) be a \( CA \)-2-group of order \( 2^n \) and \( n \geq 6 \). If there is a maximal subgroup \( M \) in \( G \) such that \( M \cong \langle a, b | a^8 = b^{2n-3} = 1, a^4 = b^{2n-4}, [b, a] = b^{-2} \rangle \), then \( G \) is one of the following pairwise non-isomorphic groups:

1. \( \langle a, b | a^8 = b^{2n-2} = 1, a^4 = b^{2n-3}, [b, a] = b^{-2} \rangle ; \)
2. \( \langle a, b, c | a^8 = b^2 = c^{2n-3} = 1, a^4 = c^{2n-4}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle . \)

**Proof.** Since \( \langle a^2, b \rangle \) is the unique abelian maximal subgroup of \( M \), \( G' \leq \langle a^2, b \rangle \).

Take \( d \in G \setminus M \). Since \( M' = \langle b^2 \rangle \) and \( Z(M) = \langle a^2 \rangle \), we see \([b^{2n-5}, d] = 1 \) or \( b^{2n-4} \), and \([a^2, d] = 1 \) or \( a^4 \). Thus \([a^2b^{2n-5}, d] = 1 \) or \( b^{2n-4} \). We may assume \([a^2b^{2n-5}, d] = 1 \). By Lemma 5.1, \( b \in \langle a^4, a^2b^{2n-5}, d \rangle \) and \( \langle b, d, a^2b^{2n-5} \rangle \) is abelian. Without loss of generality, we may assume \( d^2 = b \) or \( b^2 \). Then \([d^2, a] = b^{-2} \). By calculation, \([d, a] = b^{-1} \) or \( b^{2n-1} \). If \( d^2 = b \) and \([d, a] = b^{-1} \), then \( G = \langle a, d \rangle \) is the type (1). Let \( b_1 = a^3d, c_1 = b^{-1} \) if \( d^2 = ba^2 \), \([d, a] = b^{-1} \), and let \( b_1 = ad, c_1 = b^{2n-1} \) if \( d^2 = ba^2 \). In either case, we get \( G = \langle a, b_1, c_1 | a^8 = b_1^2 = c_1^{-1} = 1, a^4 = c_1^{-1} \rangle = 1, b_1 = c_1^{-1}, [b_1, a] = c_1, [c_1, a] = [c_1, b_1] = c_1^{-2} \rangle \). Thus \( G \) is the type (2).

\( \square \)

**Lemma 5.12.** Let \( G \) be a \( CA \)-2-group of order \( 2^n \) and \( n \geq 6 \). Then there is not a maximal subgroup \( M \) in \( G \) such that \( M \cong \langle a, b, c | a^4 = b^2 = c^{2n-4} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle . \)

**Proof.** Otherwise, it is easy to see \( G' \leq \langle a^2, ab \rangle \). If \( (ab)^j a^{2j} \in G' \), where \( j \) is odd, then \( |G'| = |(ab, a^2)| = 2^{n-2} \). Thus \( G \) is of maximal class, a contradiction. It follows that \( G' \leq \langle c, a^2 \rangle \). Take a suitable \( d \in G \setminus M \) such that \([c^{2n-6}, d] = 1 \). It is easy to see \([a^2, d] = 1 \). Thus \( a^2 \in Z(G) \). By Lemma 5.1, \( ab \in \langle a^2, c^{2n-6}, d \rangle \). It follows that \([a, d^2] = c^k \), where \( k \) is odd. On the other hand, we assume \([a, d] = a^{2k}c^k \) and so \([a, d^2] \in \langle c^2 \rangle \), a contradiction.

\( \square \)

By similar arguments as in Lemma 5.12, we have the following result:
Lemma 5.13. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 6$. Then there is not a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^4 = b^4 = c^{2^{n-4}} = 1, b^2 = c^{2^{n-5}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$ or $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-4}} = 1, a^4 = c^{2^{n-5}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$.

Lemma 5.14. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 6$. If there is a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^{2^{n-3}} = b^2 = c^4 = 1, c^2 = a^{2^{n-4}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$, then $G$ is one of the following pairwise non-isomorphic groups:

1. $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$;
2. $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-3}} = 1, a^4 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$.

Proof. It is easy to see $G' \leq \langle a, c \rangle$ and $\langle a^{2^{n-4}} \rangle = \langle c^2 \rangle \leq Z(G)$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, c, d] = 1$. Then, by Lemma 5.1, $\langle a, c, d \rangle$ is abelian and $a \in \langle a^{2^{n-4}}, d, a^{2^{n-5}} \rangle$. Without loss of generality, we may assume $d^2 = a$ or $ac$. Then $[d^2, b] = [a, b] = a^{-2}$. By calculation, $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$. If $d^2 = a$ and $[d, b] = a^{-1}$ or $a^{2^{n-4}-1}$, then $G = \langle b, c, d \rangle$ is isomorphic to the group of type (1). Let $d_1 = bd$, $c_1 = a^{-1}$ if $d^2 = ac$, $[d, b] = a^{-1}$, and let $d_1 = bd$, $c_1 = a^{2^{n-4}-1}$ if $d^2 = ac$, $[d, b] = a^{2^{n-4}-1}$. In either case, we have $G = \langle b, c_1, d_1 \mid b^2 = d_1^3 = c_1^{2^{n-3}} = 1, d_1^4 = c_1^{2^{n-4}}, [d_1, b] = c_1, [c_1, b] = [c_1, d_1] = c_1^{-2} \rangle$. Thus $G$ is the type (2).

Lemma 5.15. Let $G$ be a CAC-2-group of order $2^n$ and $n \geq 7$. Then there is not a maximal subgroup $M$ in $G$ such that $M \cong \langle a, b, c \mid a^{2^{n-5}} = b^2 = c^4 = 1, c^2 = a^{2^{n-4}}, [b, a] = c, [c, b] = c^2, [c, a] = 1 \rangle$.

Proof. Otherwise, $G' \leq \langle a, c \rangle$. Take a suitable $d \in G \setminus M$ such that $[a^{2^{n-5}}, c, d] = 1$. Then, by Lemma 5.1, $\langle a, c, d \rangle$ is abelian and $a \in \langle a^{2^{n-5}}, c, a^{2^{n-4}}, d \rangle$. It follows that $[b, d^2] = c^i$, where $i$ is odd. However it is impossible.

By checking the list of groups of order $2^i$, we get the following result:

Theorem 5.16. Let $G$ be a group of order $2^5$. Then $G$ is a CAC-2-group if and only if $G$ is one of the following pairwise non-isomorphic groups:

1. metacyclic minimal non-abelian 2-groups;
2. 2-groups of maximal class;
3. $D_{24} \times C_2$;
4. $SD_{24} \times C_2$;
5. $Q_{24} \times C_2$;
6. $M_2(2, 2, 1)$;
7. $M_2(2, 2) \times C_2$;
8. $D_8 \times C_{16}$;
9. $\langle a, b \mid a^4 = b^8 = 1, [b, a] = b^{-2} \rangle$.
Theorem 5.17. Let $G$ be a group of order $2^n$ and $n \geq 6$. Then $G$ is a $C\text{AC}-2$-group if and only if $G$ is one of the following pairwise non-isomorphic groups:

1. metacyclic minimal non-abelian 2-groups;
2. $2$-groups of maximal class;
3. $D_{2^{n-1}} \times C_2$;
4. $SD_{2^{n-1}} \times C_2$;
5. $Q_{2^{n-1}} \times C_2$;
6. $D_8 \ast C_{2^{n-2}}$;

7. $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{-2} \rangle$;
8. $\langle a, b \mid a^4 = b^{2^{n-2}} = 1, [b, a] = b^{2^{n-3} - 2} \rangle$;
9. $\langle a, b \mid a^8 = b^{2^{n-2}} = 1, a^4 = b^{2^{n-3}}, [b, a] = b^{-2} \rangle$;
10. $\langle a, b, c \mid a^{2^{n-3}} = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = [c, b] = 1 \rangle$;
11. $\langle a, b, c \mid a^{2^{n-3}} = b^4 = 1, c^2 = a^2b^2, [b, a] = b^2, [c, a] = a^{2^{n-4}}, [c, b] = 1 \rangle$;
12. $\langle a, b, c \mid a^4 = b^2 = c^{2^{n-3}} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
13. $\langle a, b, c \mid a^4 = b^2 = c^{2^{n-3}} = 1, b^2 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
14. $\langle a, b, c \mid a^8 = b^2 = c^{2^{n-3}} = 1, a^4 = c^{2^{n-4}}, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$;
15. $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [b, a] = a^2, [c, a] = [c, b] = 1 \rangle$;
16. $\langle a, b, c \mid a^{2^{n-2}} = b^2 = c^4 = 1, c^2 = a^{2^{n-3}}, [b, a] = c, [c, b] = c^2, [c, a] = 1 \rangle$;
17. $\langle a, b, c, d, e \mid a^4 = b^2 = c^2 = 1, a^2 = b^2 = c^2, [b, a] = a^2, [c, a] = d, [c, b] = e, [d, a] = [e, a] = [d, b] = [e, b] = [c, d] = [c, e] = [d, e] = 1 \rangle$;
18. $\langle a, b, c, d \mid a^4 = b^4 = d^2 = 1, b^2 = c^2, [b, a] = b^2, [c, a] = a^2, [c, b] = d, [d, a] = [d, b] = [c, d] = 1 \rangle$;
19. $\langle a, b, c, d \mid a^4 = b^4 = d^2 = 1, a^2 = c^2, [b, a] = a^2, [c, a] = b^2c^2, [c, b] = d, [d, a] = [d, b] = [c, d] = 1 \rangle$;
20. $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, [b, a] = c^2, [c, a] = b^2c^2, [c, b] = a^2b^2 \rangle$;
Assume each maximal subgroup of $G$ is abelian. Then $G$ is minimal non-abelian. If $G$ is not metacyclic, then we may assume $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, [a, b] = [a, c] = [c, b] = 1 \rangle$, where $u \geq v \geq 1$. Since $n \geq 6$, $u \geq 3$.

Noticing that $\langle a^2, b, c \rangle \leq C_G(\langle a^2, b \rangle)$ and $\langle a^2, b, c \rangle/\langle a^2, b \rangle$ is not cyclic, we see $C_G(\langle a^2, b \rangle)/\langle a^2, b \rangle$ is not cyclic, in contradiction to the hypothesis. Thus $G$ is of the type (1).

If there exists a $M \leq G$ such that $M$ is not abelian and $M$ is of maximal class, then there exists a $M_1 \leq G$ such that $M_1$ is not of maximal class by Lemma 2.7. If $M_1$ is abelian, then $G$ is of the type $(2), (3), (4), (5)$, or $(15)$ according to Lemma 5.7. Without loss of generality, we may assume that $M$ is not abelian and $M$ is not of maximal class. By Lemma 3.7, $M$ is a $\mathcal{CAC}$-2-group.

If $n \geq 8$, then, by induction hypothesis, $M$ is a group of types $(1)$ and $(3) - (16)$. By Lemma 5.3–5.6 and Lemma 5.8–5.15, $G$ is a group of types $(3) - (16)$.

Now we consider $n = 6$ and $n = 7$.

**Case 1.** $n = 6$

In this case, $M$ is one of the groups listed in Theorem 5.16 except the type (2). If $M$ is a group of types $(1), (3) - (5)$ and $(8) - (17)$ listed in Theorem 5.16, then $G$ is of the type $(3) - (16)$ or (21) according to Lemma 5.3–5.6 and Lemma 5.8–5.14. Thus, we only need to consider that $M$ is a group of the types $(6), (7), (18), (19), (20)$, and (21) listed in Theorem 5.16.

If $M$ is of the type (6) in Theorem 5.16, then we may assume $M = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = [c, a] = [c, b] = 1 \rangle$. Then $Z(M) = \langle a^2, b^2, c \rangle \cong C_2^3$.

By Lemma 5.2, $\Phi(G) = Z(M)$ and for any $g \in G \setminus M$, $x \in M \setminus Z(M)$, we have $[x, g] \neq 1$ and $o(g) = 4$. It follows from $M' = \langle c \rangle$ that $[x, g] \notin \langle c \rangle$. Thus $|G'| > 4$ and so $G' = Z(M)$. Without loss of generality, we may assume $g^2 = a^2$, $c$ or $a^2c$.

If $g^2 = a^2$, then $o(bg) = 2$ if $[b, g] = a^2b^2$ and $o(abg) = 2$ if $[ab, g] = b^2c$. Thus $[b, g] \neq a^2b^2$ and $[ab, g] \neq b^2c$. Without loss of generality, we may assume $[a, g] = a^2$, $b^2$, $a^2c$ or $b^2c$. If $[a, g] = a^2$, then $[b, g] = b^2$ or $b^2c$. If $[b, g] = b^2$, then $G = \langle a, b, g \mid a^4 = b^4 = c^2 = 1, a^2 = b^2, [a, b] = c, [a, g] = a^2, [b, g] = b^2, [a, c] = [c, b] = 1 \rangle$. By a simple checking, $G$ is the type (17). If $[b, g] = b^2c$, then $G$ is also the type (17). If $[a, g] = b^2$, then $[b, g] = a^2$, $a^2c$ or $a^2b^2c$. Then $G$ is the type (18) if $[b, g] = a^2$ or $a^2c$, and $G$ is the type (19) if $[b, g] = a^2b^2c$. If $[a, g] = a^2c$, then $[b, g] = b^2$, $b^2c$ or $a^2b^2c$. It is easy to see that $G$ is the type (17) if $[b, g] = b^2$, $G$ is the type (18) if $[b, g] = b^2c$ and $G$ is the type (19) if $G = a^2b^2c$. If $[a, g] = b^2c$, then $[b, g] = a^2c$ or $a^2b^2c$. Thus $G$ is the type (19) if $[b, g] = a^2b^2c$, and $G$ is the type (18) if $[b, g] = a^2$.

If $g^2 = c$, then $o(bg) = 2$ if $[a, g] = a^2c$ and $(ag)^2 = (ab)^2$ if $[a, g] = b^2$. Thus, without loss of generality, we may assume $[a, g] = a^2c$ or $b^2c$. Similarly,
we may assume $[b, g] = b^2$, $a^2b^2$ or $a^2c$ and $[ab, g] = a^2c$, $b^2c$ or $a^2b^2c$. It follows that $[a, g] = b^2c$ and $[b, g] = a^2b^2$. Thus $G$ is the type (20).

If $g^2 = a^2c$, then, without loss of generality, we may assume $[a, g] = a^2$ or $b^2$ and $[b, g] = a^2$ or $b^2$. It follows that $[ab, g] = a^2b^2$. Thus $(abg)^2 = a^2$, which is reduced to the case of $g^2 = a^2$.

If $M$ is of the type (7) in Theorem 5.16, then, by using the similar arguments as that $M$ is of the type (6), we have that $G$ is of the type (17), (18) or (19).

If $M$ is of the type (21) in Theorem 5.16, then, by using the similar arguments as that $M$ is of the type (6), we have that $G$ is of the type (17).

If $M$ is of the type (18), (19) or (20) in Theorem 5.16, then, by the same arguments as in Lemma 5.6, $Z(M) = \langle b^2, c^2 \rangle \leq Z(G)$ and $G' = M'$. Since $\langle a, b^2, c^2 \rangle \cong C_2^3$ and $a \notin Z(G)$, we see $C_G(a) = \langle a, b^2, c^2 \rangle$ by Lemma 3.2. Noticing that $[a, M] = G'$, we may take a suitable $d \in G \setminus M$ such that $[a, d] = 1$. Then $d \in C_G(a)$, a contradiction.

**Case 2.** $n = 7$

We only need to consider $M$ is a group of types (17), (18), (19), (20) and (21) listed in theorem.

If $M$ is of the type (17), then $M' = Z(M) = \langle a^2, d, e \rangle \cong C_2^3$. By Lemma 5.2, $G' = M'$ and for any $g \in G \setminus M$, $x \in M \setminus Z(M)$, we have $[x, g] \neq 1$. It follows from $[a, M] = \langle a^2, d \rangle$ that $[a, g] \notin \langle a^2, d \rangle$. Similarly, $[b, g] \notin \langle a^2, e \rangle$ and $[c, g] \notin \langle d, e \rangle$. We may take a suitable $h \in G \setminus M$ such that $[a, h] = e$. Then $[b, h] = d$, $da^2$, $de$ or $da^2e$ and $[c, h] = a^2$, $a^2d$, $a^2e$ or $a^2de$. It follows that $[ac, bh] = 1$ if $[c, h] = a^2$ and $[ac, bh] = 1$ if $[c, h] = a^2d$. Thus $[c, h] = a^2e$ or $a^2de$. If $[c, h] = a^2e$, then $[ab, ch] = 1$ if $[b, h] = d$, $[ab, ach] = 1$ if $[b, h] = da^2$, $[bc, ah] = 1$ if $[b, h] = de$ and $[abc, ch] = 1$ if $[b, h] = da^2e$. If $[c, h] = a^2de$, then, by letting $h_1 = ah$, we see $[a, h_1] = e$ and $[c, h_1] = a^2e$. So we may also have a contradiction.

If $G$ has a maximal subgroup which is isomorphic to type (18), (19) or (20), then, by using the similar arguments as that $M$ is of the type (17), we may have a contradiction.

If $M$ is of the type (21), then $\Omega_1(M) = Z(M) = M' = \Phi(M) = \langle a^2, b^2 \rangle \leq Z(G)$. We claim $exp(G) = 4$. Otherwise, there exists an element $g \in G \setminus M$ such that $o(g) = 8$. Assume $g^2 = x_1$. It is clear that there exist $x_2 \in M \setminus \langle a^2, b^2, x_1 \rangle$ and $x_3 \in \langle a^2, b^2 \rangle$ such that $[x_1, x_2] = 1$ and $[x_1, x_3]$ is not cyclic. By Lemma 5.1, we see $x_2 \in C_M(x_1) \leq \langle x_3, g \rangle$ and so $x_2 \in \langle x_1, x_3 \rangle$, a contradiction. Thus the claim holds. Hence for any $x \in G \setminus M$, $x^2 \in \Omega_1(M) \leq Z(G)$ and therefore $\Phi(G) \leq Z(G)$. So $G' = M'$. Noticing that $[c, M] = G'$, we may take a suitable $x \in G \setminus M$ such that $[c, x] = 1$. By Lemma 5.1, we see $b \in C_M(c) \leq \langle c, x, a^2 \rangle$. However it is impossible. So we may not have a $CAC$-group.

It is easy to see that those groups in theorem are pairwise non-isomorphic. In following we prove those groups in theorem are $CAC$-groups.

If $G$ is of the type (1), then $G$ is a $CAC$-group by Lemma 3.4.
If $G$ is of the type (2), then $G$ is metacyclic and $\Phi(G)$ is cyclic. Let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\subseteq Z(G)$, then $H \not\subseteq \Phi(G)$ and so $H \not\subseteq \Phi(C_G(H))$. Since $G$ is metacyclic, $C_G(H)$ is metacyclic. Thus $d(C_G(H)) \leq 2$. It follows that there exists an element $g \in G$ such that $C_G(H) = \langle H, g \rangle$. Hence $C_G(H)/H = \langle \bar{g} \rangle$ is cyclic. So $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group.

If $G$ is of the type (3), then assume $G = \langle a, b, c \mid a^{2^n-2} = b^2 = 1, [a, b] = a^{-2}, [c, a] = [c, b] = 1 \rangle$. Let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\subseteq Z(G)$, then there exists an element $x \in H$ with $x \not\in Z(G)$. Assume $x = a^ib^jc^k$ with $j = 1$ or 2. If $j = 2$, then $H \leq C_G(H) \leq C_G(a^ib^jc^k) = \langle a, c \rangle \cong C_{2^{n-2}} \times C_2$. Thus $C_G(H)/H$ is cyclic. If $j = 1$, then $C_G(H) \leq C_G(a^ib^jc^k) = \langle a^ib, c, a^{2^{n-3}} \rangle$. Thus $|C_G(H)| \leq 8$. Since $|H| \geq 4$, $C_G(H)/H$ is cyclic. So $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group.

Similarly, if $G$ is a group of types (4), (5), (7) – (9), and (12) – (15), then $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group.

If $G$ is of the type (6), then $Z(G)$ is a cyclic subgroup of index 4. So $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group by Lemma 3.8.

If $G$ is a group of types (10), (11) and (16), then $|Z(G)| \geq 2^{n-3}$, $\Phi(G) \leq Z(G)$ or $\Phi(G)$ is cyclic. Let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\subseteq Z(G)$. Noticing that $HZ(G) \leq Z(C_G(H))$ and $|C_G(H)/HZ(G)| \leq 2$, we see $C_G(H)$ is abelian. It is easy to check $r(G) = 2$. Then $d(C_G(H)) \leq 2$. Since $\Phi(G) \leq Z(G)$ or $\Phi(G)$ is cyclic, we have $H \not\subseteq \Phi(G)$ and so $H \not\subseteq \Phi(C_G(H))$. Thus there exists an element $g \in C_G(H)$ such that $C_G(H) = \langle H, g \rangle$. Hence $C_G(H)/H = \langle \bar{g} \rangle$. So $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group.

If $G$ is a group of types (17) – (21), then $\Omega_1(G) = Z(G)$ and $G$ has no abelian maximal subgroup. Let $H$ be a non-cyclic abelian subgroup of $G$ and $H \not\subseteq Z(G)$. Then there exists an element $x \in H$ such that $o(x) = 4$. Thus $|H| \geq 8$. If $G$ is a group of types (17) – (20), then, since $|Z(G)| = 8$, we see $|Z(G)H| \geq 16$. It follows that $|C_G(H)| = 16$. If $G$ is the type (21), then $|Z(G)| = 4$. It is easy to check $Z(M) = Z(G)$ for all subgroups $M$ of order 32. It follows that $|C_G(H)| \leq 16$. Thus $|C_G(H)/H| \leq 2$ and therefore $G$ is a $\mathcal{C}\mathcal{A}\mathcal{C}$-2-group.

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\section*{References}

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(Jiao Wang) DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444, P.R. CHINA.

E-mail address: wangjiotiedan@163.com

(Xiuyun Guo) DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, SHANGHAI 200444, P.R. CHINA.

E-mail address: xyguo@staff.shu.edu.cn