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On semi-II-property of subgroups of finite group

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ON SEMI-II-PROPERTY OF SUBGROUPS OF FINITE GROUP

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ABSTRACT. Let $G$ be a group and $H$ a subgroup of $G$. Then $H$ is said to have semi-II-property in $G$ if there is a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ has II-property in $T$. In this paper, investigating on semi-II-property of subgroups, we shall obtain some new description of finite groups.

Keywords: Finite group, semi-II-property, SE subgroup, $p$-nilpotent.


1. Introduction

Throughout this paper, all groups are finite. We use standard terminology, as in Huppert [7] or Guo [5]. $G$ always is a group, and $|G|$ is the order of $G$; $\pi(G)$ denotes the set of all primes dividing $|G|$. Also $\mathbb{P}$ is the set of all primes and $\pi$ denotes a subset of $\mathbb{P}$; $\pi'$ is the complement of $\pi$ in $\mathbb{P}$. A group $G$ is said to be a $\pi$-group if $\pi(G)$ is a subset of $\pi$.

Subgroups play a very important role in group theory and different properties of subgroups have been studied by mathematicians, such as normality, quasinormality [10], S-quasinormality (cf. [3], etc), C-normality [14], weakly s-permutability [12], s-embedded and n-embedded property [6] and cover-avoidance property (cf. [4, A(10.8)]). A property of subgroups was proposed as the following in [8], to uniform some recent results.

Definition 1.1. Let $H$ be a subgroup of $G$. $H$ is said to have II-property in $G$ if for any $G$-chief factor $L/K$, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi$ $(HK/K \cap L/K)$-number.

Li proved in [8] that there are many examples of embedding properties of subgroups implying the possession of the II-property. After the work in [8],...
some new research has been done by many mathematicians (cf. [2, 13], etc). Let \( H \) and \( T \) be two subgroups of \( G \). Recall that \( T \) is called a supplement of \( H \) in \( G \) if \( G=HT \), and if furthermore \( H \cap T=1 \), then \( T \) is said to be a complement of \( H \) in \( G \). To develop the work of \( \Pi \)-property of subgroups, we introduce the following new concept in this paper.

**Definition 1.2.** Let \( H \) be a subgroup of \( G \). Then \( H \) is said to have \( \Pi \)-property in \( G \), if there is a subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \) has \( \Pi \)-property in \( T \).

**Remark 1.3.** (1) If \( H \) is a complement, then \( H \) has \( \Pi \)-property in \( G \) (cf. Lemma 2.2 in Section 2).
(2) It is clear that if \( H \) has \( \Pi \)-property in \( G \) then \( H \) has \( \Pi \)-property, but the reverse is not true. For example, the Sylow 5-subgroups of \( A_5 \) are complement in \( A_5 \) and hence have \( \Pi \)-property in \( A_5 \), but there is no non-trivial subgroup of \( A_5 \) with \( \Pi \)-property.
(3) If \( H \) has a supersolvable supplement in \( G \), then \( H \) has \( \Pi \)-property in \( G \) (cf. Lemma 2.2 in Section 2).
(4) In [8], if \( HT = G \) and \( H \cap T \leq I \leq H \), where \( I \) is a subgroup having \( \Pi \)-property in \( G \), then \( H \) is called \( \Pi \)-supplemented in \( G \). The following example shows that a subgroup \( H \) satisfying \( \Pi \)-property in \( G \) can not be \( \Pi \)-supplemented in \( G \).

**Example 1.4.** Let \( X = \langle x \rangle \times \langle y \rangle \), where \( |x| = |y| = 25 \). The maps \( \alpha : x \mapsto x^7 \), \( y \mapsto y^{-7} \) and \( \beta : x \mapsto y^{-1} \), \( y \mapsto x \) are automorphisms of \( X \) and generate a subgroup \( A \leq \text{Aut}(X) \) of order 8 (\( A \) is isomorphic with the quaternion group). Let \( G = [X]A \). Then the subgroup \( H = \langle x^5, \alpha \rangle \) has a supplement \( T = \langle X, \beta \rangle \) in \( G \). Since \( T \) is supersolvable, \( H \) has \( \Pi \)-property in \( G \). On the other hand, since \( x^5 \) belongs to \( \Phi(X) \) and \( X \) is the normal Sylow 5-subgroup of \( G \) and \( x^5 \in T \) for any supplement \( T \) of \( H \) in \( G \). That is \( \langle x^5 \rangle \leq H \cap T \leq H \). But neither \( \langle x^5 \rangle \) nor \( H \) has the \( \Pi \)-property in \( G \), so \( H \) is not a \( \Pi \)-supplement in \( G \).

Recall that a normal subgroup \( H \) of \( G \) is said to be \( SE \) in \( G \) if every chief factor of \( G \) lying in \( H \) is cyclic, and, there is a unique maximal \( SE \) subgroup of \( G \), which is denoted by \( SE(G) \). It is (cf. [15, 1.7]). Similarly, we call that a normal subgroup \( H \) of \( G \) is \( SE_p \) in \( G \) if every \( pd \)-chief factor of \( G \) which lies in \( H \) is cyclic. The unique maximal \( SE_p \) subgroup of \( G \) is denoted by \( SE_p(G) \). If \( G \neq 1 \) is \( p \)-solvable, then \( G \) has a nontrivial \( p \)-nilpotent normal subgroup. The product of all \( p \)-nilpotent normal subgroup of \( G \) is denoted by \( F_p(G) \). A group \( G \) is said quasinilpotent if all of its elements induce an inner automorphism on each chief factor of \( G \). In a group \( G \), the product of all quasinilpotent normal subgroups is called the generalized Fitting subgroup of \( G \) is denoted by \( F^*(G) \).

Based on the concept of semi-\( \Pi \)-property, we shall mainly prove the following theorems.
Theorem A. Let $E$ be a $p$-solvable normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $F_p(E)$. Then $E \subseteq SE_p(G)$ if and only if every cyclic subgroup of $P$ of order $p$ or 4 (if $P$ is a non-abelian 2-group) has semi-$\Pi$-property in $G$.

Theorem 1.5 (Theorem B). Let $E$ be a normal subgroup of $G$. Then $E \subseteq SE(G)$ if and only if every cyclic subgroup of $F^*(E)$ of prime order or of order 4 (if the Sylow 2-subgroup is non-abelian) has semi-$\Pi$-property in $G$.

2. Preliminaries

Lemma 2.1. Let $H$ be a subgroup of $G$ and $N$ a normal subgroup of $G$.
(1) If $H \leq T \leq G$ and $H$ has $\Pi$-property in $T$, then $HN/N$ has $\Pi$-property in $TN/N$.
(2) If $H$ has $\Pi$-property in $G$, then $H$ has semi-$\Pi$-property in $G$.
(3) If $H$ has semi-$\Pi$-property in $G$, then $HN/N$ has semi-$\Pi$-property in $G/N$ when $H \subseteq N$ or $(|H|, |N|) = 1$.

Proof. (1) Since $H$ has $\Pi$-property in $T$, hence by [8, Proposition 2(1)] $H(T \cap N)/(T \cap N)$ has $\Pi$-property in $T/T \cap N$. On the other hand, by using the isomorphism

$$\sigma : T/T \cap N \rightarrow TN/N$$

$$t(T \cap N) \mapsto tN$$

we may replace $H(T \cap N)/(T \cap N)$ by $HN/N$. So $HN/N$ has $\Pi$-property in $TN/N$.

(2) It is obvious by choosing $T = G$.

(3) Suppose that $H$ has semi-$\Pi$-property in $G$, then there is a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ has $\Pi$-property in $T$. If $N \subseteq H$, then $(H/N) \cap (TN/N) = (H \cap T)N/N$, and $(H \cap T)N/N$ has $\Pi$-property in $TN/N$ by (1). Thus $H/N$ has semi-$\Pi$-property in $G/N$. If $(|H|, |N|) = 1$, then $N \subseteq T$ since $N$ is normal in $G$. Similarly as above, we have $H/N$ has semi-$\Pi$-property in $G/N$.

□

Lemma 2.2. Let $H$ be a subgroup of $G$. Then $H$ has semi-$\Pi$-property in $G$ if one of the following holds:
(1) $H$ is complement in $G$; (2) $H$ has a supersolvable supplement in $G$.

Proof. (1) Assume that $T$ is a complement of $H$ in $G$. Then, $H \cap T = 1$ has $\Pi$-property in $T$ and hence $H$ has semi-$\Pi$-property in $G$.

(2) Assume that $T$ is supersolvable and $G = HT$. Then every subgroup of $T$ has $\Pi$-property in $T$ by [8, Proposition 2.11]. In particular, $H \cap T$ has $\Pi$-property in $T$ and therefore, $H$ has semi-$\Pi$-property in $G$. □
Lemma 2.3. ([8, Proposition 2.9]) Let $H$ be a $p$-subgroup of $G$ for some prime divisor $p$ of $|G|$, and assume that $H$ has II-property in $G$. Then any $G$-chief factor $L/K$ which does not avoid $H$ is a $p$-factor and hence is abelian.

Lemma 2.4. ([8, Proposition 2.7]) Let $H$ be a $p$-group of $G$ and $N$ a minimal normal subgroup of $G$. Assume that $H$ has II-property in $G$. If there is a Sylow $p$-subgroup $G_p$ of $G$ such that $H \leq G_p$, then $H \cap N = N$ or $1$.

Lemma 2.5. Let $N$ be a normal subgroup of order $p$ in $G$ and $a \in G$ is an element of order $p$. If $H = \langle N, a \rangle$ has II-property in $G$ then so does $A = \langle a \rangle$.

Proof. Let $L/K$ be an arbitrary chief factor of $G$. By the definition, we only need to prove that $|G/K : N_{G/K}((A \cap L)K/K)|$ is a $p$-number. If $A \leq K$, then it is clear. Assume that $A \not\leq K$. By [8, Proposition 2.1 (1)], $HN/N$ has II-property in $G/N$. If $N \leq K$ then $H \leq AK$ and so, $(A \cap L)K = (H \cap L)K$. It follows that $|G/K : N_{G/K}((A \cap L)K/K)| = |G/K : N_{G/K}((H \cap L)K/K)|$ is a $p$-number since $H$ has II-property in $G$. If $N \not\leq K$, then the hypotheses still hold for $G/K$. By induction, if $K \neq 1$, then $AK/K$ has II-property in $G/K$. This induces that $|G/K : N_{G/K}((A \cap L)K/K)|$ is a $p$-number. Assume that $K = 1$ and hence $L$ is a minimal normal subgroup of $G$. Since $N$ is also minimal normal in $G$, we see that $L = N$ or $L \cap N = 1$. If $L = N$, then $A \cap L = 1$ or $A = N$ and thus $|G : N_G(A \cap L)| = 1$. Assume that $L \cap N = 1$. Since $H$ has order $p^2$, $H \cap L = 1$ or cyclic of order $p$. On the other hand, since $A$ is cyclic of order $p$, $A \cap L = 1$ or is cyclic of order $p$, too. If $A \cap L = 1$, then $|G : N_G(A \cap L)| = 1$. If $A \cap L$ is of order $p$, we should have $A \cap L = H \cap L$ and hence $|G : N_G(A \cap L)| = |G : N_G(H \cap L)|$ is a $p$-number. This shows that the lemma holds.

Lemma 2.6. ([9, Lemma 2.7]) Let $P$ be a Sylow $p$-subgroup of $G$ and $N$ a normal subgroup of $G$ with $G = NP$. Assume that $N_G(P)$ is $p$-nilpotent and all subgroups of order $p$ in $N$ are complemented in $G$. Then $G$ is $p$-nilpotent.

3. Proofs of Theorems A and B

Lemma 3.1. Let $P$ be a normal $p$-subgroup of $G$. If every cyclic subgroup of $P$ of order $p$ or $4$ (if $P$ is a non-abelian 2-group) has semi-II-property in $G$ then $P \leq S(E(G))$.

Proof. Assume that this lemma does not hold. Then there is a $G$-chief factor in $P$ which is not of prime order. Choose a $G$-chief factor $L/K$ in $P$ such that $|L/K|$ is not prime but $|U/V|$ is prime for any chief factor $U/V$ of $G$ in $P$ with $|U| < |L|$.

Let $W = \bigcap_{U \leq K} C_G(U/V)$, where $U/V$ is a $G$-chief factor. Then, by [4, A(12.3)], all elements in $W$ of $p'$-order act trivially on $K$. Let $C = C_G(K)$ and assume $L \not\leq C$. If $L \subseteq KC$, then $(L \cap C)/(K \cap C) \cong L/K$ is chief factor of $G$. By the choice of $L/K$, $|L/K| = |(L \cap C)/(K \cap C)|$ is prime, a contradiction. If
$L \nsubseteq KC$, then it is easy to see that $LC/K = L/K \times KC/K$, and thereby all $p'$-elements in $C$ act trivially on $L/K$. It follows that all $p'$-elements in $W$ act trivially on $L/K$. Hence $W \subseteq C_G(L/K)$. Since $G/W = G/\bigcap_{U \subseteq K} C_G(U/V)$ is an abelian group of exponent dividing $p - 1$ and $W \subseteq C_G(L/K)$, $G/C_G(L/K)$ is an prime order by [15, I, Lemma 1.3], a contradiction.

Now assume $L \subseteq C$. Then $K \subseteq Z(L)$. Let $a, b$ be elements of order $p$ in $L$. Suppose $p > 2$ or $P$ is abelian. Then $(ab)^p = a^p b^p [b, a]^{a^p b^{-1}} = 1$. Hence the product of elements of order $p$ is of order $p$ or $1$ and hence $\Omega = \{a \in L | a^p = 1\}$ is a subgroup of $L$. If $\Omega \nsubseteq K$, then all elements of $W$ with $p'$-order act trivially on all elements of $L$ with order $p$ since they act trivially on $K$. It follows from [7, IV, Satz 5.12] that all elements in $W$ of order $p'$ act trivially on $L$. Thus $W \subseteq C_G(L/K)$ and, as above argument, $L/K$ is of prime order, a contradiction. If $\Omega \subseteq K$, then $L = \Omega K$. Choose an element $a$ in $\Omega \setminus K$ such that $\langle a \rangle K/K \subseteq L/K \cap Z(G_p/K)$. Let $H = \langle a \rangle$. Then $H$ has semi-II-property in $G$ and so there is a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ has II-property in $T$. If $T = G$, then $H \cap T = H$ has II-property in $T = G$. By Lemma 2.1 (1), $HK/K$ has II-property in $G/K$. It follows from Lemma 2.4 that $L/K = HK/K \cap L/K = HK/K$ is cyclic, a contradiction.

Assume that $T < G$. Clearly, $T$ is maximal in $G$. If $K \nsubseteq T$, then $KT_G/T_G$ is nontrivial. By Bare’s Theorem (cf. [1, A(15.2)]), $G/T_G$ has a unique minimal normal subgroup $R/T_G$ which is contained in $KT_G/T_G$ and is self-centralized. Clearly $R/T_G \leq KT_G/T_G \leq LT_G/T_G$. Since $R/T_G \leq Z(LT_G/T_G)$ by the property of $p$-group, $KT_G = LT_G$. It follows that $\frac{|K|}{|R|} = \frac{|L|}{|U|}$ and hence $|L/K| = (|L \cap T_G|/(K \cap T_G))$. Since $K \nsubseteq T_G$, $L \nsubseteq T_G$ and so, $|L \cap T_G| < |L|$. By the choice of $L/K$, $(L \cap T_G)/(K \cap T_G)$ is of order $p$ and so is $L/K$, a contradiction. Assume that $K \subseteq T_G \leq T$. Then $T/K$ is maximal in $G/K$ and $G/K = ((a)K/K)(T/K) = (L/K)(T/K)$. It follows that $T/K$ is isomorphic to $L/K$ in $G/K$ and $|L/K| = |G/K : T/K| = |G : T| = |x| = p$. Thus $L/K$ is cyclic. It can be proved that $L/K$ is cyclic similarly when $p = 2$ and $P$ is a non abelian 2-group. This contradiction shows $P \leq SE(G)$ and the lemma holds.

Proof of Theorem A. The “if” part: assume that $O_{p'}(E) \neq 1$. Then $F_p(E/O_{p'}(E)) = F_p(E)/O_{p'}(E)$. By Lemma 2.1, the hypotheses still hold on $E/O_{p'}(E)$. Then, by induction on $|E|$, $E/O_{p'}(E) \leq SE_p(G/O_{p'}(E))$ and hence every $pd$-G-chief factor which lies in $E$ is cyclic, that is $E \leq SE_p(G)$.

Assume that $O_{p'}(E) = 1$. Then $F_p(E) = F(E) = O_p(E)$ is a $p$-group. By Lemma 3.1, $F(E) \leq SE(G)$. Let $M_i/N_i$, $i = 1, \ldots, n$, be all $G$-chief factor in $F(E)$ and $C = \bigcap_{i=1}^n C_G(M_i/N_i)$. Then $F(E) \leq C$. We claim that $F(E) = C$. Otherwise, let $R/F(E)$ be a $G$-chief factor with $R \leq C$. Since $E$ is $p$-solvable, $R/F(E)$ is a $p$-factor or $p'$-factor. In particular, $R/F(E)$ is $p$-nilpotent. But $R \leq C$, so $R$ is $p$-nilpotent and hence $R \leq F_p(E) = F(E)$, a contradiction.
Thus our claim holds and $F(E) = C$. If $E \not\leq \text{SE}_p(G)$, then there is a $G$-chief factor $L/K$ in $E$ such that $L/K$ is noncyclic, but any $G$-chief factor $U/V$ in $E$ with $|V| < |K|$ is cyclic. Let $M/N$ be an arbitrary $G$-chief factor lying in $F(E)$ and put $C_1 = C_E(M/N)$. Since $M/N$ is of prime order, $E/C_1$ is cyclic and hence $L/L \cap C_1 \cong LC_1/C_1 \leq E/C_1$ is cyclic. It follows that $L \cap C_1 \not\leq K$ and so $L = (L \cap C_1)K$. Therefore, $L/K = (L \cap C_1)K/K \cong L \cap C_1/K \cap C_1$ is a $G$-chief factor. By the choice of $K$, we have $K \leq C_1$ and so $L = K(L \cap C_1) = L \cap KC_1 = L \cap C_1$. This induces that $L \leq C_1$ and consequently $L \leq C_E(M/N)$ for any $G$-chief factor $M/N$ of $F(E)$. Thus $L \leq C = F(E)$, a contradiction and hence $E \leq \text{SE}_p(G)$.

The “only if” part: we shall prove that every $p$-subgroup of $E$ has II-property in $G$ and hence the “only if” part holds. To prove this, by [8, Proposition 2.3], we only need to prove that every $p$-subgroup of $E$ is a CAP-subgroup of $G$.

Let $H$ be a $p$-subgroup of $E$ and $L/K$ be a $G$-chief factor. Since $E$ is normal in $G$, $E$ covers or avoids $L/K$. If $E$ avoids $L/K$ then so does $H$ since $H \leq E$. Assume that $E$ covers $L/K$. Then $L \leq KE$ and hence $L = L \cap KE = (L \cap E)K$. It follows that $L/K = (L \cap E)K/K \cong (L \cap E)/(K \cap E) \leq E/(K \cap E)$. Since $E \leq \text{SE}_p(G)$, $L/K$ is either of $p$-order or of order $p$. If $L/K$ is of $p$-order, then clearly, $H$ avoids $L/K$. If $L/K$ is of order $p$, then $(H \cap L)K/K = L/K$ or $1$. If $(H \cap L)K/K = L/K$ then $L = (H \cap L)K = L \cap HK$ and hence $H$ covers $L/K$. If $(H \cap L)K/K = 1$ then $H \cap L \leq K$ and hence $H$ avoids $L/K$. This means that $H$ is a CAP-subgroup of $G$ and hence the theorem holds.

\textbf{Proof of Theorem B}. The “only if” part can be proved similarly to Theorem A and we only prove the “if” part.

We claim that $F^*(E)$ is solvable. Let $H$ be a subgroup of $F^*(E)$ with order 2 and let $T$ be a supplement of $H$ in $G$. If $H \cap T = 1$ then $|G : T| = 2$ and hence $T \leq G$. It follows that $F^*(E) \cap T \leq G$ and $F^*(E) \cap T < F^*(E) \leq E$. Clearly, the hypotheses still hold for $(G, F^*(E) \cap T)$ and, by induction on $|E|$, we have that $F^*(E) \cap T \leq \text{SE}(G)$. In particular, $T \cap F^*(E)$ is solvable. Since $F^*(E)/F^*(E) \cap T$ is of order 2, $F^*(E)$ is solvable. Assume that $H \cap T = H$ for any supplement $T$, then $H$ has II-property in $G$ by the hypotheses. If the Sylow 2-subgroup of $F^*(E)$ is abelian. Then $F^*(E)$ is 2-nilpotent and hence is solvable by [8, Lemma 3.2]. Assume that the Sylow 2-subgroup of $F^*(E)$ is nonabelian. If $F^*(E)$ is a 2-group, then it is solvable. Assume that $F^*(E)$ is not a 2-group. Then $O_2(F^*(E)) < F^*(E)$. Let $R/O_2(F^*(E))$ be a $G$-chief factor in $F^*(E)$. Then $|R|$ is even and $R$ has a subgroup $H$ of order 2. By above argument, $H$ has II-property in $G$. Clearly, $H$ does not avoid $R/O_2(F^*(E))$. By Lemma 2.3, $R/O_2(F^*(E))$ is a 2-group and so $R$ is solvable. Since $R \leq F^*(E)$ is quasinilpotent, $R$ is nilpotent and hence $O_2(R) \neq 1$. It follows that $O_2(E) \neq 1$ and by Lemma 3.1, $O_2(E) \leq \text{SE}(G)$. Thus, every $G$-chief factor in $O_2(E)$ is cyclic. Therefore, $O_2(E) \leq Z(G)$. Let $X/F(E)$ be a $G$-chief factor in $F^*(E)$. If $X$ is solvable, then $X$ is nilpotent since $X \leq F^*(E)$, a contradiction.
Thus $X$ and so $X/F(E)$ is not solvable. Since $X$ is not solvable, there is a minimal non-$2$-nilpotent subgroup $M$ in $X$. By the structure of a minimal non-$p$-nilpotent group, $M = A \times B$, where $A$ is a $2$-group of exponent $2$ or $4$ (when $A$ is nonabelian $2$-group) and $B$ is a $p'$-group. If all elements of $X$ of order $2$ and $4$ are in $O_2(E)$, then all such elements are in $O_2(E) \leq Z_{\infty}(G)$. Thus $A \leq Z_{\infty}(G)$. It follows that $A \leq M \cap Z_{\infty}(G) \leq Z_{\infty}(M)$ and so $M$ is nilpotent, a contradiction. Hence there must be some element $x$ of order $2$ or $4$ such that $x \in X$ and $x \notin O_2(E)$. Furthermore, we can choose that $x^2 \in O_2(E)$. By the hypotheses, $H = \langle x \rangle$ has semi-$II$-property in $G$. Let $T$ be a supplement of $H$ in $G$. Assume $T < G$. If $O_2(E) \leq T$, then $|G : T| = 2$ since $x^2 \in O_2(E) \leq T$ and $HT = G$. Thus $T \leq G$. By a similar argument as above, $F^*(E)$ is solvable. If $O_2(E) \nsubseteq T$, then there must be a subgroup $D$ of $O_2(E)$ such that $DT$ is a subgroup of $G$ and $|G : DT| = 2$ since $O_2(E) \leq Z_{\infty}(G)$ and $|G : T| = 2$ or $4$. Then $DT \leq G$ and similarly as above, $F^*(E)$ is solvable. Finally, assume that $G$ is the only supplement of $H$ in $G$. Then $H$ has $II$-property in $G$. By Lemma 2.3, $X/F(E)$ is abelian, a contradiction. This contradiction shows that $F^*(E)$ is solvable and our claim holds.

Now, let $F^*(E) = F(E)$ be the direct product of primary subgroups. By Lemma 3.1, $F^*(E) \leq SE(G)$. Similar to the proof of Theorem A, $E \leq SE(G)$ and the theorem holds.

\section{4. On $p$-nilpotency of groups}

\textbf{Theorem 4.1.} Let $G$ be a group and $p$ a prime with $([G], p-1) = 1$. Assume that $E$ is a normal subgroup of $G$ with $p$-nilpotent quotient. Let $P$ be a Sylow $p$-subgroup of $E$. If every subgroup of $P$ of order $p$ or $4$ (if $P$ is a nonabelian $2$-group) has semi-$II$-property in $G$, then $G$ is $p$-nilpotent.

\textit{Proof.} By a similar argument as in the proof of Theorem B, we can obtain that $E$ is solvable. Then, it follows from Theorem A that $E \leq SE_p(G)$. Thus every $pd$-chief factor $H/K$ of $G$ in $E$ is cyclic of order $p$. Since $G/C_G(H/K)$ is isomorphic to some subgroup of $\text{Aut}(H/K)$, which is cyclic of order $p-1$, and $([G], p-1) = 1$, we see that $G/C_G(H/K) = 1$ and $H/K$ is central, that is, every $G$-chief factor in $E$ is either of $p'$-order or central in $G$. Since $G/E$ is $p$-nilpotent, we obtain that $G$ is also $p$-nilpotent. \hfill $\square$

It is easy to show that if $([G], p^2-1) = 1$, then $G$ has no chief factor of order $p^2$ and so if $p^3 \mid |G|$ then $G$ is $p$-nilpotent. A more general result can be found in [1, Lemma 2.12]. Considering groups in which every subgroup of order $p^2$ has semi-$II$-property, we obtain the following theorem.

\textbf{Theorem 4.2.} Let $G$ be a group and $p$ a prime with $([G], p^2-1) = 1$. Assume that $E$ is a normal subgroup of $G$ with $p$-nilpotent quotient. Let $P$ be a Sylow
p-subgroup of $E$. If every subgroup of $P$ of order $p^2$ has semi-$\Pi$-property in $G$, then $G$ is p-nilpotent.

**Proof.** Assume that the theorem is not true, and $G$ is a counterexample of minimal order. We prove the theorem via the following steps:

1. $O_p'(G) = 1$.

By Lemma 2.1, the hypotheses still hold on $G/O_p'(G)$. If $O_p'(G) \neq 1$, then we can assume that $G/O_p'(G)$ is p-nilpotent by the choice of $G$. It follows that $G$ is p-nilpotent, a contradiction. Hence $O_p'(G) = 1$.

2. Let $N$ be a minimal normal subgroup of $G$. If $N \leq O_p(G)$, then $N$ is cyclic of order $p$. Since $N \not\leq E$, then $NE/E$ is a chief factor of $G/E$. But $G/E$ is p-nilpotent, so $N \cong NE/E$ is cyclic of order $p$. Assume that $N \subseteq E$. If $|N| > p^2$, then $N$ has a proper subgroup $H$ of order $p^2$ with $H \lneq G_p$, a Sylow $p$-subgroup of $G$. By the hypotheses, $H$ has semi-$\Pi$-property in $G$, and so there is a subgroup $T$ of $G$ such that $G = TH$ and $H \cap T$ has $\Pi$-property in $T$. Clearly, $G = NT$. Thus $N \cap T \leq G$ since $N$ is abelian. If $T \neq G$, then $N \cap T \neq N$. It follows that $N \cap T = 1$ since $N$ is minimal normal in $G$. Hence $|N| = |G : T| \leq |H| = p^2$, a contradiction. If $T = G$, then $H \cap T = H$ has $\Pi$-property in $T = G$. By Lemma 2.4, $N = H$ is of order $p^2$, which contradicts to $|N| > p^2$ and thus $|N| \leq p^2$.

If $|N| = p^2$, then $\Aut(N)$ is of order $(p^2 - 1)(p^2 - p)$. Since $G/C_G(N)$ is isomorphic to some subgroup of $\Aut(N)$, $|G/C_G(N)|$ is a divisor of $(p^2 - 1)(p^2 - p)$. But $(|G|, p^2 - 1) = 1$, so $|G/C_G(N)|$ is a $p$-number. It follows from [5, Lemma 1.7.11] that $|G/C_G(N)| = 1$ and hence $N \subseteq Z(G)$. It follows that all subgroups of $N$ are normal in $G$. This is not true for $|N| = p^2$ and $N$ is a minimal normal subgroup $G$. Therefore, $|N| \neq p^2$ and so $|N| = p$.

3. $p = 2$.

If $p \neq 2$, then $G$ is of odd order since $(|G|, p^2 - 1) = 1$. Thus $G$ is solvable and so $O_p(G) \neq 1$ by (1). Then (2) implies that $G$ has a minimal normal subgroup $N$ of order $p$. By Lemma 2.1(3) and the hypotheses, every subgroup of order $p$ in $E/N$ has semi-$\Pi$-property in $G/N$. By Theorem A, $E/N \leq \SE_p(G/N)$. Since $N$ is cyclic, $E \leq \SE_p(G)$. Similar to the proof of Theorem 4.1, we have that $G$ is $p$-nilpotent, contrary to the choice of $G$. Thus $p = 2$ and (3) holds.

4. Let $N \leq E$ be a minimal normal subgroup of $G$, then $N$ is of order 2 and hence is contained in $Z(G)$.

Assume that $L$ is a minimal normal subgroup of $N$. If $N$ is not a 2-group, then, since $O_p'(G) = 1$ by (1), $L$ is a nonabelian simple group. By [11, (10.1.9)] and $(|G|, p^2 - 1) = 1$, the order of a Sylow 2-subgroup of $L$ is greater than 4. Choose $H$ to be a subgroup of $L$ of order 4. We claim that $G$ is the only supplement of $H$ in $G$. In fact, if $H$ has a proper supplement $T$ in $G$, then $|G : T| = 2$ or 4. If $|G : T| = 2$, then $T$ is normal in $G$. Since $LT = HT = G$, $|L : L \cap T| = |G : T| = 2$. This shows that $L \cap T \leq L$, a contradiction. If
If the theorem is not true and let $j$, $t$ be a counter example of minimal order, we prove the theorem via the following steps.

(1) $O_p'(G) = 1$.

If $O_p'(G) \neq 1$, then the hypotheses still hold on $G/O_p'(G)$. Hence we can assume that $G/O_p'(G)$ is $p$-nilpotent by the choice of $G$. It follows that $G$ is $p$-nilpotent, a contradiction and then $O_p'(G) = 1$.

(2) $O_p(E) \neq 1$ and $O_p(E) = F(E) = F^*(E)$.

Let $N$ be a minimal normal subgroup of $G$ contained in $E$. Then, by (1), $p$ divides $|N|$. By the hypotheses, every subgroup of order $p$ in $N$ has semi-II-property in $G$. If all subgroups of order $p$ in $N$ are complemented in $G$,
by Lemma 2.6, $NP$ is $p$-nilpotent and so is $N$. Since $O_p'(G) = 1$ by (1), $N$ is a $p$-group. Assume there is a subgroup $H$ of order $p$ in $N$ not complemented in $G$. Then $G$ is the only supplement of $H$ in $G$. Hence $H$ must have II-property in $G$. By Lemma 2.3, $N$ is a $p$-group and so $O_p(E) \neq 1$.

Since $O_p'(E) \leq O_p'(G) = 1$, it is easy to see that $F(E) = O_p(E)$. If $F(E) \neq F^*(E)$, then we can choose a $G$-chief factor $R/F(E)$ with $R \subseteq F^*(E)$. Let $Q = O^p(R) = \langle a \in R \mid p \mid [a] \rangle$ and $O = O_p(E) \cap O^p(R)$. Then $Q/O \cong R/O_p(G)$ is a chief factor of $G$ and is characteristically simple. Choose $M/O$ to be a minimal normal subgroup of $Q/O$. Then $Q/O \cong M/O \times \cdots \times M/O$. Clearly, $M$ is not solvable, otherwise, $M \subseteq F(E)$. Let $x$ be an element of $F^*(E)$ of order $p'$. Then $x$ induces an inner automorphism on each chief factor of $F^*(E)$ and so acts trivially on all abelian chief factors. In particular, $x$ acts trivially on all $G$-chief factors of $F(E)$. By [4, A.12.3], $x$ acts trivially on $O_p(E)$. Thus all $p'$-elements in $F^*(E)$ act trivially on $O_p(E)$. It follows that $Q$ and so $O^p(M)$ act trivially on $O_p(E)$. If all elements of order $p$ in $A$ are contained in $O_p(E)$, then $B$ acts trivially on $O_p(E)$.

Now, we claim that every subgroup of order $p$ in $Q/O$ is complemented in $G/O$. Choose $\overline{H} = HO/O$ to be a subgroup of order $p$ in $Q/O$, where $H$ is a subgroup of order $p$. Then by above argument, $H$ is complemented in $G$. Assume that $T$ is a complement of $H$ in $G$. Then $T$ is maximal in $G$. Let $Q_1$ be
a minimal supplement of $O$ in $Q$. Then, by [5, Lemma 2.3.4], $Q_1 \cap O \subseteq \Phi(Q_1)$. So, $Q_1$ is generated by all its $p'$-elements. Since, by the above argument, $O \leq Z_\infty(Q)$, we have $O \leq C_Q(Q_1)$. Hence $Q_1 \unlhd Q$. Thus $Q = O^p(Q) \leq Q_1$ and so $Q = Q_1$. It follows that $O = O \cap Q \subseteq \Phi(Q) \subseteq \Phi(G) \leq T$. Hence $HO/O$ is complemented in $G/O$ and $T/O$ is a complement of it. Our claim holds.

It is easy to see that $N_{G/O}(P/O)$ is $p$-nilpotent. Hence, by Lemma 2.6, $QP/O$ is $p$-nilpotent and so is $Q$. But this is not true and so (2) holds.

(3) The final contradiction.

By Theorem B and Step (2), we have that $E \leq S(E)$. In particular, $E$ is supersolvable. Since $O_{p'}(G) = 1$ by (1), $P$, the Sylow $p$-subgroup of $E$, is normal in $E$ and hence is normal in $G$. Thus $G = N_G(P)$ is $p$-nilpotent. The theorem holds. \qed

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