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ON SEMI-Π-PROPERTY OF SUBGROUPS OF FINITE GROUP

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Abstract. Let G be a group and H a subgroup of G. Then H is said to have semi-Π-property in G if there is a subgroup T of G such that $G = HT$ and $H \cap T$ has Π-property in T. In this paper, investigating on semi-Π-property of subgroups, we shall obtain some new description of finite groups.
Keywords: Finite group, semi-Π-property, SE subgroup, p-nilpotent.
MSC(2010): Primary: 20D10; Secondary: 20D20.

1. Introduction

Throughout this paper, all groups are finite. We use standard terminology, as in Huppert [7] or Guo [5]. G always is a group, and $|G|$ is the order of G; $\pi(G)$ denotes the set of all primes dividing $|G|$. Also $\mathbb{P}$ is the set of all primes and π denotes a subset of $\mathbb{P}$; π' is the complement of π in $\mathbb{P}$. A group G is said to be a π-group if $\pi(G)$ is a subset of π.

Subgroups play a very important role in group theory and different properties of subgroups have been studied by mathematicians, such as normality, quasinormality [10], S-quasinormality (cf. [3], etc), C-normality [14], weakly s-permutability [12], s-embedded and n-embedded property [6] and cover-avoidance property (cf. [4, A(10.8)]). A property of subgroups was proposed as the following in [8], to uniform some recent results.

Definition 1.1. Let H be a subgroup of G. H is said to have Π-property in G if for any G-chief factor L/K, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a π $(HK/K \cap L/K)$-number.

Li proved in [8] that there are many examples of embedding properties of subgroups implying the possession of the Π-property. After the work in [8],

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some new research has been done by many mathematicians (cf. [2, 13], etc). Let H and T be two subgroups of G. Recall that T is called a supplement of H in G if $G{=}HT$, and if furthermore $H \cap T{=}1$, then T is said to be a complement of H in G. To develop the work of Π-property of subgroups, we introduce the following new concept in this paper.

Definition 1.2. Let H be a subgroup of G. Then H is said to have semi-Π-property in G, if there is a subgroup T of G such that $G = HT$ and $H \cap T$ has Π-property in T.

Remark 1.3. (1) If H is a complement, then H has semi-Π-property in G (cf. Lemma 2.2 in Section 2).
(2) It is clear that if H has Π-property in G then H has semi-Π-property, but the reverse is not true. For example, the Sylow 5-subgroups of A_5 are complement in A_5 and hence have semi-Π-property in A_5, but there is no non-trivial subgroup of A_5 with Π-property.
(3) If H has a supersolvable supplement in G, then H has semi-Π-property in G (cf. Lemma 2.2 in Section 2).
(4) In [8], if $HT = G$ and $H \cap T \leq I \leq H$, where I is a subgroup having Π-property in G, then H is called Π-supplemented in G. The following example shows that a subgroup H satisfying semi-Π-property in G can not be Π-supplemented in G.

Example 1.4. Let $X = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = 25$. The maps $\alpha : x \mapsto x^7,\ y \mapsto y^{-7}$ and $\beta : x \mapsto y^{-1},\ y \mapsto x$ are automorphisms of X and generate a subgroup $A \leq \mathrm{Aut}(X)$ of order 8 (A is isomorphic with the quaternion group). Let $G = [X]A$. Then the subgroup $H = \langle x^5, \alpha \rangle$ has a supplement $T = \langle X, \beta \rangle$ in G. Since T is supersolvable, H has semi-Π-property in G. On the other hand, since x^5 belongs to $\Phi(X)$ and X is the normal Sylow 5-subgroup of G and $x^5 \in T$ for any supplement T of H in G. That is $\langle x^5 \rangle \leq H \cap T \leq H$. But neither $\langle x^5 \rangle$ nor H has the Π-property in G, so H is not a Π-supplement in G.

Recall that a normal subgroup H of G is said to be SE in G if every chief factor of G lying in H is cyclic, and, there is a unique maximal SE subgroup of G, which is denoted by $\mathrm{SE}(G)$. It is (cf. [15, 1.7]). Similarly, we call that a normal subgroup H of G is SE_p in G if every pd-chief factor of G which lies in H is cyclic. The unique maximal SE_p subgroup of G is denoted by $\mathrm{SE}_p(G)$. If $G \neq 1$ is p-solvable, then G has a nontrivial p-nilpotent normal subgroup. The product of all p-nilpotent normal subgroup of G is denoted by $F_p(G)$. A group G is said quasinilpotent if all of its elements induce an inner automorphism on each chief factor of G. In a group G, the product of all quasinilpotent normal subgroups is called the generalized Fitting subgroup of G is denoted by $F^*(G)$.

Based on the concept of semi-Π-property, we shall mainly prove the following theorems.

Theorem A. Let E be a p-solvable normal subgroup of G and P a Sylow p-subgroup of $F_p(E)$. Then $E \leq \mathrm{SE}_p(G)$ if and only if every cyclic subgroup of P of order p or 4(if P is a non-abelian 2-group) has semi-Π-property in G.

Theorem 1.5 (Theorem B)**.** *Let E be a normal subgroup of G. Then $E \leq SE(G)$ if and only if every cyclic subgroup of $F^*(E)$ of prime order or of order 4 (if the Sylow 2-subgroup is non-abelian) has semi-Π-property in G.*

2. **Preliminaries**

Lemma 2.1. *Let H be a subgroup of G and N a normal subgroup of G.*
(1) *If $H \leq T \leq G$ and H has Π-property in T, then HN/N has Π-property in TN/N.*
(2) *If H has Π-property in G, then H has semi-Π-property in G.*
(3) *If H has semi-Π-property in G, then HN/N has semi-Π-property in G/N when $H \subseteq N$ or $(|H|, |N|) = 1$.*

Proof. (1) Since H has Π-property in T, hence by [8, Proposition 2(1)] $H(T \cap N)/(T \cap N)$ has Π-property in $T/T \cap N$. On the other hand, by using the isomorphism

$$\sigma : T/T \cap N \longrightarrow TN/N$$

$$t(T \cap N) \longmapsto tN$$

we may replace $H(T \cap N)/(T \cap N)$ by HN/N. So HN/N has Π-property in TN/N.

(2) It is obvious by choosing $T = G$

(3) Suppose that H has semi-Π-property in G, then there is a subgroup T of G such that $G = HT$ and $H \cap T$ has Π-property in T. If $N \subseteq H$, then $(H/N) \cap (TN/N) = (H \cap T)N/N$, and $(H \cap T)N/N$ has Π-property in TN/N by (1). Thus H/N has semi-Π-property in G/N. If $(|H|, |N|) = 1$, then $N \subseteq T$ since N is normal in G. Similarly as above, we have H/N has semi-Π-property in G/N.

□

Lemma 2.2. *Let H be a subgroup of G. Then H has semi-Π-property in G if one of the following holds:*
(1) *H is complement in G;* (2) *H has a supersolvable supplement in G.*

Proof. (1) Assume that T is a complement of H in G. Then, $H \cap T = 1$ has Π-property in T and hence H has semi-Π-property in G.

(2) Assume that T is supersolvable and $G = HT$. Then every subgroup of T has Π-property in T by [8, Proposition 2.11]. In particular, $H \cap T$ has Π-property in T and therefore, H has semi-Π-property in G. □

Lemma 2.3. ([8, Proposition 2.9]) *Let H be a p-subgroup of G for some prime divisor p of $|G|$, and assume that H has Π-property in G. Then any G-chief factor L/K which does not avoid H is a p-factor and hence is abelian.*

Lemma 2.4. ([8, Proposition 2.7]) *Let H be a p-group of G and N a minimal normal subgroup of G. Assume that H has Π-property in G. If there is a Sylow p-subgroup G_p of G such that $H \trianglelefteq G_p$, then $H \cap N = N$ or 1.*

Lemma 2.5. *Let N be a normal subgroup of order p in G and $a \in G$ is an element of order p. If $H = \langle N, a\rangle$ has Π-property in G then so does $A = \langle a\rangle$.*

Proof. Let L/K be an arbitrary chief factor of G. By the definition, we only need to prove that $|G/K : N_{G/K}((A \cap L)K/K)|$ is a p-number. If $A \le K$, then it is clear. Assume that $A \not\le K$. By [8, Proposition 2.1 (1)], HN/N has Π-property in G/N. If $N \le K$ then $H \subseteq AK$ and so, $(A\cap L)K = (H \cap L)K$. It follows that $|G/K : N_{G/K}((A \cap L)K/K)| = |G/K : N_{G/K}((H \cap L)K/K)|$ is a p-number since H has Π-property in G. If $N \not\le K$, then the hypotheses still hold for G/K. By induction, if $K \ne 1$, then AK/K has Π-property in G/K. This induces that $|G/K : N_{G/K}((A \cap L)K/K)|$ is a p-number. Assume that $K = 1$ and hence L is a minimal normal subgroup of G. Since N is also minimal normal in G, we see that $L = N$ or $L \cap N = 1$. If $L = N$, then $A \cap L = 1$ or $A = N$ and thus $|G : N_G(A \cap L)| = 1$. Assume that $L \cap N = 1$. Since H has order p^2, $H \cap L = 1$ or cyclic of order p. On the other hand, Since A is cyclic of order p, $A \cap L = 1$ or is cyclic of order p, too.. If $A \cap L = 1$, then $|G : N_G(A \cap L)| = 1$. If $A \cap L$ is of order p, we should have $A \cap L = H \cap L$ and hence $|G : N_G(A \cap L)| = |G : N_G(H \cap L)|$ is a p-number. This shows that the lemma holds. □

Lemma 2.6. ([9, Lemma 2.7]) *Let P be a Sylow p-subgroup of G and N a normal subgroup of G with $G = NP$. Assume that $N_G(P)$ is p-nilpotent and all subgroups of order p in N are complemented in G. Then G is p-nilpotent.*

3. **Proofs of Theorems A and B**

Lemma 3.1. *Let P be a normal p-subgroup of G. If every cyclic subgroup of P of order p or 4(if P is a non-abelian 2-group) has semi-Π-property in G then $P \le SE(G)$.*

Proof. Assume that this lemma does not hold. Then there is a G-chief factor in P which is not of prime order. Choose a G-chief factor L/K in P such that $|L/K|$ is not prime but $|U/V|$ is prime for any chief factor U/V of G in P with $|U| < |L|$.

Let $W = \bigcap_{U\subseteq K} C_G(U/V)$, where U/V is a G-chief factor. Then, by [4, A(12.3)], all elements in W of p'-order act trivially on K. Let $C = C_G(K)$ and assume $L \nsubseteq C$. If $L \subseteq KC$, then $(L \cap C)/(K \cap C) \cong L/K$ is chief factor of G. By the choice of L/K, $|L/K| = |(L \cap C)/(K \cap C)|$ is prime, a contradiction. If

$L \nsubseteq KC$, then it is easy to see that $LC/K = L/K \times KC/K$, and thereby all p'-elements in C act trivially on L/K. It follows that all p'-elements in W act trivially on L/K. Hence $W \subseteq C_G(L/K)$. Since $G/W = G/\bigcap_{U\subseteq K} C_G(U/V)$ is an abelian group of exponent dividing $p-1$ and $W \subseteq C_G(L/K)$, $G/C_G(L/K)$ is an prime order by [15, I,Lemma 1.3], a contradiction.

Now assume $L \subseteq C$. Then $K \subseteq Z(L)$. Let a, b be elements of order p in L. Suppose $p > 2$ or P is abelian. Then $(ab)^p = a^p b^p [b,a]^{\frac{p(p-1)}{2}} = 1$. Hence the product of elements of order p is of order p or 1 and hence $\Omega = \{a \in L | a^p = 1\}$ is a subgroup of L. If $\Omega \subseteq K$, then all elements of W with p'-order act trivially on all elements of L with order p since they act trivially on K. It follows from [7, IV,Satz 5.12] that all elements in W of order p' act trivially on L. Thus $W \subseteq C_G(L/K)$ and, as above argument, L/K is of prime order, a contradiction. If $\Omega \nsubseteq K$, then $L = \Omega K$. Choose an element a in $\Omega \backslash K$ such that $\langle a\rangle K/K \subseteq L/K \cap Z(G_p/K)$. Let $H = \langle a\rangle$. Then H has semi-Π-property in G and so there is a subgroup T of G such that $G = HT$ and $H \cap T$ has Π-property in T. If $T = G$, then $H \cap T = H$ has Π-property in $T = G$. By Lemma 2.1 (1), HK/K has Π-property in G/K. It follows from Lemma 2.4 that $L/K = HK/K \cap L/K = HK/K$ is cyclic, a contradiction. Assume that $T < G$. Clearly, T is maximal in G. If $K \nsubseteq T_G$, then KT_G/T_G is nontrivial. By Bare's Theorem (cf. [4, A(15.2)]), G/T_G has a unique minimal normal subgroup R/T_G which is contained in KT_G/T_G and is self centralized. Clearly $R/T_G \leq KT_G/T_G \leq LT_G/T_G$. Since $R/T_G \leq Z(LT_G/T_G)$ by the property of p-group, $KT_G = LT_G$. It follows that $\frac{|K|}{|K\cap T_G|} = \frac{|L|}{|L\cap T_G|}$ and hence $|L/K| = |(L\cap T_G)/(K\cap T_G)|$. Since $K \nsubseteq T_G$, $L \nsubseteq T_G$ and so, $|L \cap T_G| < |L|$. By the choice of L/K, $(L\cap T_G)/(K\cap T_G)$ is of order p and so is L/K, a contradiction. Assume that $K \leq T_G \leq T$. Then T/K is maximal in G/K and $G/K = (\langle a\rangle K/K)(T/K) = (L/K)(T/K)$. It follows that T/K is a complement of L/K in G/K and $|L/K| = |G/K : T/K| = |G : T| = |x| = p$. Thus L/K is cyclic. It can be proved that L/K is cyclic similarly when $p = 2$ and P is a non abelian 2-group. This contradiction shows $P \leq SE(G)$ and the lemma holds. □

Proof of Theorem A. The "if" part: assume that $O_{p'}(E) \neq 1$. Then $F_p(E/O_{p'}(E)) = F_p(E)/O_{p'}(E)$. By Lemma 2.1, the hypotheses still hold on $E/O_{p'}(E)$. Then, by induction on $|E|$, $E/O_{p'}(E) \leq \mathrm{SE}_p(G/O_{p'}(E))$ and hence every pd-G-chief factor which lies in E is cyclic, that is $E \leq \mathrm{SE}_p(G)$.

Assume that $O_{p'}(E) = 1$. Then $F_p(E) = F(E) = O_p(E)$ is a p-group. By Lemma 3.1, $F(E) \leq \mathrm{SE}(G)$. Let M_i/N_i, $i = 1, \cdots, n$, be all G-chief factor in $F(E)$ and $C = \bigcap_{i=1}^{n} C_G(M_i/N_i)$. Then $F(E) \leq C$. We claim that $F(E) = C$. Otherwise, let $R/F(E)$ be a G-chief factor with $R \leq C$. Since E is p-solvable, $R/F(E)$ is a p-factor or p'-factor. In particular, $R/F(E)$ is p-nilpotent. But $R \leq C$, so R is p-nilpotent and hence $R \leq F_p(E) = F(E)$, a contradiction.

Thus our claim holds and $F(E) = C$. If $E \not\leq \mathrm{SE}_p(G)$, then there is a G-chief factor L/K in E such that L/K is noncyclic, but any G-chief factor U/V in E with $|V| < |K|$ is cyclic. Let M/N be an arbitrary G-chief factor lying in $F(E)$ and put $C_1 = C_E(M/N)$. Since M/N is of prime order, E/C_1 is cyclic and hence $L/L \cap C_1 \cong LC_1/C_1 \leq E/C_1$ is cyclic. It follows that $L \cap C_1 \not\leq K$ and so $L = (L \cap C_1)K$. Therefore, $L/K = (L \cap C_1)K/K \cong L \cap C_1/K \cap C_1$ is a G-chief factor. By the choice of K, we have $K \leq C_1$ and so $L = K(L \cap C_1) = L \cap KC_1 = L \cap C_1$. This induces that $L \leq C_1$ and consequently $L \leq C_E(M/N)$ for any G-chief factor M/N of $F(E)$. Thus $L \leq C = F(E)$, a contradiction and hence $E \leq \mathrm{SE}_p(G)$.

The "only if" part: we shall prove that every p-subgroup of E has Π-property in G and hence the "only if" part holds. To prove this, by [8, Proposition 2.3], we only need to prove that every p-subgroup of E is a CAP-subgroup of G.

Let H be a p-subgroup of E and L/K be a G-chief factor. Since E is normal in G, E covers or avoids L/K. If E avoids L/K then so does H since $H \leq E$. Assume that E covers L/K. Then $L \leq KE$ and hence $L = L \cap KE = (L \cap E)K$. It follows that $L/K = (L \cap E)K/K \cong (L \cap E)/(K \cap E) \leq E/(K \cap E)$. Since $E \leq \mathrm{SE}_p(G)$, L/K is either of p'-order or of order p. If L/K is of p'-order, then clearly, H avoids L/K. If L/K is of order p. Then $(H \cap L)K/K = L/K$ or 1. If $(H \cap L)K/K = L/K$ then $L = (H \cap L)K = L \cap HK$ and hence H covers L/K. If $(H \cap L)K/K = 1$ then $H \cap L \leq K$ and hence H avoids L/K. This means that H is a CAP-subgroup of G and hence the theorem holds. □

Proof of Theorem B. The "only if" part can be proved similarly to Theorem A and we only prove the "if" part.

We claim that $F^*(E)$ is solvable. Let H be a subgroup of $F^*(E)$ with order 2 and let T be a supplement of H in G. If $H \cap T = 1$ then $|G : T| = 2$ and hence $T \trianglelefteq G$. It follows that $F^*(E) \cap T \trianglelefteq G$ and $F^*(E) \cap T < F^*(E) \leq E$. Clearly, the hypotheses still hold for $(G, F^*(E) \cap T)$ and, by induction on $|E|$, we have that $F^*(E) \cap T \leq \mathrm{SE}(G)$. In particular, $T \cap F^*(E)$ is solvable. Since $F^*(E)/F^*(E) \cap T$ is of order 2, $F^*(E)$ is solvable. Assume that $H \cap T = H$ for any supplement T, then H has Π-property in G by the hypotheses. If the Sylow 2-subgroup of $F^*(E)$ is abelian. Then $F^*(E)$ is 2-nilpotent and hence is solvable by [8, Lemma 3.2]. Assume that the Sylow 2-subgroup of $F^*(E)$ is nonabelian. If $F^*(E)$ is a $2'$-group, then it is solvable. Assume that $F^*(E)$ is not a $2'$-group. Then $O_{2'}(F^*(E)) < F^*(E)$. Let $R/O_{2'}(F^*(E))$ be a G-chief factor in $F^*(E)$. Then $|R|$ is even and R has a subgroup H of order 2. By above argument, H has Π-property in G. Clearly, H does not avoid $R/O_{2'}(F^*(E))$. By Lemma 2.3, $R/O_{2'}(F^*(E))$ is a 2-group and so R is solvable. Since $R \leq F^*(E)$ is quasinilpotent, R is nilpotent and hence $O_2(R) \neq 1$. It follows that $O_2(E) \neq 1$ and by Lemma 3.1, $O_2(E) \leq \mathrm{SE}(G)$. Thus, every G-chief factor in $O_2(E)$ is cyclic. Therefore, $O_2(E) \leq Z_\infty(G)$. Let $X/F(E)$ be a G-chief factor in $F^*(E)$. If X is solvable, then X is nilpotent since $X \leq F^*(E)$, a contradiction.

Thus X and so $X/F(E)$ is not solvable. Since X is not solvable, there is a minimal non-2-nilpotent subgroup M in X. By the structure of a minimal non-p-nilpotent group, $M = A \rtimes B$, where A is a 2-group of exponent 2 or 4(when A is nonabelian 2-group) and B is a p'-group. If all elements of X of order 2 and 4 are in $O_2(E)$, then all such elements are in $O_2(E) \leq Z_\infty(G)$. Thus $A \leq Z_\infty(G)$. It follows that $A \leq M \cap Z_\infty(G) \leq Z_\infty(M)$ and so M is nilpotent, a contradiction. Hence there must be some element x of order 2 or 4 such that $x \in X$ and $x \notin O_2(E)$. Furthermore, we can choose that $x^2 \in O_2(E)$. By the hypotheses, $H = \langle x \rangle$ has semi-Π-property in G. Let T be a supplement of H in G. Assume $T < G$. If $O_2(E) \leq T$, then $|G : T| = 2$ since $x^2 \in O_2(E) \leq T$ and $HT = G$. Thus $T \trianglelefteq G$. By a similar argument as above, $F^*(E)$ is solvable. If $O_2(E) \nsubseteq T$, then there must be a subgroup D of $O_2(E)$ such that DT is a subgroup of G and $|G : DT| = 2$ since $O_2(E) \leq Z_\infty(G)$ and $|G : T| = 2$ or 4. Then $DT \trianglelefteq G$ and similarly as above, $F^*(E)$ is solvable. Finally, assume that G is the only supplement of H in G. Then H has Π-property in G. By Lemma 2.3, $X/F(E)$ is abelian, a contradiction. This contradiction shows that $F^*(E)$ is solvable and our claim holds.

Now, let $F^*(E) = F(E)$ be the direct product of primary subgroups. By Lemma 3.1, $F^*(E) \leq \mathrm{SE}(G)$. Similar to the proof of Theorem A, $E \leq \mathrm{SE}(G)$ and the theorem holds.

□

4. On p-nilpotency of groups

Theorem 4.1. *Let G be a group and p a prime with $(|G|, p-1) = 1$. Assume that E is a normal subgroup of G with p-nilpotent quotient. Let P be a Sylow p-subgroup of E. If every subgroup of P of order p or 4 (if P is a nonabelian 2-group) has semi-Π-property in G, then G is p-nilpotent.*

Proof. By a similar argument as in the proof of Theorem B, we can obtain that E is solvable. Then, it follows from Theorem A that $E \leq \mathrm{SE}_p(G)$. Thus every pd-chief factor H/K of G in E is cyclic of order p. Since $G/C_G(H/K)$ is isomorphic to some subgroup of $\mathrm{Aut}(H/K)$, which is cyclic of order $p-1$, and $(|G|, p-1) = 1$, we see that $G/C_G(H/K) = 1$ and H/K is central, that is, every G-chief factor in E is either of p'-order or central in G. Since G/E is p-nilpotent, we obtain that G is also p-nilpotent. □

It is easy to show that if $(|G|, p^2-1) = 1$, then G has no chief factor of order p^2 and so if $p^3 \nmid |G|$ then G is p-nilpotent. A more general result can be found in [1, Lemma 2.12]. Considering groups in which every subgroup of order p^2 has semi-Π-property, we obtain the following theorem.

Theorem 4.2. *Let G be a group and p a prime with $(|G|, p^2-1) = 1$. Assume that E is a normal subgroup of G with p-nilpotent quotient. Let P be a Sylow*

*p-subgroup of E. If every subgroup of P of order p^2 has semi-*Π*-property in G, then G is p-nilpotent.*

Proof. Assume that theorem is not true, and G is a counterexample of minimal order. We prove the theorem via the following steps:

(1) $O_{p'}(G) = 1$.

By Lemma 2.1, the hypotheses still hold on $G/O_{p'}(G)$. If $O_{p'}(G) \neq 1$, then we can assume that $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction. Hence $O_{p'}(G) = 1$.

(2) *Let N be a minimal normal subgroup of G. If $N \leq O_p(G)$, then N is cyclic of order p.*

Since $N \leq O_p(G)$, N is a p-group. If $N \nsubseteq E$, then NE/E is a chief factor of G/E. But G/E is p-nilpotent, so $N \cong NE/E$ is cyclic of order p. Assume that $N \subseteq E$. If $|N| > p^2$, then N has a proper subgroup H of order p^2 with $H \trianglelefteq G_p$, a Sylow p-subgroup of G. By the hypotheses, H has semi-Π-property in G, and so there is a subgroup T of G such that $G = TH$ and $H \cap T$ has Π-property in T. Clearly, $G = NT$. Thus $N \cap T \trianglelefteq G$ since N is abelian. If $T \neq G$, then $N \cap T \neq N$. It follows that $N \cap T = 1$ since N is minimal normal in G. Hence $|N| = |G : T| \leq |H| = p^2$, a contradiction. If $T = G$, then $H \cap T = H$ has Π-property in $T = G$. By Lemma 2.4, $N = H$ is of order p^2, which contradicts to $|N| > p^2$ and thus $|N| \leq p^2$.

If $|N| = p^2$, then $\mathrm{Aut}(N)$ is of order $(p^2 - 1)(p^2 - p)$. Since $G/C_G(N)$ is isomorphic to some subgroup of $\mathrm{Aut}(N)$, $|G/C_G(N)|$ is a divisor of $(p^2 - 1)(p^2 - p)$. But $(|G|, p^2 - 1) = 1$, so $|G/C_G(N)|$ is a p-number. It follows from [5, Lemma 1.7.11] that $|G/C_G(N)| = 1$ and hence $N \subseteq Z(G)$. It follows that all subgroups of N are normal in G. This is not true for $|N| = p^2$ and N is a minimal normal subgroup G. Therefore, $|N| \neq p^2$ and so $|N| = p$.

(3) $p = 2$.

If $p \neq 2$, then G is of odd order since $(|G|, p^2 - 1) = 1$. Thus G is solvable and so $O_p(G) \neq 1$ by (1). Then (2) implies that G has a minimal normal subgroup N of order p. By Lemma 2.1(3) and the hypotheses, every subgroup of order p in E/N has semi-Π-property in G/N. By Theorem A, $E/N \leq \mathrm{SE}_p(G/N)$. Since N is cyclic, $E \leq \mathrm{SE}_p(G)$. Similar to the proof of Theorem 4.1, we have that G is p-nilpotent, contrary to the choice of G. Thus $p = 2$ and (3) holds.

(4) *Let $N \leq E$ be a minimal normal subgroup of G, then N is of order 2 and hence is contained in $Z(G)$.*

Assume that L is a minimal normal subgroup of N. If N is not a 2-group, then, since $O_{p'}(G) = 1$ by (1), L is a nonabelian simple group. By [11, (10.1.9)] and $(|G|, p^2 - 1) = 1$, the order of a Sylow 2-subgroup of L is greater than 4. Choose H to be a subgroup of L of order 4. We claim that G is the only supplement of H in G. In fact, if H has a proper supplement T in G, then $|G : T| = 2$ or 4. If $|G : T| = 2$, then T is normal in G. Since $LT = HT = G$, $|L : L \cap T| = |G : T| = 2$. This shows that $L \cap T \trianglelefteq L$, a contradiction. If

$|G:T| = 4$, then $|L:T\cap L| = 4$. Considering the permutation of L on the right coset of $L\cap T$, we can see that $L/(L\cap T)_G$ is isomorphic to some subgroup of the symmetric group S_4 of degree 4. Since $(|G|, p^2-1) = 1$, $L/(L\cap T)_G$ is a 2-group. But L is simple, a contradiction. Thus our claim holds and therefore, H has Π-property in G. Applying Lemma 2.3, N is abelian and hence is a 2-group. By Theorem A, N is cyclic and thus (4) holds.

(5) $\mathrm{Soc}(E)\cap\Phi(G) \neq 1$.

If $\mathrm{Soc}(E)\cap\Phi(G) = 1$ then $\mathrm{Soc}(E)$ is complement in G and hence is complement in E. By (4), $\mathrm{Soc}(E) \leq Z(G)$. If $E = \mathrm{Soc}(E)$, then $E \leq Z(G)$ and hence G is p-nilpotent since G/E is. If $\mathrm{Soc}(E) < E$ and let M be a complement of $\mathrm{Soc}(E)$ in E, then $M \neq 1$. Moreover, since $\mathrm{Soc}(E) \leq Z(G)$, $M \trianglelefteq E$ and hence $M\cap\mathrm{Soc}(E) \neq 1$, a contradiction. Thus (5) holds.

(6) *Every cyclic subgroup of order 2 or 4 in E has semi-Π-property in G.*

It follows directly and Step (3) that every subgroup of order 4 has semi-Π-property in G. Let $A = \langle a\rangle$ be a subgroup of order 2. By (4) and (5), G has a minimal normal subgroup $N \leq E\cap\Phi(G)$ and $N = \langle b\rangle$ is of order 2. Obviously, $N \leq Z(G)$. If $A = N$, then it is clear that A has semi-Π-property. Assume $A \neq N$. Then $H = AN$ is of order 4. By hypotheses, H has semi-Π-property in G. Suppose $G = HT$ and $H\cap T$ has Π-property in T. If $T = G$, then H has Π-property in G. It follows from Lemma 2.5 that A has Π-property in G. If $T < G$, then $|G:T| = 2$ or 4 since $|H| = 4$. If $|G:T| = 2$, then T is maximal in G and hence $N \leq T$. Thereby, $A\cap T = 1$, otherwise $H = AN \leq T$ and $G = HT = T$. Also, $AT = ANT = G$. Thus A has semi-Π-property in G in this case. If $|G:T| = 4$, then $N \not\leq T$ and $T_1 = NT$ is maximal in G, and then T_1 is a complement of A in G and A has semi-Π-property in G. Thus (6) holds.

(7) *The final contradiction.*

By (6) and Theorem A, $E \leq \mathrm{SE}_p(G)$. It follows from G/E is p-nilpotent that G is p-nilpotent. This is the final contradiction and the theorem holds. □

Theorem 4.3. *Let G be a group and p an odd prime divisor of $|G|$. Assume that E is a normal subgroup of G with p-nilpotent quotient. Suppose that P is a Sylow p-subgroup of E and $N_G(P)$ is p-nilpotent. If every minimal subgroup of P has semi-Π-property in G, then G is p-nilpotent.*

Proof. Assume that the theorem is not true and let G be a counter example of minimal order, we prove the theorem via the following steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then the hypotheses still hold on $G/O_{p'}(G)$. Hence we can assume that $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction and then $O_{p'}(G) = 1$.

(2) $O_p(E) \neq 1$ and $O_p(E) = F(E) = F^*(E)$.

Let N be a minimal normal subgroup of G contained in E. Then, by (1), p devides $|N|$. By the hypotheses, every subgroup of order p in N has semi-Π-property in G. If all subgroups of order p in N are complemented in G, then by Lemma 2.6, NP is p-nilpotent and so is N. Since $O_{p'}(G) = 1$ by (1), N is a p-group. Assume there is a subgroup H of order p in N not complemented in G. Then G is the only supplement of H in G. Hence H must have Π-property in G. By Lemma 2.3, N is a p-group and so $O_p(E) \neq 1$.

Since $O_{p'}(E) \leq O_{p'}(G) = 1$, it is easy to see that $F(E) = O_p(E)$. If $F(E) \neq F^*(E)$, then we can choose a G-chief factor $R/F(E)$ with $R \subseteq F^*(E)$. Let $Q = O^p(R) = \langle a \in R | p \nmid |a| \rangle$ and $O = O_p(E) \cap O^p(R)$. Then $Q/O \cong R/O_p(G)$ is a chief factor of G and is characteristically simple. Choose M/O to be a minimal normal subgroup of Q/O. Then $Q/O \cong M/O \times \cdots \times M/O$. Clearly, M is not solvable, otherwise, $M \subseteq F(E)$. Let x be an element of $F^*(E)$ of order p'. Then x induces an inner automorphism on each chief factor of $F^*(E)$ and so acts trivially on all abelian chief factors. In particular, x acts trivially on all G-chief factors of $F(E)$. By [4, A,12.3], x acts trivially on $O_p(E)$. Thus all p'-elements in $F^*(E)$ act trivially on $O_p(E)$. It follows that Q and so $O^p(M)$ act trivially on $O_p(E)$. Clearly, M is not p-nilpotent, so M has a subgroup X, which is not p-nilpotent, but all of its proper subgroups are p-nilpotent. Then $X = A \rtimes B$, where A is a p-group and B is a cyclic p'-group. Assume $B = \langle x \rangle$. Then $x \in F^*(E)$ and x acts trivially on $O_p(E)$. If all elements of order p in A are contained in $O_p(E)$, then B acts trivially on A by [7, Satz IV.5.12]. This is contrary to the choice of X. Thus there must be elements of order p in $A \backslash O_p(E)$. Then $\langle x \rangle$ does not avoid Q/O. If $\langle x \rangle$ has Π-property in G, then, by Lemma 2.3, Q/O is a p-factor. This is not true and hence $\langle x \rangle$ does not have Π-property in G. On the other hand, since $|x| = p$, if $\langle x \rangle$ is not complemented in G, then $\langle x \rangle$ is contained in every supplement of it in G and it follows that $\langle x \rangle$ has Π-property in G since $\langle x \rangle$ has semi-Π-property in G by the hypotheses. Therefore, $\langle x \rangle$ is complemented in G and so is in M. Choose T to be a complement of $\langle x \rangle$ in M. Then $|M : T| = p$. By considering the action of M on the right coset of T in M, one can find that M/T_M is isomorphic to a subgroup of S_p, the symmetric group of degree p. Hence the Sylow p-subgroups of M/T_M are of order p. If T_M is not contained in $O_p(E)$, then M/T_M is a p-group since $M/M \cap O_p(E)$ is simple. It follows that $x \in O^p(M) \subseteq T$, a contradiction. Thus $T_M \subseteq O_p(E)$ and so the Sylow p-subgroups of $M/M \cap O_p(E) = M/O$ are of order p. Since $x \in M \setminus O_p(E)$ and $|x| = p$, every non trivial p-group of M/O is a conjugate of $\langle x \rangle O/O$, that is, for every subgroup of $\overline{H}$ of order p in M/O there is a subgroup H of order p in M such that $\overline{H} = HO/O$. Furthermore, by the choice of M, we see that for every subgroup $\overline{H}$ of order p in Q there is also a subgroup H of order p in Q such that $\overline{H} = HO/O$.

Now, we claim that every subgroup of order p in Q/O is complemented in G/O. Choose $\overline{H} = HO/O$ to be a subgroup of order p in Q/O, where H is a subgroup of order p. Then by above argument, H is complemented in G. Assume that T is a complement of H in G. Then T is maximal in G. Let Q_1 be a minimal supplement of O in Q. Then, by [5, Lemma 2.3.4], $Q_1 \cap O \subseteq \Phi(Q_1)$. So, Q_1 is generated by all its p'-elements. Since, by the above argument, $O \leq Z_\infty(Q)$, we have $O \leq C_Q(Q_1)$. Hence $Q_1 \trianglelefteq Q$. Thus $Q = O^p(Q) \leq Q_1$ and so $Q = Q_1$. It follows that $O = O \cap Q \subseteq \Phi(Q) \subseteq \Phi(G) \subseteq T$. Hence HO/O is complemented in G/O and T/O is a complement of it. Our claim holds.

It is easy to see that $N_{G/O}(P/O)$ is p-nilpotent. Hence, by Lemma 2.6. QP/O is p-nilpotent and so is Q. But this is not true and so (2) holds.

(3) *The final contradiction.*

By Theorem B and Step (2), we have that $E \leq$SE(E). In particular, E is supersolvable. Since $O_{p'}(G) = 1$ by (1), P, the Sylow p-subgroup of E, is normal in E and hence is normal in G. Thus $G = N_G(P)$ is p-nilpotent. The theorem holds. □

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