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INFINITELY MANY SOLUTIONS FOR A CLASS OF p-BIHARMONIC EQUATION IN $\mathbb{R}^N$

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Abstract. Using variational arguments, we prove the existence of infinitely many solutions to a class of p-biharmonic equation in $\mathbb{R}^N$. The existence of nontrivial solution is established under a new set of hypotheses on the potential $V(x)$ and the weight functions $h_1(x), h_2(x)$.
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MSC(2010): Primary: 35J30; Secondary: 35J35, 35J91.

1. Introduction and main result

In this paper, we are interested in the existence of multiple solutions for the p-biharmonic equation

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) + V(x)|u|^{p-2}u = f(x,u), & x \in \mathbb{R}^N, \\ u \in E = \mathcal{D}^{2,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V), \end{cases} \tag{1.1}$$

where $f(x,u) = h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $2p < N, 1 < p < q < m < p_* = \frac{pN}{N-2p}$ and $V(x) > 0$ is a potential function.

When Ω is a bounded domain of $\mathbb{R}^N$, the problem

$$\begin{cases} \Delta^2 u = h(x,u), & x \in \Omega \\ u = \Delta u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

has been studied extensively in recent years. Problem (1.2) arises in the study of traveling waves in suspension bridges and the study of the static deflection of an elastic plate in a fluid, see [10,16,27]. Since then, more nonlinear biharmonic equations and p-biharmonic equations have been studied, and the existence of

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solutions for nonlinear fourth order differential equations have been paid a great deal of interest, see [3, 12, 13, 17]. In those papers, the equation

$$\Delta(|\Delta u|^{p-2}\Delta u) = h(x,u), \quad x \in \Omega \tag{1.3}$$

with suitable boundary condition is studied. For $p = 2$ and $\Omega = \mathbb{R}^N$, fourth-order elliptic problem (1.3) also attracts a lot of attention, see [5,8,19,26,28,29].

However, to the author's knowledge, it seems that very few results are devoted to the elliptic equations of p-biharmonic type in unbounded domain $\mathbb{R}^N$.

The main purpose of this paper is to investigate the existence of multiple solutions of problem (1.1). Different from some known works, the equation that we considered is quasilinear, which might have degeneracy or singularity. In fact, if $p > 2$, the equation is degenerate at the points where $\Delta u = 0$; while if $1 < p < 2$, then the equation has singularity at the points where $\Delta u = 0$. Since the nonlinear term $f(x,u)$, in general, is not radially symmetric with respect to x, it is inappropriate to seek for the radial solutions. Here, we use the variational methods to study the existence of nontrivial solutions for problem (1.1). In particular, we are interested in the existence of solutions depending on the potential function $V(x)$ and the weight functions $h_1(x), h_2(x)$ in (1.1).

Many authors have also considered the existence and multiplicity of solutions for the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{m-2}u, \quad x \in \mathbb{R}^N \tag{1.4}$$

and we observe that interesting conditions on $V(x)$ have been studied. For examples, the paper due to Berestycki and Lions [4] with the potential $V(x) = b > 0$ in $\mathbb{R}^N$ was considered, also see [9, 14, 27]. In Jeanjean and Tanaka [15], Liu [18], the potential $V(x)$ has been assumed asymptotic to a positive constant, that is, there is $\alpha > 0$ such that

$$0 < V(x) \le \alpha \ \ \forall x \in \mathbb{R}^N \ \text{ and } \ \lim_{|x|\to\infty} V(x) = \alpha. \tag{1.5}$$

In Miyagaki [23], Mihăilescu and Rădulescu [22], the authors have focused attention on the case in that $V(x)$ is coercive, that is, $V(x) \to \infty$ as $|x| \to \infty$. In Barstch and Wang [2], a more weak condition than coercivity on $V(x)$ has been assumed, more precisely, it was supposed that for all $M > 0$,

$$\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) \le M\}) < \infty. \tag{1.6}$$

This assumption guarantees that the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \le s < 2^* = 2N/(N-2)$. The potential $V(x)$ vanishing at infinity, that is, $V(x) \to 0$ as $|x| \to \infty$, has also received much attention, see [1, 7, 20] and the references therein.

In this paper, we are motivated by the above papers and interested in the existence of multiple solutions for problem (1.1) by considering a new set of hypotheses on the potential $V(x)$ and the weight functions $h_1(x), h_2(x)$. Throughout this paper, we make the following hypotheses.

(H_1) The power parameters p, q, m satisfy $2p < N$ and $1 < p < q < m < p_* = \frac{pN}{N-2p}$.

(H_2) $V(x)$ is continuous and positive in $\mathbb{R}^N$.

(H_3) The functions $h_1, h_2 \in L^\infty(\mathbb{R}^N)$, $h_1(x) > 0$ a.e. in $\mathbb{R}^N$ and $h_2(x) \geq 0$ in $\mathbb{R}^N$. In addition,

$$\frac{h_1}{V},\ \frac{h_2}{V} \in L^\infty(\mathbb{R}^N) \text{ and } h_1(x)|V(x)|^{\frac{m-p_*}{p_*-p}} \to 0,\ \ h_2(x)|V(x)|^{\frac{q-p_*}{p_*-p}} \to 0 \text{ as } |x| \to \infty.$$

(H_4) The functions $h_1, h_2 \in L^\infty(\mathbb{R}^N)$, $h_1(x) > 0$ a.e. in $\mathbb{R}^N$ and $h_2(x) \geq 0$ in $\mathbb{R}^N$. In addition,

$$h_1(x) \in L^\alpha(\mathbb{R}^N),\ \ h_2(x) \in L^\beta(\mathbb{R}^N),\ \text{ where } \alpha = \frac{p_*}{p_* - m},\ \beta = \frac{p_*}{p_* - q}.$$

Remark 1.1. Assumptions $(H_3) - (H_4)$ are independent. For example, let $V(x) = 1$ and $k > N$, then the function

$$h_1(x) = \begin{cases} 1, & 0 < |x| < 1, \\ \exp(-|x|^k |\sin \pi |x||^{1/\alpha}), & |x| \geq 1 \end{cases} \tag{1.7}$$

satisfies (H_4), but $h_1(x) \not\to 0$ as $|x| \to \infty$. On the other hand, the function $h_1(x) = 1$ for $|x| \leq 1$ and $h_1(x) = |x|^{-\delta}$ for $|x| \geq 1$ and any $\delta > 0$ satisfies (H_3), but does not necessarily verify (H_4).

Remark 1.2. The authors in [14] studied the existence of infinitely many solutions of (1.4) with $p = 2, \lambda = 1, V(x) = 1$ and different assumptions on h_1 and h_2. Our assumption (H_3) is different from their assumptions.

In order to state our main result, we introduce some Sobolev spaces and norms. Let

$$X = \mathcal{D}^{2,p}(\mathbb{R}^N) = \{u \in L^{p_*}(\mathbb{R}^N) :\ \Delta u \in L^p(\mathbb{R}^N)\} \tag{1.8}$$

with the norm $\|u\|_X = \|\Delta u\|_p$ and

$$E = \left\{u \in \mathcal{D}^{2,p}(\mathbb{R}^N) :\ \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\right\} \tag{1.9}$$

with the norm

$$\|u\|_E = (\int_{\mathbb{R}^N} (|\Delta u|^p + V(x)|u|^p) dx)^{1/p}. \tag{1.10}$$

We denote by S_* the Sobolev constant, that is,

$$S_* = \inf_{u \in \mathcal{D}^{2,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\Delta u|^p dx}{(\int_{\mathbb{R}^N} |u|^{p_*} dx)^{p/p_*}} \tag{1.11}$$

and

$$S_*(\int_{\mathbb{R}^N}|u|^{p_*}dx)^{p/p_*} \le \int_{\mathbb{R}^N}|\Delta u|^p dx, \quad \forall u \in \mathcal{D}^{2,p}(\mathbb{R}^N). \tag{1.12}$$

Obviously, the embedding $X \hookrightarrow L^{p_*}(\mathbb{R}^N)$ is continuous.

It is well known that S_* is achieved by a (unique, up to a multiplicative constant, and up to dilations and translations in $\mathbb{R}^N$) positive and radially symmetric function, see [25].

Definition 1.3. A function $u \in E$ is said to be a (weak) solution of (1.1) if for any $\varphi \in E$, there holds

$$\int_{\mathbb{R}^N}(|\Delta u|^{p-2}\Delta u\Delta\varphi + V|u|^{p-2}u\varphi)dx = \int_{\mathbb{R}^N}(h_1|u|^{m-2}u + h_2|u|^{q-2}u)\varphi dx. \tag{1.13}$$

Our main result in this paper is as follows.

Theorem 1.4. *Let $(H_1)-(H_2)$ hold. In addition, suppose that either (H_3) or (H_4) is satisfied. Then problem (1.1) admits infinitely many nonnegative solutions $u_n \in E$ such that $J(u_n) \to \infty$ as $n \to \infty$.*

This paper is organized as follows. In section 2, we set up the variational framework and establish some lemmas, and then we prove theorem 1.3.

2. Proof of Theorem 1.4

In this section, we first set up the variational framework for problem (1.1), and in the position of the hypotheses in Theorem 1.4, we derive some lemmas and finally give the proof of Theorem 1.4.

Let $J(u): E \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$J(u) = \frac{1}{p}\|u\|_E^p - \frac{1}{m}\int_{\mathbb{R}^N} h_1|u|^m dx - \frac{1}{q}\int_{\mathbb{R}^N} h_2|u|^q dx, \quad \forall u \in E. \tag{2.1}$$

From the hypotheses on h_1 and h_2, we see that the functional $J \in C^1(E,\mathbb{R})$ and its Gateaux derivative is given by

$$J'(u)\varphi = \int_{\mathbb{R}^N}(|\Delta u|^{p-2}\Delta u\Delta\varphi + V|u|^{p-2}u\varphi)dx - \int_{\mathbb{R}^N}(h_1|u|^{m-2}u + h_2|u|^{q-2}u)\varphi dx, \forall\varphi \in E. \tag{2.2}$$

In order to prove the main theorem, we recall some useful concepts and results.

Definition 2.1. Let E be a real Banach space, $J \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. We say that J satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c, \quad J'(u_n) \to 0 \ \ in \ \ E^* \quad \text{as} \ \ n \to \infty \tag{2.3}$$

has a convergent subsequence in E.

Proposition 2.2. (Theorem 6.5, [24]). *Let E be an infinite dimensional real Banach space, $J \in C^1(E, \mathbb{R})$ be even and satisfy the $(PS)_c$ condition and $J(0) = 0$. If $E = Y \oplus Z$, in which Y is finite dimensional, and J satisfies*

(J_1) there exist constants $\rho, \alpha_0 > 0$ such that $J(z) \geq \alpha_0$ on $\partial B_\rho \cap Z$;

(J_2) for each finite dimensional subspace $E_0 \subset E$, there is an $R = R(E_0)$ such that $J(z) \leq 0$ on $E_0 \setminus B_R$, where $B_R = \{z \in E : \|z\|_E < R\}, \partial B_R = \{z \in E : \|z\|_E = R\}$.

Then J possesses an unbounded sequence of critical values, i.e. there exists a sequence $\{u_n\} \subset E$ such that $J'(u_n) = 0$ and $J(u_n) \to \infty$ as $n \to \infty$.

In the following, we let all hypotheses in Theorem 1.4 hold.

Lemma 2.3. *Any $(PS)_c$ sequence $\{u_n\}$ of J is bounded in E.*

Proof. Let $J(u_n) \to c, J'(u_n) \to 0$ in E^* as $n \to \infty$. Then, for large n, one sees that

$$c + 1 + \|u_n\|_E \geq J(u_n) - \frac{1}{q} J'(u_n) u_n \geq (\frac{1}{p} - \frac{1}{q}) \|u_n\|_E^p. \tag{2.4}$$

This implies that the sequence $\{u_n\}$ is bounded in E and the proof is finished. □

Lemma 2.4. *The space E is compactly embedded in $L^m(\mathbb{R}^N, h_1)$ and $L^q(\mathbb{R}^N, h_2)$.*

Proof. Let $\{u_n\}$ be a bounded sequence in E. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $v \in E$ such that as $n \to \infty$,

$$\begin{aligned} &\|u_n\|_E \leq M,\ u_n \rightharpoonup v \text{ weakly in } E,\ u_n \to v \text{ in } L^s_{loc}(\mathbb{R}^N), \\ &\qquad 1 \leq s < p_*,\ u_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N \end{aligned} \tag{2.5}$$

with some constant $M > 0$.

We begin with assuming (H_3). First it is important to observe that for each fixed $x \in \mathbb{R}^N$, the function

$$g(s) = V(x) s^{p-m} + s^{p_*-m}, \quad \forall s > 0 \tag{2.6}$$

attains its minimum $\lambda_1 |V(x)|^{\frac{p_*-m}{p_*-p}}$ at $s_0 = \frac{m-p}{p_*-m} V(x)$, where

$$\lambda_1 = \frac{p_* - p}{p_* - m} \left(\frac{m-p}{p_* - m} \right)^{\frac{p-m}{p_*-p}}. \tag{2.7}$$

Hence,

$$\lambda_1 |V(x)|^{\frac{p_*-m}{p_*-p}} \leq g(s) = V(x) s^{p-m} + s^{p_*-m}, \quad \forall x \in \mathbb{R}^N,\ \forall s > 0. \tag{2.8}$$

By (H_3), for any small $\epsilon > 0$, there exists $a > 0$ such that

$$0 \leq h_1(x) \leq \epsilon \lambda_1 [V(x)]^{\frac{p_*-m}{p_*-p}}, \quad \forall |x| \geq a \tag{2.9}$$

and so

$$(2.10)\quad 0 \le h_1(x)|s|^m \le \epsilon\lambda_1(V(x)|s|^p + |s|^{p_*}) \quad \forall |x| \ge a, \text{ and } \forall s \in \mathbb{R}.$$

Then inequality (2.10) gives

$$\int_{B_a^c} h_1(x)|u_n|^m dx \le \epsilon\lambda_1 \int_{B_a^c} (V(x)|u_n|^p + |u_n|^{p_*})dx. \tag{2.11}$$

where $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ and $B_r^c = \{x \in \mathbb{R}^N : |x| \ge r\}$ for $r > 0$.

On the other hand, it follows from the Young inequality with $\varepsilon > 0$ and (1.12) that

$$\begin{aligned} 0 &\le \int_{B_a} h_1|u_n|^m dx \\ &\le \epsilon \int_{B_a} h_1|u_n|^p dx + C_\varepsilon \int_{B_a} h_1|u_n|^{p_*} dx \\ &\le \epsilon M_1 \|u_n\|_{p,V}^p + M_2 C_\varepsilon \|u_n\|_E^{p_*}, \end{aligned} \tag{2.12}$$

where

(2.13)

$$M_1 = \max\{\lambda_1, \|\frac{h_1}{V}\|_\infty\},\ M_2 = \max\{\lambda_1, \|h_1\|_\infty\},\ \|u\|_{p,V}^p = \int_{\mathbb{R}^N} V|u|^p dx.$$

Then we get from (2.11) and (2.12) that

$$0 \le \int_{\mathbb{R}^N} h_1(x)|u_n|^m dx \le \epsilon M_1 \|u_n\|_{p,V}^p + M_2 C_\varepsilon \|u_n\|_E^{p_*}. \tag{2.14}$$

Similarly, for any $\varepsilon > 0$, there is $b > 0$ such that

$$\begin{aligned} 0 \le \int_{B_b} h_2|u_n|^q dx &\le \epsilon M_3 \|u_n\|_{p,V}^p + M_4 C_\varepsilon \|u_n\|_E^{p_*}, \\ \int_{B_b^c} h_2|u_n|^q dx &\le \epsilon \int_{B_b^c} (V(x)|u_n|^p + |u_n|^{p_*})dx \end{aligned} \tag{2.15}$$

and

$$\int_{\mathbb{R}^N} |h_2(x)||u_n|^q dx \le \epsilon M_3 \|u_n\|_{p,V}^p + M_4 C_\varepsilon \|u_n\|_E^{p_*}, \tag{2.16}$$

where

$$M_3 = \max\{1, \|\frac{h_2}{V}\|_\infty\},\quad M_4 = \max\{1, \|h_2\|_\infty\}. \tag{2.17}$$

Choosing $M_0 = \max\{M_1, M_3\}$ and $\varepsilon = \frac{1}{4pM_0}$, we obtain from (2.14) and (2.16) that

$$0 \le \int_{\mathbb{R}^N} h_1|u_n|^m dx + \int_{\mathbb{R}^N} h_2|u_n|^q dx \le \frac{1}{2p}\|u_n\|_E^p + C_0\|u_n\|_E^{p_*}, \tag{2.18}$$

with some constant $C_0 > 0$.

On the other hand, it follows from (2.5) and (2.11) that

$$0 \le \int_{B_a^c} h_1|u_n|^m dx \le 2\varepsilon M^p \ \text{ and } \ 0 \le \int_{B_a^c} h_1|v|^m dx \le 2\varepsilon M^p, \tag{2.19}$$

where M is the constant in (2.5). Since $m \in (p, p_*)$ and $h_1 \in L^\infty(\mathbb{R}^N)$, it follows from Sobolev compact embedding in bounded domain that

$$\lim_{n\to\infty} \int_{B_a} h_1(x)|u_n|^m dx = \int_{B_a} h_1(x)|v|^m dx. \tag{2.20}$$

Then, combining (2.19) with (2.20) yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h_1(x)|u_n|^m dx = \int_{\mathbb{R}^N} h_1(x)|v|^m dx. \tag{2.21}$$

Using Brezis-Lieb Lemma [6], we derive

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h_1(x)|u_n - v|^m dx = 0. \tag{2.22}$$

Similarly, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h_2(x)|u_n - v|^q dx = 0. \tag{2.23}$$

Arguing as in [30], if (H_4) is true, then for every $\varepsilon > 0$, there exists $\rho_0 > 0$ such that

$$\int_{B_\rho^c} |h_1(x)|^\alpha dx < \varepsilon, \ \text{ for } \rho > \rho_0. \tag{2.24}$$

Note that h_1 is bounded in B_ρ and $\{u_n\}$ is also bounded in $L^{p_*}(\mathbb{R}^N)$. Thus we have from Hölder's inequality that

$$\int_{\mathbb{R}^N} |h_1(x)||u_n-v|^m dx \le \|h_1\|_\infty \int_{B_\rho} |u_n-v|^m dx + \Big(\int_{B_\rho^c} |h_1(x)|^\alpha dx\Big)^{\frac{1}{\alpha}} \Big(\int_{B_\rho^c} |u_n-v|^{p_*} dx\Big)^{\frac{m}{p_*}}. \tag{2.25}$$

Using the fact $u_n \to v$ in $L^m(B_\rho)$ and (2.24), we obtain (2.22) from (2.25). Similarly we have (2.23). This completes the proof of Lemma 2.4. ☐

Lemma 2.5. *Let $\{u_n\}$ be a $(PS)_c$ sequence and satisfy (2.5). Then the following statements hold.*

- (i) *The weak limit $v \in E$ is a critical point of the functional J.*
- (ii) *$u_n \to v$ in E, that is, the functional J satisfies $(PS)_c$ condition.*

Proof. (i). From (2.5) and Lemma 2.3, one sees that as $n \to \infty$,

$$\int_{\mathbb{R}^N} |\Delta u_n|^{p-2}\Delta u_n \Delta\varphi dx \to \int_{\mathbb{R}^N} |\Delta v|^{p-2}\Delta v \Delta\varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N) \tag{2.26}$$

and

$$\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |v|^{p-2}v)\varphi dx \to 0, \quad \int_{\mathbb{R}^N}(f(x,u_n) - f(x,v))\varphi dx \to 0. \tag{2.27}$$

Then, it follows from $J'(u_n) \to 0$ in E^* and (2.22)-(2.27) that

$$0 = \lim_{n\to\infty} J'(u_n)\varphi = J'(v)\varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.28}$$

Since the set $C_0^\infty(\mathbb{R}^N)$ is dense in E, we have $J'(v)\varphi = 0, \forall \varphi \in E$. In particular, $J'(v)v = 0$. Hence, v is a critical point of J in E.

(ii). Denote $J'(u_n)(u_n - v) := P_n - R_n$, where

$$\begin{aligned} P_n &= \int_{\mathbb{R}^N}[|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - v) + V(x)|u_n|^{p-2}u_n(u_n - v)]dx \\ R_n &= \int_{\mathbb{R}^N}[h_1(x)|u_n|^{m-2}u_n + h_2(x)|u_n|^{q-2}u_n](u_n - v)dx. \end{aligned} \tag{2.29}$$

Then the fact $J'(u_n) \to 0$ in E^* implies that $J'(u_n)(u_n - v) \to 0$ as $n \to \infty$. Moreover, the fact $u_n \rightharpoonup v$ in E gives $Q_n \to 0$, where

$$Q_n := \int_{\mathbb{R}^N}[|\Delta v|^{p-2}\Delta v \Delta(u_n - v) + V(x)|v|^{p-2}v(u_n - v)]dx. \tag{2.30}$$

Furthermore, we have from (1.12), (2.5), (2.22) and (2.23) that as $n \to \infty$,

$$\int_{\mathbb{R}^N}|h_1(x)||u_n|^{m-1}|u_n - v|dx \le C_1\|h_1\|_\alpha \|u_n\|_E^m \Big(\int_{\mathbb{R}^N}|h_1(x)||u_n - v|^m dx\Big)^{1/m} \to 0 \tag{2.31}$$

with some constant $C_1 > 0$. Similarly, we can derive that as $n \to \infty$,

$$\int_{\mathbb{R}^N}|h_2(x)||u_n|^{q-1}|u_n - v|dx \le C_1\|h_2\|_\beta \|u_n\|_E^q \Big(\int_{\mathbb{R}^N}|h_2(x)||u_n - v|^q dx\Big)^{1/q} \to 0 \tag{2.32}$$

Hence, we have that for large n, $P_n - Q_n = o_n(1)$, where

$$P_n - Q_n = \int_{\mathbb{R}^N}\big[(|\Delta u_n|^{p-2}\Delta u_n - |\Delta v|^{p-2}\Delta v)\Delta(u_n - v) + V(x)(|u_n|^{p-2}u_n - |v|^{p-2}v)(u_n - v)\big]dx. \tag{2.33}$$

For any $k \in \mathbb{N}$, using the standard inequality [11] given by

$$\begin{aligned} (|x|^{p-2}x - |y|^{p-2}y)(x - y) &\ge c_p|x - y|^p, \ \ p \ge 2, \ \ \forall x, y \in \mathbb{R}^k \\ (|x|^{p-2}x - |y|^{p-2}y)(x - y) &\ge \frac{c_p|x - y|^p}{(|x| + |y|)^{2-p}}, \ \ 1 < p < 2, \ \ \forall x, y \in \mathbb{R}^k, \end{aligned} \tag{2.34}$$

we have from (2.33) and (2.34) that $\|u_n - v\|_E \to 0$ as $n \to \infty$. Thus $J(u)$ satisfies $(PS)_c$ condition on E and the proof of Lemma 2.5 is finished. $\square$

Proof of Theorem 1.4. Clearly, the functional J defined by (2.1) is even in E and $J(0)=0$. By Lemma 2.4, the functional J satisfies the $(PS)_c$ condition.

By dint of Proposition 2.1, we verify the conditions (J_1) and (J_2). For assumption (H_3), it follows from (2.18) that

$$J(u) \geq \frac{1}{2p}\|u\|_E^p - C_0\|u\|_E^{p_*}, \quad u \in E. \tag{2.35}$$

Let $\|u\|_E = \rho = (\frac{1}{4pC_0})^{1/(p_*-p)}$ and then

$$J(u) \geq \frac{1}{4p}\rho^p \equiv \alpha_0 > 0, \quad \text{with } \|u\|_E = \rho. \tag{2.36}$$

So, the condition (J_1) is satisfied.

For assumption (H_4), we have from Hölder inequality and (1.12) that

$$0 \leq \int_{\mathbb{R}^N}(h_1|u|^m + h_2|u|^q)dx \leq C_2(\|h_1\|_\alpha\|u\|_E^m + \|h_2\|_\beta\|u\|_E^q), \quad u \in E, \tag{2.37}$$

and

$$J(u) \geq \frac{1}{p}\|u\|_E^p - C_2(\|h_1\|_\alpha\|u\|_E^m + \|h_2\|_\beta\|u\|_E^q), \quad u \in E, \tag{2.38}$$

with some constant $C_2 > 0$. Similarly, (J_1) is satisfied.

We now verify condition (J_2). For any finite dimensional subspace $E_0 \subset E$, we assert that there holds $J(u_n) \to -\infty$ as $\|u_n\|_E \to \infty$, $u_n \in E_0$. Arguing by contradiction, suppose that for some sequence $\{u_n\} \subset E_0$ with $\|u_n\|_E \to \infty$, there is $\eta > 0$ such that $J(u_n) \geq -\eta, \forall n \in \mathbb{N}$. Set $v_n(x) = \frac{u_n(x)}{\|u_n\|_E}$, then $\|v_n\|_E = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n(x) \to v(x)$ a.e on $\mathbb{R}^N$. Since E_0 is finite dimensional, then $v_n \to v$ in E_0, $v \not\equiv 0$ in $\mathbb{R}^N$ and $\|v\|_E = 1$. Set $A = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Then $\text{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty}|u_n(x)| = \infty$. Hence, $A \subset Z_n = \{x \in \mathbb{R}^N : |u_n(x)| \geq 1\}$ for large $n \in \mathbb{N}$. Then, it follows from (2.1) that

$$\begin{aligned} 0 = \lim_{n\to\infty}\frac{-\eta}{\|u_n\|_E^p} &\leq \lim_{n\to\infty}\frac{J(u_n)}{\|u_n\|_E^p} \leq \lim_{n\to\infty}\Big[\frac{1}{p} - \frac{1}{m}\int_{\mathbb{R}^N} h_1|u_n|^{m-p}|v_n|^p dx\Big] \\ &\leq \frac{1}{p} - \int_{\mathbb{R}^N}\liminf_{n\to\infty}\Big(h_1(x)|u_n(x)|^{m-p}\chi_{Z_n}(x)|v_n(x)|^p\Big)dx = -\infty. \end{aligned} \tag{2.39}$$

Hence we conclude a contradiction. So, there is $R = R(E_0) > 0$ such that $J(u) < 0$ for $u \in E_0$ and $\|u\|_E \geq R$. Then the application of Proposition 2.1 proves that problem (1.1) admits infinitely many solutions $u_n \in E$ with $J(u_n) \to \infty$ as $n \to \infty$. The solutions u_n can be supposed nonnegative since $J(u_n) = J(|u_n|)$. Then, we complete the proof of Theorem 1.4. $\square$

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