Title:
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ON THE DIAMETER OF THE COMMUTING GRAPH OF THE FULL MATRIX RING OVER THE REAL NUMBERS

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Abstract. In a recent paper C. Miguel proved that the diameter of the commuting graph of the matrix ring $M_n(\mathbb{R})$ is equal to 4 if either $n = 3$ or $n \geq 5$. But the case $n = 4$ remained open, since the diameter could be 4 or 5. In this work we close the problem showing that also in this case the diameter is 4.

Keywords: Commuting graph, diameter, idempotent matrix.

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1. Introduction

For a ring $R$, the commuting graph of $R$, denoted by $\Gamma(R)$, is a simple undirected graph whose vertices are all non-central elements of $R$, and two distinct vertices $a$ and $b$ are adjacent if and only if $ab = ba$. The commuting graph was introduced in [1] and has been extensively studied in recent years by several authors [2–7, 12, 13].

In a graph $G$, a path $P$ is a sequence of distinct vertices $(v_1, \ldots, v_k)$ such that every two consecutive vertices are adjacent. The number $k - 1$ is called the length of $P$. For two vertices $u$ and $v$ in a graph $G$, the distance between $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path between $u$ and $v$, if such a path exists. Otherwise, we define $d(u, v) = \infty$. The diameter of a graph $G$ is defined

$$\text{diam}(G) = \sup\{d(u, v) : u \text{ and } v \text{ are vertices of } G\}.$$ 

A graph $G$ is called connected if there exists a path between every two distinct vertices of $G$, equivalently, $\text{diam}(G) < \infty$.

Most research has been conducted regarding the diameter of commuting graphs of certain classes of rings [3, 7–10]. Here, we deal with the full matrix rings over fields. Let $F$ be an arbitrary field. We known that $\Gamma(M_2(F))$ is never

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connected. It was proved in [4] that $\Gamma(M_n(\mathbb{F}))$ is connected if and only if every field extension of $\mathbb{F}$ of degree $n$ contains a proper intermediate field. Moreover, it was shown in [3] that if $\Gamma(M_n(\mathbb{F}))$ is connected, then $4 \leq \text{diam}(\Gamma(M_n(\mathbb{F}))) \leq 6$ and it is conjectured that $\text{diam}(\Gamma(M_n(\mathbb{F}))) \leq 5$. Let $\mathbb{Q}$ and $\mathbb{R}$ be the fields of rational and real numbers, respectively. We know from [3, 4] that $\Gamma(M_n(\mathbb{Q}))$ is disconnected for any $n \geq 2$ and $\text{diam}(\Gamma(M_n(\mathbb{F}))) = 4$ for every algebraically closed field $\mathbb{F}$ and $n \geq 3$. Quite recently, C. Miguel [11] has verified this conjecture for $\mathbb{R}$, proving the following result.

**Theorem 1.1.** Let $n \geq 3$ be any integer. Then, $\text{diam}(\Gamma(M_n(\mathbb{R}))) = 4$ for $n \neq 4$ and $4 \leq \text{diam}(\Gamma(M_4(\mathbb{R}))) \leq 5$.

Unfortunately, this result left open the question whether $\text{diam}(\Gamma(M_4(\mathbb{R})))$ is 4 or 5. In this paper we solve this open problem. Namely we will prove the following result.

**Theorem 1.2.** The diameter of $\Gamma(M_4(\mathbb{R}))$ is equal to 4.

2. **On the diameter of $\Gamma(M_n(\mathbb{R}))$**

Before we proceed, let us introduce some notation. If $a, b \in \mathbb{R}$, we define the matrix $A_{a,b}$ as

$$A_{a,b} := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$  

Now, given two matrices $X, Y \in M_n(\mathbb{R})$, we define

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

Finally, two matrices $A, B \in M_n(\mathbb{R})$ are called similar and are written as $A \sim B$ if there exists an invertible matrix $P$ such that $P^{-1}AP = B$.

The proof of Theorem 1.1 in [11] relies on the possible forms of the Jordan canonical form of a real matrix. In particular, it is proved that the distance between two matrices $A, B \in M_4(\mathbb{R})$ is at most 4 unless we are in the situation where $A$ and $B$ have no real eigenvalues and only one of them is diagonalizable over $\mathbb{C}$. In other words, the case when

$$A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}, \quad B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix},$$

The following result will provide us the main tool to prove that the distance between $A$ and $B$ is at most 4 also in the previous setting. It is true for any division ring $D$. In what follows, given a matrix $A$, $L_A$ and $R_A$ will denote the left and right multiplication by $A$, respectively.

**Proposition 2.1.** Let $A, B \in M_n(D)$ matrices such that $A^2 = A$ and $B^2 = 0$. Then, there exists a non-scalar matrix commuting with both $A$ and $B$. 
Proof. Since $A^2 = A$; i.e., $A(I - A) = (I - A)A = 0$, then one of nullity $A$ or nullity $(I - A)$ is at least $n/2$. Since $I - A$ is also idempotent and a matrix commutes with $A$ if and only if it commutes with $I - A$ we can assume that nullity $A \geq n/2$. Moreover, since $B^2 = 0$, it follows that nullity $B \geq n/2$.

Now, if $\text{Ker}L_A \cap \text{Ker}L_B \neq \{0\}$ and $\text{Ker}R_A \cap \text{Ker}R_B \neq \{0\}$ we can apply [3, Lemma 4] and the result follows. Hence, we assume that $\text{Ker}L_A \cap \text{Ker}L_B = \{0\}$, since in the case $\text{Ker}R_A \cap \text{Ker}R_B = \{0\}$ we can consider the transposes of $A$ and $B$ instead of $A$ and $B$, respectively. Note that, in these conditions, $n = 2r$ and the nullities of $A$ and $B$ are equal to $r$.

Let $B_1$ and $B_2$ be bases for $\text{Ker}L_A$ and $\text{Ker}L_B$, respectively, and consider $B = B_1 \cup B_2$ a basis for $D^n$. Since $A$ is idempotent, it follows that $D^n = \text{Ker}L_A \oplus \text{Im}L_A$.

We want to find the representation matrix of $L_A$ in the basis $B$. To do so, if $v \in B_2$, we write $v = a + a'$ with $a \in \text{Ker}L_A$ and $a' \in \text{Im}L_A$. If $a' = Aa''$ for some $a'' \in D^n$, then $Av = Aa + Aa' = 0 + A(Aa'') = Aa'' = a' = -a + v$. Since $Av = 0$ for every $v \in B_1$, we get that the representation matrix of $L_A$ in the basis $B$ is of the form
\[
\begin{pmatrix}
0 & A' \\
0 & I_r
\end{pmatrix},
\]
with $A' \in M_r(D)$.

Now, we want to find the representation matrix of $L_B$ in the basis $B$. Clearly, $Bv = 0$ for every $v \in B_2$. Let $w \in B_1$. Then, $Bw = w_1 + w_2$ with $w_1 \in \text{Ker}L_A$ and so $w_2 \in \text{Ker}L_B$. Hence, $0 = B^2w = Bw_1$ and $w_1 \in \text{Ker}L_A \cap \text{Ker}L_B = \{0\}$. Thus, the representation matrix of $L_B$ in the basis $B$ is of the form
\[
\begin{pmatrix}
0 & 0 \\
B' & 0
\end{pmatrix},
\]
with $B' \in M_r(D)$.

As a consequence of the previous work we can find a regular matrix $P$ such that:
\[
PAP^{-1} = \begin{pmatrix}
0 & A' \\
0 & I_r
\end{pmatrix}, \quad PBP^{-1} = \begin{pmatrix}
0 & 0 \\
B' & 0
\end{pmatrix}.
\]

Now, if $A'B' \neq B'A'$, then $P^{-1}(A'B' \oplus B'A')P$ is a non-scalar matrix commuting with $A$ and $B$. If $A'$ and $B'$ commute, we can find a non-scalar matrix $S \in M_r(D)$ commuting with both $A'$ and $B'$. Therefore $P^{-1}(S \oplus S)P$ commutes with both $A$ and $B$ and the proof is complete. \qed

We are now in the condition to prove the main result of the paper.

**Theorem 2.2.** The diameter of $\Gamma(M_4(\mathbb{R}))$ is four.

**Proof.** In [11] it was proved that $d(A, B) \leq 4$ for every $A, B \in M_4(\mathbb{R})$, unless
\[
A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix},
\]
for some real numbers $a, b, c, d, s, t$. Hence, we only focus on this case. Assume that
\[
A = P^{-1} \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} P \quad \text{and} \quad B = Q^{-1} \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix} Q,
\]
for some invertible matrices $P$ and $Q$. Let
\[
M = P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} P \quad \text{and} \quad N = Q^{-1} \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix} Q.
\]
It is straightforwardly checked that $M^2 = M$, $N^2 = 0$, $AM = MA$, and $BN = NB$. Furthermore, Proposition 2.1 implies that there exists a non-scalar matrix $X$ that commutes both with $M$ and $N$.

Thus, we have found a path $(A, M, X, N, B)$ of length 4 connecting $A$ and $B$ and the result follows. \hfill $\Box$

**Corollary 2.3.** For every $n \geq 3$, $\text{diam}(\Gamma(M_4(\mathbb{R}))) = 4$.

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**References**

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