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EXISTENCE OF SOLUTIONS FOR A VARIATIONAL INEQUALITY ON THE HALF-LINE

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Abstract. In this paper we study the existence of nontrivial solutions for a variational inequality on the half-line. Our approach is based on the non-smooth critical point theory for Szulkin-type functionals.

Keywords: Variational inequality, critical point, mountain pass theorem, minimization, Szulkin-type functionals.
MSC(2010): Primary: 47J20; Secondary: 49J40.

1. Introduction

Variational inequalities have applications in physics, mechanics, engineering and optimization (see [3–6] and [10]) and they arise for example in obstacle problems (see [6, 12, 14] and the references therein). We note that variational inequalities are generalizations of integral equations.

In [8], the author developed a theory of variational inequalities for demicontinuous S-contractive maps in reflexive smooth Banach spaces and studied the existence of nonzero positive weak solutions for p-Laplacian elliptic inequalities. In [9], the author introduced a new class of operators and established some existence results for general variational inequalities.

In this paper, we consider the variational inequality, denoted by (P):
Find $u \in K$ such that

$$\int_0^{+\infty} u'(t)(v'(t)-u'(t))\mathrm{d}t + \int_0^{+\infty} u(t)(v(t)-u(t))\mathrm{d}t$$
$$- \int_0^{+\infty} q(t)f(t,u(t))(v(t)-u(t))\mathrm{d}t \geq 0, \quad \forall v \in K,$$

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where $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which satisfies the following condition:
(H_f) For any constant $R > 0$, we assume $\sup\{|f(t, \frac{1}{p(t)}y)| : t \in [0, \infty), y \in [-R, R]\} < \infty$.
Here K is a closed convex set in the Sobolev space $H_0^1(0, +\infty)$ with $0 \in K$, and $p : [0, +\infty) \longrightarrow (0, +\infty)$ is continuously differentiable and bounded, $q : [0, +\infty) \longrightarrow \mathbb{R}_+$ with $\frac{q}{p} \in L^1[0, +\infty)$ and

$$M = 2\max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty.$$

In Section 3, we use the abstract theory from [13] using some motivating ideas initiated in [7]. In particular we use non-smooth critical point theory for Szulkin-type functionals to obtain nontrivial solutions for (P). In our analysis we will use a new compactness result (the embedding $H_0^1(0, +\infty) \hookrightarrow C_{l,p}[0, +\infty)$ is compact) obtained in Section 2.

The authors in [7] studied a variational inequality posed on a very special set K in $W^{1,2}(0, \infty)$, namely

$$K = \{u \in W^{1,2}(0, \infty) : u \geq 0, \ u \text{ is nonincreasing on } (0, \infty)\}.$$

Our variational inequality is posed on a very general set, namely on any convex closed subset in $W_0^{1,2}(0, \infty)$. The results in [7] do not extend to general convex closed subsets of $W^{1,2}(0, \infty)$. Moreover the authors in [7] considered nonlinear terms of the form $f(t, u) = f(u)$ whereas in our paper we consider the general form $f(t, u)$. Also the hypotheses in our paper are quite different from those in [7]; see for example (f3) in [7] and our hypothesis (H_f).

2. Preliminaries

We endow the space $H_0^1(0, +\infty)$ with its natural norm

$$\|u\| = \left(\int_0^{+\infty} u^2(t)dt + \int_0^{+\infty} u'^2(t)dt \right)^{\frac{1}{2}},$$

associated with the scalar product

$$(u, v) = \int_0^{+\infty} u(t)v(t)dt + \int_0^{+\infty} u'(t)v'(t)dt.$$

Note that if $u \in H_0^1(0, +\infty)$, then $u(0) = u(+\infty) = 0$, (see [1, Corollary 8.9]).
Let

$$C_{l,p}[0, +\infty) = \{u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} p(t)u(t) \text{ exists } \}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{t \in [0, +\infty)} p(t)|u(t)|.$$

Definition 2.1. A Banach space X is embedded continuously in a Banach space Y $(X \hookrightarrow Y)$ if
(i) $X \subseteq Y$,
(ii) the canonical injection $j : X \longrightarrow Y$ is a continuous (linear) operator.
Moreover, if the canonical injection $j : X \longrightarrow Y$ is compact, then we say that X is compactly embedded in Y.

Lemma 2.2. *$H_0^1(0,+\infty)$ embeds continuously in $C_{l,p}[0,+\infty)$.*

Proof. For $u \in H_0^1(0,+\infty)$, we have

$$\begin{aligned} |p(t)u(t)| &= |p(t)u(t) - p(0)u(0)| \\ &= \left| \int_0^t (pu)'(s)ds \right| \\ &\leq \left| \int_0^t p'(s)u(s)ds \right| + \left| \int_0^t p(s)u'(s)ds \right| \\ &\leq \left(\int_0^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u^2(s)ds \right)^{\frac{1}{2}} \\ &+ \left(\int_0^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} u'^2(s)ds \right)^{\frac{1}{2}} \\ &\leq 2\max(\|p'\|_{L^2}, \|p\|_{L^2})\|u\|. \end{aligned}$$

Hence

$$\|u\|_{\infty,p} \leq M\|u\|.$$

□

Let

$$C_l[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to+\infty} u(t) \text{ exists } \},$$

endowed with the norm $\|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|$. Note if $p(t) = 1,\ \forall t \in [0,+\infty)$ then $C_{l,p}[0,+\infty) = C_l[0,+\infty)$.

To prove that $H_0^1(0,+\infty)$ embeds compactly in $C_{l,p}[0,+\infty)$, we need the following Corduneanu compactness criterion.

Lemma 2.3. ([2]) *Let $D \subset C_l([0,+\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.

$$\forall\, J \subset [0,+\infty) \text{ compact}, \forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, t_1, t_2 \in J : \\ |t_1 - t_2| < \delta \Longrightarrow |u(t_1) - u(t_2)| \leq \varepsilon, \forall\, u \in D,$$

(b) D is equiconvergent at $+\infty$ *i.e.,*

$$\begin{gathered}\forall\,\varepsilon>0, \exists\, T=T(\varepsilon)>0 \text{ such that}\\ \forall\, t: t\geq T(\varepsilon) \Longrightarrow |u(t)-u(+\infty)|\leq\varepsilon,\ \forall\, u\in D.\end{gathered}$$

Lemma 2.4. *Let* $D\subset C_{l,p}([0,+\infty),\mathbb{R})$ *be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, *i.e.*

$$\begin{gathered}\forall\, J\subset[0,+\infty) \text{ compact}, \forall\,\varepsilon>0,\ \exists\,\delta>0,\ \forall\, t_1,t_2\in J:\\ |t_1-t_2|<\delta\Longrightarrow |p(t_1)u(t_1)-p(t_2)u(t_2)|\leq\varepsilon, \forall\, u\in D,\end{gathered}$$

(b) D is equiconvergent at $+\infty$ *i.e.,*

$$\begin{gathered}\forall\,\varepsilon>0, \exists\, T=T(\varepsilon)>0 \text{ such that}\\ \forall\, t: t\geq T(\varepsilon) \Longrightarrow |p(t)u(t)-(pu)(+\infty)|\leq\varepsilon,\ \forall\, u\in D.\end{gathered}$$

Proof. It is easy to see that $D'=\{v:\ \ v(t)=p(t)u(t),\ u\in D\}\subseteq C_l$ satisfies the conditions of Lemma 2.3. Thus there exists a sequence $(v_n)\subset D'$ and $v_0\in C_l$ such that $\lim\limits_{n\to+\infty}\|v_n-v_0\|_{C_l}=0$. Let $u_n(t)=\frac{1}{p(t)}v_n(t)$ for $n=1,2,\ldots,$ and $u_0(t)=\frac{1}{p(t)}v_0(t)$. Obviously, $(u_n)\subset D$, $u_0\in C_{l,p}$ and $\lim\limits_{n\to+\infty}\|u_n-u_0\|_{C_{l,p}}=\lim\limits_{n\to+\infty}\|v_n-v_0\|_{C_l}=0$. □

Lemma 2.5. *The embedding*

$$H_0^1(0,+\infty)\hookrightarrow C_{l,p}[0,+\infty)$$

is compact.

Proof. Let $D\subset H_0^1(0,+\infty)$ be a bounded set. Then it is bounded in $C_{l,p}[0,+\infty)$ by Lemma 2.2. Let $R>0$ be such that for all $u\in D$, $\|u\|\leq R$. We will apply Lemma 2.4

(a) D is equicontinuous on every compact interval of $[0,+\infty)$. Let $u\in D$ and $t_1,t_2\in J\subset[0,+\infty)$ where J is a compact sub-interval. By using the

Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|p(t_1)u(t_1) - p(t_2)u(t_2)| &= \left| \int_{t_2}^{t_1} (pu)'(s)ds \right| \\
&= \left| \int_{t_2}^{t_1} p'(s)u(s) + u'(s)p(s)ds \right| \\
&\leq \left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u^2(s)ds \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \left(\int_{t_2}^{t_1} u'^2(s)ds \right)^{\frac{1}{2}} \\
&\leq 2\max \left[\left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\leq 2R\max \left[\left(\int_{t_2}^{t_1} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t_2}^{t_1} p^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}$$

as $|t_1 - t_2| \to 0$.

(b) D is equiconvergent at $+\infty$. For $t \in [0, +\infty)$ and $u \in D$, using the fact that $(pu)(+\infty) = 0$ (note $u(\infty) = 0$ and p is bounded) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|(pu)(t) - (pu)(+\infty)| &= \left| \int_{t}^{+\infty} (pu)'(s)ds \right| \\
&= \left| \int_{t}^{+\infty} p'(s)u(s) + u'(s)p(s)ds \right| \\
&\leq 2\max \left[\left(\int_{t}^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \right] \|u\| \\
&\leq 2R\max \left[\left(\int_{t}^{+\infty} p'^2(s)ds \right)^{\frac{1}{2}}, \left(\int_{t}^{+\infty} p^2(s)ds \right)^{\frac{1}{2}} \right] \longrightarrow 0,
\end{aligned}$$

as $t \to +\infty$. □

3. **Szulkin-type functionals**

Let X be a real Banach space, X^* its dual and let $E \in C^1(X, \mathbb{R})$ (the space of continuously differentiable functions from X to $\mathbb{R}$). Also let $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper (i.e., $\psi \neq +\infty$), convex, lower semicontinuous functional. We say then that, $I = E + \psi$ is a Szulkin-type functional, (see [13]).

Definition 3.1. A functional $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called lower semicontinuous at a point u_0 if for every sequence $\{u_n\} \subset X$ with $u_n \to u_0$, we have $\psi(u_0) \leq \liminf_{n \to +\infty} \psi(u_n)$.

Definition 3.2. An element $u \in X$ is called a critical point of $I = E + \psi$ if

$$E'(u)(v - u) + \psi(v) - \psi(u) \geq 0 \text{ for all } v \in X, \tag{3.1}$$

which is equivalent to

$$0 \in E'(u) + \partial\psi(u) \text{ in } X^*;$$

here $\partial\psi(u)$ is the subdifferential of the convex functional ψ at $u \in X$ (see [11]).

Definition 3.3. A functional $I : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called coercive if $\lim_{\|u\|_X \to +\infty} I(u) = +\infty$.

Definition 3.4. The functional $I = E + \psi$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(PSZ)_c$ if every sequence $\{u_n\} \subset X$ such that $\lim_{n \to \infty} I(u_n) = c$ and

$$\langle E'(u_n), v - u_n \rangle_X + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\| \text{ for all } v \in \text{X},$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Theorem 3.5. ([13]) *Let X be a Banach space, $I = E + \psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a Szulkin-type functional and is bounded from below. If I satisfies the $(PSZ)_c$-condition for*

$$c = \inf_{u \in X} I(u),$$

then c is a critical value.

Szulkin proved the following version of the Mountain Pass theorem.

Theorem 3.6. ([13]) *Let X be a Banach space, $I = E + \psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a Szulkin-type functional and we assume that*
(i) $I(u) \geq \alpha$ for all $\|u\| = \rho$ with $\alpha, \rho > 0$, and $I(0) = 0$;
(ii) there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.
If I satisfies the $(PSZ)_c$-condition for

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \alpha$.

4. **Main results**

We give now the main results of this paper. We denote by F the primitive of f with respect to its second variable, i.e., $F(t,x)=\int_0^x f(t,s)ds$.

Theorem 4.1. *Let f satisfy (H_f) and the following condition:*

(f_1) there exists positive functions $\beta_1, \beta_2 \in L^\infty(0,+\infty)$ with $\beta_1^* = \sup\limits_{t\in[0,+\infty)} \beta_1(t)$ < 1 and $\beta_2^* = \sup\limits_{t\in[0,+\infty)} \beta_2(t) < +\infty$ such that

$$|f(t,x)| \leq \frac{\beta_1(t)}{q(t)}|x| + \beta_2(t), \text{ for all } t\in[0,+\infty) \text{ and all } x\in\mathbb{R}.$$

Then, problem (P) has at least one solution $u \in K$.

Theorem 4.2. *Let f satisfy (H_f) and the following conditions:*

(h_1) there exist positive functions r_1, r_2, with $r_1 q, r_2 q \in L^1(\mathbb{R}^+,\mathbb{R}^+)$, and $\nu > 2$ such that

(1) $F(t,x) \geq r_1(t)|x|^\nu - r_2(t)$ for all $t\in[0,+\infty)$, $\forall x \in \mathbb{R}\setminus\{0\}$.

(2) $\nu F(t,x) \leq x f(t,x)$, for all $t\in[0,+\infty)$, $\forall x\in\mathbb{R}$.

(h_2) There exists a function $\gamma \in L^\infty(0,+\infty)$ with $\gamma^* = \sup\limits_{t\in[0,+\infty)} |(p^2\gamma)(t)| <$ $\frac{1}{2}$ such that

$$\limsup_{|x|\to 0} \frac{F(t,\frac{1}{p(t)}x)}{\frac{1}{q(t)}|x|^2} \leq \gamma(t), \text{ uniformly with respect to } t\in[0,+\infty).$$

Then, problem (P) has at least one nontrivial solution $u \in K$.

We define the functional $E : H_0^1(0,+\infty) \longrightarrow \mathbb{R}$ by

$$E(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(t)F(t,u(t))\mathrm{d}t.$$

Since $f : [0,+\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous, using the Lebesgue dominated convergence theorem and the compact embedding of $H_0^1(0,+\infty)$ in $C_{l,p}[0,+\infty)$, (Lemma 2.5) and (H_f) we have that $E \in C^1(H_0^1(0,+\infty),\mathbb{R})$.
Define the indicator functional of the set K by

$$\psi_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K. \end{cases}$$

We remark that the functional ψ_K is convex, proper, and lower semicontinuous. Then, $I = E + \psi_K$ is a Szulkin-type functional.

Proposition 4.3. *Every critical point $u \in H_0^1(0,+\infty)$ of $I = E + \psi_K$ is a solution of (P).*

Proof. Since $u \in H_0^1(0,+\infty)$ is a critical point of $I = E + \psi_K$, then

$$E'(u)(v-u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in H_0^1(0,+\infty).$$

We claim u belongs to K. If not, then $\psi_K(u) = +\infty$ and taking then, $v = 0 \in K$ in the above inequality, we obtain a contradiction. Thus $u \in K$. Fix $v \in K$. Since

$$\begin{aligned} E'(u)(v-u) &= \int_0^{+\infty} u'(t)(v'(t) - u'(t))dt + \int_0^{+\infty} u(t)(v(t) - u(t))dt \\ &- \int_0^{+\infty} q(t)f(t,u(t))(v(t) - u(t))\mathrm{d}t, \end{aligned}$$

then u is a solution of (P). □

5. **Proof of Theorem 4.1**

Assume the conditions of Theorem 4.1 are satisfied. We prove the existence of a solution for problem (P) using Theorem 3.5.

Proposition 5.1. *If the function f satisfies the hypothesis (f_1), then $I = E + \psi_K$ is coercive and bounded from below in $H_0^1(0,+\infty)$.*

Proof. We have

$$I(u) = E(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(t)F(t,u(t))\mathrm{d}t$$

for every $u \in K$. From hypothesis (f_1), we have

$$|F(t,x)| \leq \frac{1}{2}\frac{\beta_1(t)}{q(t)}|x|^2 + \beta_2(t)|x|.$$

Using the continuous embedding of $H_0^1(0,+\infty)$ in $L^2[0,+\infty)$ with constant of embedding $N = 1$ (see [1]) and Lemma 2.2, we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty}\left[\frac{1}{2}\beta_1(t)u^2(t) + q(t)\beta_2(t)|u(t)|\right]dt \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\sup_{t\in[0,+\infty)}\beta_1(t)\int_0^{+\infty}|u(t)|^2 dt \\ &- \sup_{t\in[0,+\infty)}\beta_2(t)\int_0^{+\infty}\frac{q(t)}{p(t)}p(t)|u(t)|dt \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}\|u\|^2 - \frac{\beta_1^*}{2}\|u\|_{L^2}^2 - \beta_2^*\|u\|_{\infty,p}\|\frac{q}{p}\|_{L^1} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\beta_1^*}{2}\|u\|^2 - \beta_2^* M\|u\|\|\frac{q}{p}\|_{L^1} \\ &= \frac{1}{2}(1-\beta_1^*)\|u\|^2 - \beta_2^* M\|\frac{q}{p}\|_{L^1}\|u\|. \end{aligned}$$

Since $\beta_1^* < 1$, this implies that the functional $I = E + \psi_K$ is coercive. We claim it is bounded from below on $H_0^1(0,+\infty)$. If this is not true, there exists a sequence $\{u_n\}$ in $H_0^1(0,+\infty)$ such that $\|u_n\| \to +\infty$ and $I(u_n) \to -\infty$, which is a contradiction with the coerciveness of I. □

Proposition 5.2. *If the function f satisfies (H_f), then $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ be fixed. Let $\{u_n\}$ be a sequence in $H_0^1(0,+\infty)$ such that

$$I(u_n) = E(u_n) + \psi_K(u_n) \to c \tag{5.1}$$

and

$$E'(u_n)(v-u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n\|v-u_n\|, \tag{5.2}$$

where $\{\varepsilon_n\}$ a sequence in $[0,\infty)$ with $\varepsilon_n \to 0$. By (5.1), we obtain that the sequence $\{u_n\}$ is in K. From Proposition 5.1, since I is coercive on $H_0^1(0,+\infty)$, the sequence $\{u_n\}$ is bounded in K. Since the sequence $\{u_n\}$ is bounded in $H_0^1(0,+\infty)$, there exists a subsequence still denoted by $\{u_n\}$ which converges weakly in $H_0^1(0,+\infty)$. Then there exists $u \in H_0^1(0,+\infty)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(0,+\infty), \tag{5.3}$$

$$u_n \to u \quad \text{in} \quad C_{l,p}(0,+\infty). \tag{5.4}$$

Since K is weakly closed, $u \in K$. Setting $v = u$ in (5.2), we obtain

$$\begin{aligned} \int_0^{+\infty} u_n'(t)(u'(t)-u_n'(t))dt + \int_0^{+\infty} u_n(t)(u(t)-u_n(t))dt \\ + \int_0^{+\infty} q(t)f(t,u_n(t))(u_n(t)-u(t))dt \\ \geq -\varepsilon_n\|u-u_n\|. \end{aligned}$$

Therefore, for large $n \in \mathbb{N}$, we have

$$\begin{aligned}\|u-u_n\|^2 &\leq \int_0^{+\infty} u'(t)(u'(t)-u_n'(t))dt + \int_0^{+\infty} u(t)(u(t)-u_n(t))dt \\ &\quad + \int_0^{+\infty} q(t)f(t,u_n(t))(u_n(t)-u(t))\mathrm{d}t + \varepsilon_n\|u-u_n\| \\ &\leq (u,u-u_n)_{H_0^1} + \|u-u_n\|_{\infty,p}\int_0^{+\infty}\frac{q(t)}{p(t)}f(t,\frac{1}{p(t)}p(t)u_n(t))dt \\ &\quad +\varepsilon_n\|u-u_n\|.\end{aligned}$$

Since $\{u_n\}$ is bounded in $H_0^1(0,+\infty)$, then it is bounded in $C_{l,p}[0,+\infty)$. From (H_f) we obtain that

$$\|u-u_n\|^2 \leq (u,u-u_n)_{H^1_{0,p}}+\|u-u_n\|_{\infty,p}\|\frac{q}{p}\|_{L^1}\sup_{t\in[0,\infty),\, y\in[-R_0,R_0]}|f(t,\frac{1}{p(t)}y)|+\varepsilon_n\|u-u_n\|,$$

where $R_0=\|u\|_{\infty,p}+1$. From (5.3) we have

$$\lim_n (u,u-u_n)_{H_0^1}=0.$$

From (5.4), the second term in the last inequality also converges to 0. Since $\varepsilon_n\to 0^+$, $\{u_n\}$ converges strongly to u in $H_0^1(0,+\infty)$. This completes the proof. □

From Proposition 5.2, the functional I satisfies the $(PSZ)_c$ condition, and by Proposition 5.1, the functional I is bounded from below. Therefore (Theorem 3.5), the number

$$c_1=\inf_{u\in H_0^1(0,+\infty)} I(u)$$

is a critical value of I. Proposition 4.3 concludes that the critical point $u_1\in H_0^1(0,+\infty)$ which corresponds to c_1, is actually an element of K and a solution of problem (P).

Example 5.3. Consider the function f defined by

$$f(t,x)=\frac{1}{2}e^{-t}x+\sin t,$$

and $q(t)=e^{-2t}$, $p(t)=e^{-t}$ (note $\frac{q}{p}\in L^1$). Let $\beta_1(t)=\frac{1}{2}e^{-3t}$, $\beta_2(t)=|\sin t|$ (note $\beta_1^*<1, \beta_2^*=1<+\infty$). From Theorem 4.1, problem (P) has at least one solution $u\in K$.

6. **Proof of Theorem 4.2**

Assume the conditions of Theorem 4.2 are satisfied. Now we prove the existence of a nontrivial solution for problem (P) using the Mountain Pass theorem of Szulkin type (see Theorem 3.6).

Proposition 6.1. *If the function f satisfies (H_f) and (h_1), then the functional $I = E + \psi_K$ satisfies $(PSZ)_c$ for every $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ be a fixed number. Let $\{u_n\}$ be a sequence in $H_0^1(0, +\infty)$ such that

$$I(u_n) = E(u_n) + \psi_K(u_n) \to c \tag{6.1}$$

and

$$E'(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n \|v - u_n\|, \tag{6.2}$$

where $\{\varepsilon_n\}$ is a sequence in $[0, \infty)$ with $\varepsilon_n \to 0$. From (6.1), we obtain that the sequence $\{u_n\}$ belongs to K. We put $v = 2u_n$ in (6.2), and we obtain

$$E'(u_n)(u_n) \geq -\varepsilon_n \|u_n\|.$$

Thus

$$\|u_n\|^2 - \int_0^{+\infty} q(t) f(t, u_n(t)) u_n(t) \mathrm{d}t \geq -\varepsilon_n \|u_n\|. \tag{6.3}$$

From (6.1) for large $n \in \mathbb{N}$, we obtain

$$c + 1 \geq \frac{1}{2}\|u_n\|^2 - \int_0^{+\infty} q(t) F(t, u_n(t)) \mathrm{d}t. \tag{6.4}$$

Multiplying (6.3) by ν^{-1} and adding this to (6.4) (note $\varepsilon_n \to 0$) and using $(h_1)(2)$, for large $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} c + 1 + \frac{1}{\nu}\|u_n\| &\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \int_0^{+\infty} q(t)\left(F(t, u_n(t)) - \frac{1}{\nu} f(t, u_n(t)) u_n(t)\right) dt \\ &= \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \frac{1}{\nu}\int_0^{+\infty} q(t)(\nu F(t, u_n(t)) - f(t, u_n(t)) u_n(t)) dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2. \end{aligned}$$

Since $\nu > 2$, we deduce that the sequence $\{u_n\}$ is bounded in K. Then there exits a subsequence which converges weakly in $H_0^1(0, +\infty)$. There exists $u \in H_0^1(0, +\infty)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(0, +\infty), \tag{6.5}$$

$$u_n \to u \quad \text{in} \quad C_{l,p}[0, +\infty). \tag{6.6}$$

Since K is weakly closed, $u \in K$. Put $v = u$ in (6.2), and we obtain

$$\begin{aligned}\int_0^{+\infty} p(t)u_n'(t)(u'(x) - u_n'(t))dt + \int_0^{+\infty} u(x)(u(t) - u_n(t))dt \\ + \int_0^{+\infty} q(t)f(t, u_n(t))(u_n(t) - u(t))\mathrm{d}t \\ \geq -\varepsilon_n\|u - u_n\|.\end{aligned}$$

Then, for large $n \in \mathbb{N}$, we have

$$\begin{aligned}\|u - u_n\|^2 \leq & \int_0^{+\infty} u'(x)(u'(t) - u_n'(t))dt + \int_0^{+\infty} u(x)(u(t) - u_n(t))dt \\ & + \int_0^{+\infty} q(t)f(t, u_n(t))(u_n(t) - u(t))\mathrm{d}t + \varepsilon_n\|u - u_n\| \\ \leq & (u, u - u_n)_{H_0^1} + \|u - u_n\|_{\infty,p}\int_0^{+\infty}\frac{q(t)}{p(t)}f(t, \frac{1}{p(t)}p(t)u(t))dt + \varepsilon_n\|u - u_n\|.\end{aligned}$$

Since $\{u_n\}$ is bounded in $H_0^1(0, +\infty)$, then it is bounded in $C_{l,p}[0, +\infty)$. From (H_f) we obtain that

$$\|u - u_n\|^2 \leq (u, u - u_n)_{H_0^1} + \|u - u_n\|_{\infty,p}\|\frac{q}{p}\|_{L^1} \sup_{t\in[0,\infty),\, y\in[-R,R]} |f(t, \frac{1}{p(t)}y)| + \varepsilon_n\|u - u_n\|$$

where $R = \|u\|_{\infty,p} + 1$. From (6.5), we have

$$\lim_n (u, u - u_n)_{H_{0,p}^1} = 0.$$

From (6.6), the second term in the last inequality also tends to 0. Since $\varepsilon_n \to 0^+$, $\{u_n\}$ converges strongly to u in $H_0^1(0, +\infty)$. This completes the proof. □

Proposition 6.2. *If the function f satisfies (h_1) and (h_2), then the following assertions are true:*
(i) there exist constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$;
(ii) there exists an $e \in H_0^1(0, +\infty)$ with $\|e\| > \rho$ and $I(e) \leq 0$.

Proof. (i) From condition (h_2), there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$|x| \leq \delta \Longrightarrow |F(t, \frac{1}{p(t)}x)| \leq (\gamma(t) - \varepsilon)\frac{1}{q(t)}|x|^2.$$

Therefore, by using the continuous embeddings of $H_0^1(0, +\infty)$ in $L^2[0, +\infty)$ and $H_0^1(0, +\infty)$ in $C_{l,p}[0, +\infty)$ with $\|u\|_{L^2} \leq \|u\|$, and $\|u\|_{\infty,p} \leq M\|u\|$, we have for $\|u\| = \rho$ small enough and $\alpha = (\frac{1}{2} - (\gamma^* - \varepsilon \sup_{t\in[0,+\infty)} |p(t)|^2))\rho^2 > 0$, that

$\|u\|_{\infty,p} \leq M\rho \leq \delta$ and so we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(t)F(t,u(t))\mathrm{d}t \\ &= \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(t)F(t,\frac{1}{p(t)}p(t)u(t))\mathrm{d}t \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} (\gamma(t)-\varepsilon)|p(t)u(t)|^2\mathrm{d}t \\ &\geq \frac{1}{2}\|u\|^2 - \sup_{t\in[0,+\infty)}\big(|p(t)|^2(\gamma(t)-\varepsilon)\big)\int_0^{+\infty}|u(t)|^2\mathrm{d}t \\ &\geq \frac{1}{2}\|u\|^2 - (\gamma^* - \varepsilon\sup_{t\in[0,+\infty)}|p(t)|^2)\|u\|^2_{L^2} \\ &\geq \frac{1}{2}\|u\|^2 - (\gamma^* - \varepsilon\sup_{t\in[0,+\infty)}|p(t)|^2)\|u\|^2 \\ &= (\frac{1}{2} - (\gamma^* - \varepsilon\sup_{t\in[0,+\infty)}|p(t)|^2))\|u\|^2. \end{aligned}$$

Then assertion (i) holds.

(ii) Fix $u_0 \in K\backslash\{0\}$, and let $u = su_0$ $(s > 0)$. From condition $(h_1)(1)$, we have

$$\begin{aligned} I(su_0) &= \frac{1}{2}s^2\|u_0\|^2 - \int_0^{+\infty} q(t)F(t,su_0(t))\mathrm{d}t \\ &\leq \frac{1}{2}s^2\|u_0\|^2 - s^\nu\int_0^{+\infty} q(t)r_1(t)|u_0|^\nu - \int_0^{+\infty} q(t)r_2(t)\mathrm{d}t. \end{aligned}$$

Since $\nu > 2$, we obtain that $I(su_0) \to -\infty$ as $s \to +\infty$. Thus, it is possible to take s so large such that for $e = su_0$, we have $\|e\| > \rho$ and $I(e) \leq 0$. The proof is complete. □

From Proposition 6.1, the functional I satisfies the $(PSZ)_c$-condition $c \in \mathbb{R}$, and $I(0) = 0$. From Proposition 6.2, it follows that there exist constants $\alpha, \rho > 0$ and $e \in H^1_0(0,+\infty)$ such that I satisfies the conditions of Theorem 3.6 and therefore,

$$c_2 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

is a critical value of I with $c_2 \geq \alpha > 0$, where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}.$$

We remark that the critical point $u_2 \in H^1_0(0,+\infty)$ associated to the critical value c_2 cannot be trivial because $I(u_2) = c_2 > 0 = I(0)$. From Proposition 4.3, we obtain that u_2 is an element of K and then a solution of (P).

Example 6.3. Consider the function f defined by

$$f(t,x) = e^{-3t}x|x|,$$

and $q(t) = e^{-2t}$, $p(t) = e^{-t}$ (note $\frac{q}{p} \in L^1$). Let $r_1(t) = \frac{1}{3}e^{-t}$ and $\nu = 3$ (note $r_1 q \in L^1$). From Theorem 4.2, problem (P) has at least one nontrivial solution $u \in K$.

Remark 6.4. It is possible to replace (H_f) with: For any constant $R > 0$ there exists a nonnegative function ψ_R with $\frac{q}{p}\psi_R \in L^1[0,\infty)$ and $\sup\{|f(t, \frac{1}{p(t)}y)| : y \in [-R,R]\} \leq \psi_R(t)$ for a.e. $t \geq 0$, so with obvious adjustments we see that the results in this paper can be extended.

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