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SOME ITERATIVE METHOD FOR FINDING A COMMON ZERO OF A FINITE FAMILY OF ACCRETIVE OPERATORS IN BANACH SPACES

K. SITTHITHAKERNGKIET, P. SUNTHRAYUTH AND P. KUMAM*

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Abstract. The purpose of this paper is to introduce a new mapping for a finite family of accretive operators and introduce an iterative algorithm for finding a common zero of a finite family of accretive operators in a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function invariant on $[0, \infty)$. Furthermore, we prove the strong convergence under some certain conditions. The results obtained in this paper improve and extend the corresponding ones announced by many others.
Keywords: Iterative method, accretive operator, strong convergence, common zero, Uniformly Gâteaux differentiable norm, Gauge function.
MSC(2010): Primary: 47H05; Secondary: 47H06, 47H09, 47H10.

1. Introduction

Let X be a real Banach space, and X^* be its dual space. The *duality mapping* $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^2, \|f^*\| = \|x\|\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. It is well known that, (i) if X is a Hilbert space then $J = I$ where I is the identity mapping; (ii) if X is smooth then J is single-valued which is denoted by j (see [35]).

Let C be a nonempty closed convex subset of X. Recall that a self-mapping $f : C \to C$ is said to be a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, $\forall x, y \in C$.

A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote $Fix(T)$ by the set of fixed points of T, that is,

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$Fix(T) = \{x \in C : Tx = x\}$. Let $A : D(A) \subset X \to 2^X$ be a multi-valued mapping. We denote $D(A)$ by the domain of A, $R(A)$ by the range of A and $N(A)$ by the set of zeros of A (i.e., $N(A) = \{x \in D(A) : 0 \in Ax\} = A^{-1}0$). An operator A is called *accretive* if for all $x, y \in D(A)$, there exists $j(x+y) \in J(x+y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0 \text{ for all } u \in Ax \text{ and } v \in Ay.$$

Note that the accretive mapping in a Hilbert space is called *monotone*. If A is accretive, then we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ for each $r > 0$ by $J_r := (I + rA)^{-1}$, which is called the *resolvent* of A. An operator A is said to be *m-accretive* if it is accretive and $R(I + rA)$ is X for all $r > 0$ and A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I+rA)$ for all $r > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A.

Accretive operator theory has been studied widely in nonlinear analysis. Note that most of the accretive operator theory is connected with the theory of differential equations. It is well known that many physically significant problems can be modeled by the initial value problems of the form

$$\begin{cases} x'(t) + Ax(t) = 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space (see [40]). Typical examples where such evolution equations occur can be found in the heat, wave, or Schrödinger equations. Especially, one of the fundamental results in the theory of accretive operators, which is due to Browder [7], states that if A is locally Lipschitzian and accretive, then A is m-accretive. This result was subsequently generalized by Martin [25] to the continuous accretive operators. If $x(t)$ is independent of t, then (1.1) reduces to $Au = 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, considerable research effects have been devoted, especially within the past 20 years or so, to iterative methods for approximating these equilibrium points.

In recent years, some iterative methods have been developed for finding zeros of accretive operators and related fixed points problems, see [9–11, 16–19, 26, 31–33, 37, 39] and the references therein.

In 1974, Bruck [8] introduced an iteration process and proved the convergence of the process to a zero of a maximal monotone operator in the setting of Hilbert spaces. In 1979, Reich [28] extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. Inspired by the proximal point algorithm of Rockafellar [30] and the iterative methods of Halpern [15], in 2003, Benavides et al. [13] studied the following Halpern type iteration process to find a zero of an m-accretive operator A in a uniformly smooth Banach space with a weakly continuous duality mapping J_φ with gauge function φ in virtue of the resolvent J_r of A:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n, \ \forall n \geq 1. \tag{1.2}$$

Xu [38] and Marino and Xu [24] established the convergence theorems of the iteration process (1.2) in a uniformly smooth Banach space. Song et al. [33] extended the results of Xu [38] and Marino and Xu [24] to a reflexive Banach space with a weakly continuous duality mapping J_φ.

In 2007, Aoyama et al. [3] studied the following iterative process in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm for the resolvents J_{r_n} of A such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I+rA)$:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n, \ \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$. They proved that $\{x_n\}$ generated by (1.3) converges strongly to a zero of A under certain appropriate conditions on the parameters $\{\alpha_n\}$ and $\{r_n\}$.

Recently, Hu and Liu [17] studied strong convergence theorem for a common zero of a family of accretive operators in a real strictly convex Banach space. To be more precise, they proved the following result.

Theorem HL. Let C be a nonempty closed convex subset of a real strictly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let $A_i : C \to X$ $(i = 1, 2, \ldots, N)$ be a finite family of accretive operators with $\bigcap_{i=1}^N N(A_i) \neq \emptyset$, satisfying the range condition:

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i) \text{ for } i = 1, 2, \ldots, N.$$

Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ and $\{r_n\}$ be a sequence in $\mathbb{R}^+$, satisfying the following conditions:

- (C1) $\alpha_n + \beta_n + \gamma_n = 1$;
- (C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (C4) $\lim_{n\to\infty} r_n = r \in \mathbb{R}^+$.

For any $u, x_0 \in C$, the sequence $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{r_n} x_n, \ \forall n \geq 0, \tag{1.4}$$

where $S_{r_n} := a_0 I + a_1 J^1_{r_n} + a_2 J^2_{r_n} + \ldots + a_N J^N_{r_n}$, with $J_{r_n} := (I + r_n A_i)^{-1}$ for $0 < a_i < 1$ for $i = 0, 1, \ldots, N$ and $\sum_{i=1}^N a_i = 1$. Then $\{x_n\}$ converges strongly to a common solution of the equations $A_i x = 0$ for $i = 0, 1, \ldots, N$.

On the other hand, let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. In 1999, Atsushiba and Takahashi [4] (see also

[21]) defined the mapping $W_n : C \to C$ as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})I, \end{aligned}$$

where $\{\lambda_{n,i}\}_{i=1}^N \subset [0,1]$. In 2000, Takahashi and Shimoji [36] proved that if X is a strictly convex Banach space, then $Fix(W) = \bigcap_{i=1}^N Fix(T_i)$, where $\{\lambda_{n,i}\}_{i=1}^N \subset (0,1)$.

Very recently, Kangtunyakarn and Suantai [20] introduced a mapping for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\lambda_{n,i}\}_{i=1}^N \subseteq [0,1]$, the mapping $K_n : C \to C$ is defined as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})U_1, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2}, \\ K_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}. \end{aligned}$$

This mapping is called the *K-mapping* generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$.

The present work is motivated and inspired by the idea of Kangtunyakarn and Suantai [20] and Hu and Liu [17]. First, we introduce a new mapping as follows: Let C be a nonempty closed convex subset of a real Banach space. Let $A_i : C -- > X(i = 1, 2, ..., N)$ be a finite family of accretive operators and let $\{\lambda_{n,i}\}_{i=1}^N$ in $[0,1]$ and $\{r_{n,i}\}_{i=1}^N$ be a sequence in $(0,\infty)$. We define the mapping $W_n : C \to C$ as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}J_{r_n}^1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}J_{r_n}^2U_{n,1} + (1-\lambda_{n,2})U_{n,1}, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}J_{r_n}^{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2}, \\ W_n &= U_{n,N} = \lambda_{n,N}J_{r_n}^NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}, \end{aligned} \tag{1.5}$$

where $J_{r_n}^i := (I + r_nA_i)^{-1}$ for all $i = 1, 2, \dots, N$.

Second, we introduce an iterative algorithm for finding a common zero of a finite family of accretive operators $A_i : C \to X$ $(i = 1, 2, \dots, N)$ as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n x_n, \ \forall n \ge 1, \tag{1.6}$$

where W_n is defined by (1.5). Furthermore, we obtain the strong convergence theorem under some certain conditions of the purposed iterative algorithm in a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ, where φ is a gauge function invariant on $[0,\infty)$.

2. Preliminaries

Let $U := \{x \in X : \|x\| = 1\}$ denote the unit sphere of a Banach space X. The space X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. The space X is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit is attained uniformly for $x \in U$. It is well known that if the norm of X is a *uniformly Gâteaux differentiable*, then the duality mapping is single valued and norm-weak* uniformly continuous on each bounded subset of X (see p. 111 of [35]). The space X is said to be *strictly convex* if for $a_i \in (0,1)$ $(i = 1,2,\dots,N)$ such that $\sum_{i=1}^N a_i = 1$, then $\|a_1x_1 + a_2x_2 + \cdots + a_Nx_N\| < 1$ for $x_i \in U$ $(i = 1,2,\dots,N)$ and $x_i \neq x_j$ for some $i \neq j$. In a strictly convex Banach space X, if

$$\|x_1\| = \|x_2\| = \cdots = \|x_N\| = \|a_1x_1 + a_2x_2 + \cdots + a_Nx_N\|$$

for all $x_i \in X$, $a_i \in (0,1)$ $(i = 1,2,\dots,N)$ and $\sum_{i=1}^N a_i = 1$, then $x_1 = x_2 = \cdots = x_N$ (see [35]). The space X is said to be *uniformly convex* if, for each $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\dfrac{\|x+y\|}{2} \leq 1 - \delta$. It is known that a uniformly convex Banach space is a reflexive and strictly convex Banach spaces (see [35]). Let C be a nonempty closed convex subset of X. We call that C has the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a *gauge function*. The duality mapping $J_\varphi : X \to 2^{X^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|),\ \|f^*\| = \varphi(\|x\|),\ \forall x \in X\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality paring. In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J that referred to as the normalized duality mapping. In this case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called *generalized duality mapping*. It follows from the definition that $J_\varphi(x) = \dfrac{\varphi(\|x\|)}{\|x\|}J(x)$ for each $x \neq 0$, and $J_q(x) = \|x\|^{q-2}J(x)$, $q > 1$ (see [6]).

Remark 2.1. For the gauge function φ, the function $\Phi : [0, \infty) \to [0, \infty)$ defined by $\Phi(t) = \int_0^t \varphi(\tau)d\tau$ is convex and strictly increasing continuous function on $[0, \infty)$. Then Φ has a continuous inverse function Φ^{-1}.

Remark 2.2. If a Banach space X has a uniformly Gâteaux differentiable norm, then $J\varphi$ is single-valued which is denoted by j_φ.

Lemma 2.3. ([22]) *For all $x, y \in X$, the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \; j_\varphi(x + y) \in J_\varphi(x + y).$$

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* [27] if

$$Q(Qx + t(x - Qx)) = Qx,$$

where $Qx+t(x-Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \to D$ is said to be *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

Lemma 2.4. (see [14,23,27]) Let C be a closed and convex subset of a smooth Banach space X and D be a nonempty subset of C. Let $Q : C \to D$ be a retraction and J be the normalized duality mapping on X. Then the following are equivalent:

(a) Q is sunny and nonexpansive.
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ for all $x, y \in C$.
(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in D$.

Lemma 2.5. *Let C be a closed and convex subset of a smooth Banach space X and D be a nonempty subset of C. Let $Q : C \to D$ be a retraction. Let J be the normalized duality mapping and J_φ be duality mapping with a gauge function φ. Then the following are equivalent:*

(a) $\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in D$.
(b) $\langle x - Qx, J_\varphi(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in D$.

Proof. Assume that $y \neq Qx$, from Lemma 2.4, we see that

$$\begin{aligned}\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad &\Leftrightarrow \quad \frac{\|y - Qx\|}{\varphi(\|y - Qx\|)}\langle x - Qx, J_\varphi(y - Qx)\rangle \leq 0 \\ &\Leftrightarrow \quad \langle x - Qx, J_\varphi(y - Qx)\rangle \leq 0.\end{aligned}$$

This completes the proof. □

Lemma 2.6. (*The Resolvent Identity* [5]) *Suppose that* $\lambda > 0$*,* $\mu > 0$ *and* $x \in X$*. Then*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Lemma 2.7. ([34]) *Let* $\{x_n\}$ *and* $\{l_n\}$ *be bounded sequences in a Banach space* X *and* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$*. Suppose that* $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ *for all* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$*. Then*

$$\lim_{n\to\infty}\|l_n - x_n\| = 0.$$

Lemma 2.8. ([38]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\sigma_n)a_n + \delta_n, \ \forall n \ge 0,$$

where $\{\sigma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a real sequence such that*

(i) $\sum_{n=0}^{\infty}\sigma_n = \infty$;

(ii) $\limsup_{n\to\infty}\dfrac{\delta_n}{\sigma_n} \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. ([12]) *Let* C *be a nonempty closed convex subset of a real reflexive strictly convex Banach space* X *which has a uniformly Gâteaux differentiable norm and admits the duality mapping* j_φ*. Let* $W : C \to C$ *be a nonexpansive mapping such that* $Fix(W) \neq \emptyset$ *and* $f : C \to C$ *be an* α*-contraction mapping with a constant* $\alpha \in (0,1)$*. Suppose that* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$*. Then, the sequence* $\{x_n\}$ *defined by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)Wx_n, \ \forall n \ge 1,$$

converges strongly to a point $x^* \in Fix(W)$*.*

Definition 2.10. Let C be a nonempty and convex subset of a real Banach space X. Let $A_i : C \to X$ $(i = 1, 2, \ldots, N)$ be a finite family of accretive operators and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for all $i = 1, 2, \ldots, N$. We define a mapping $W : C \to C$ as follows:

$$\begin{array}{lcl} U_1 & = & \lambda_1 J_r^1 + (1-\lambda)I, \\ U_2 & = & \lambda_2 J_r^2 U_1 + (1-\lambda_2)U_1, \\ \vdots & & \\ U_{N-1} & = & \lambda_{N-1}J_r^{N-1}U_{N-2} + (1-\lambda_{N-1})U_{N-2}, \\ W & = & U_N = \lambda_N J_r^N U_{N-1} + (1-\lambda_N)U_{N-1}, \end{array} \tag{2.2}$$

where $J_r^i := (I + rA_i)^{-1}$ for all $i = 1, 2, \ldots, N$.

Tha above mapping W is called the W*-mapping* generated by $J_r^1, J_r^2, \ldots, J_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a real strictly convex Banach space X. Let $A_i : C \to X$ $(i = 1, 2, \dots, N)$ be a finite family of accretive operators such that $\bigcap_{i=1}^N N(A_i) \neq \emptyset$, satisfying the range condition:*

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i) \text{ for } i = 1, 2, \dots, N.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for all $i = 1, 2, \dots, N$. Let W be the W-mapping generated by $J_r^1, J_r^2, \dots, J_r^N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$. Then $U_1, U_2, \dots, U_{N-1}$ and W are nonexpansive. Moreover, $Fix(W) = \bigcap_{i=1}^N N(A_i)$.

Proof. For each $i = 1, 2, \dots, N$, A_i satisfies the range condition, we have J_r^i is well defined nonexpansive mapping from $R(I + rA) \to C$ with $\bigcap_{i=1}^N Fix(J_r^i) = \bigcap_{i=1}^N N(A_i)$. Then $U_1, U_2, \dots, U_N$ and W are nonexpansive mappings.

To show that $Fix(W) = \bigcap_{i=1}^N N(A_i)$, we first show that $\bigcap_{i=1}^N N(A_i) \subseteq Fix(W)$. Let $z \in \bigcap_{i=1}^N N(A_i)$, then

$$\begin{aligned}
U_1 z &= \lambda_1 J_r^1 z + (1 - \lambda_1) z = z, \\
U_2 z &= \lambda_2 J_r^2 U_1 z + (1 - \lambda_2) U_1 z = z, \\
&\vdots \\
U_{N-1} z &= \lambda_{N-1} J_r^{N-1} U_{N-2} z + \lambda_{N-1} J_r^{N-1} U_{N-2} z = z, \\
W z &= U_N z = \lambda_N J_r^N U_{N-1} z + (1 - \lambda_N) U_{N-1} z = z.
\end{aligned}$$

Hence, $Wz = z$, i.e., $z \in Fix(W)$.

Next, we will show that $Fix(W) \subseteq \bigcap_{i=1}^N N(A_i)$. Let $w \in Fix(W)$ and $v \in \bigcap_{i=1}^N N(A_i)$. By the definition of W, we have

$$\begin{aligned}
\|w - v\| &= \|Ww - v\| \\
&= \|\lambda_N (J_r^N U_{N-1} w - v) + (1 - \lambda_N)(U_{N-1} w - v)\| \\
&\le \lambda_N \|J_r^N U_{N-1} w - v\| + (1 - \lambda_N)\|U_{N-1} w - v\| \\
&\le \|U_N w - v\| \\
&= \|\lambda_{N-1} (J_r^{N-1} U_{N-2} w - v) + (1 - \lambda_{N-1})(U_{N-2} w - v)\| \\
&\le \lambda_{N-1} \|J_r^{N-1} U_{N-2} w - v\| + (1 - \lambda_{N-1})\|U_{N-2} w - v\| \\
&\le \|U_{N-2} w - v\| \\
&\ \vdots \\
&\le \|U_1 w - v\| \\
&= \|\lambda_1 (J_r^1 w - v) + (1 - \lambda_1)(w - v)\| \\
&\le \lambda_1 \|J_r^1 w - v\| + (1 - \lambda_1)\|w - v\| \\
&\le \|w - v\|. \qquad (3.1)
\end{aligned}$$

This implies that

$$\|w-v\| = \|\lambda_1(J_r^1 w - v) + (1-\lambda_1)(w-v)\|$$

and

$$\|w-v\| = \lambda_1\|J_r^1 w - v\| + (1-\lambda_1)\|w-v\|.$$

So $\|w-v\| = \|J_r^1 w - v\|$. By the strict convexity of X, we obtain that $J_r^1 w = w$, i.e., $w \in Fix(J_r^1)$. This implies that $U_1 w = w$. From (3.1) together with $U_1 w = w$, we have

$$\begin{aligned}\|w-v\| &= \|\lambda_2(J_r^2 U_1 w - v) + (1-\lambda_2)(U_2 w - v)\| \\ &= \|\lambda_2(J_r^2 w - v) + (1-\lambda_2)(w-v)\|\end{aligned}$$

and

$$\begin{aligned}\|w-v\| &= \lambda_2\|J_r^2 U_1 w - v\| + (1-\lambda_2)\|U_1 w - v\| \\ &= \lambda_2\|J_r^2 w - v\| + (1-\lambda_2)\|w-v\|.\end{aligned}$$

So $\|w-v\| = \|J_r^2 w - v\|$. By the strict convexity of X, we obtain that $J_r^2 w = w$, i.e., $w \in Fix(J_r^2)$. This implies that $U_2 w = w$. Applying the same proof as above, we get $w = J_r^1 w = J_r^2 w = \cdots = J_r^{N-1} w$ and $w = U_1 w = U_2 w = \cdots = U_{N-1} w$. Since $w \in Fix(W) = Fix(U_N)$ and $U_{N-1} w = w$, then $w = \lambda_N J_r^N w + (1-\lambda_N) w$. This implies that $w = J_r^N w$. Therefore, $w \in \bigcap_{i=1}^N Fix(J_r^i) = \bigcap_{i=1}^N N(A_i)$. This completes the proof. □

Lemma 3.2. *For each $r, s > 0$,*

$$\|J_r x - J_s x\| \leq \left|1 - \frac{s}{r}\right| \|J_r x - x\| \text{ for all } x \in X.$$

Proof. Follows from the resolvent identity, we can conclude the desired result easily. □

Lemma 3.3. *Let C be a nonempty closed convex subset of a real Banach space X. Let $A_i : C \to X$ $(i = 1, 2, \ldots, N)$ be a finite family of accretive operators such that $\bigcap_{i=1}^N N(A_i) \neq \emptyset$, satisfying the range condition:*

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i) \text{ for } i = 1, 2, \ldots, N.$$

Let $\{\lambda_{n,i}\}$ $(i = 1, 2, \ldots, N)$ be a sequence in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that $r_n \to r$ as $n \to \infty$. Suppose that W is the mapping generated by $J_r^1, J_r^2, \ldots, J_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and for each $n \in \mathbb{N}$, W_n-mapping is the mapping generated by $J_{r_n}^1, J_{r_n}^2, \ldots, J_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Then, for each $x \in C$, we have

$$\lim_{n\to\infty} \|W_n x - W x\| = 0.$$

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Proof. Let $x \in C$. Suppose that U_k is the mapping generated by $J_r^1, J_r^2, \ldots, J_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and for each $n \in \mathbb{N}$, $U_{k,n}$ is the mapping generated by $J_{r_n}^1, J_{r_n}^2, \ldots, J_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. From Lemma 3.2, we have

$$\begin{aligned}
\|U_{n,1}x - U_1\| &= \|\lambda_{n,1}J_{r_n}^1 x + (1-\lambda_1)x - \lambda_1 J_r^1 x - (1-\lambda_1)x\| \\
&= \|\lambda_{n,1}(J_{r_n}^1 x - J_r^1 x) + (\lambda_{n,1} - \lambda_1)(J_r^1 x - x)\| \\
&\leq \lambda_{n,1}\|J_{r_n}^1 x - J_r^1 x\| + |\lambda_{n,1} - \lambda_1|\|J_r^1 x - x\| \\
&\leq \left|1 - \frac{r_n}{r}\right|\|J_r^1 x - x\| + |\lambda_{n,1} - \lambda_1|\|J_r^1 x - x\| \\
&\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)\|J_r^1 x - x\|.
\end{aligned}$$

Using the same argument as above, for each $i = 2, 3, \ldots, N$, we obtain that

$$\begin{aligned}
& \|U_{n,k}x - U_k x\| \\
= \; & \|\lambda_{n,k}J_{r_n}^k U_{n,k-1}x + (1-\lambda_{n,k})x - \lambda_k J_r^k U_{k-1}x - (1-\lambda_k)x\| \\
= \; & \|\lambda_{n,k}(J_{r_n}^k U_{n,k-1}x - J_{r_n}^k U_{k-1}x) + \lambda_{n,k}(J_{r_n}^k U_{k-1}x - J_r^k U_{k-1}x) \\
& +(\lambda_{n,k} - \lambda_k)(J_r^k U_{k-1}x - x)\| \\
\leq \; & \lambda_{n,k}\|J_{r_n}^k U_{n,k-1}x - J_{r_n}^k U_{k-1}x\| + \lambda_{n,k}\|J_{r_n}^k U_{k-1}x - J_r^k U_{k-1}x\| \\
& +|\lambda_{n,k} - \lambda_k|\|J_r^k U_{k-1}x - x\| \\
\leq \; & \|U_{n,k-1}x - U_{k-1}x\| + \left|1 - \frac{r_n}{r}\right|\|J_r^k U_{k-1}x - U_{k-1}x\| \\
& +|\lambda_{n,k} - \lambda_k|\|J_r^k U_{k-1}x - x\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|W_n x - Wx\| &= \|U_{n,N}x - U_N x\| \\
&\leq \|U_{n,1}x - U_1 x\| + \textstyle\sum_{i=1}^N |\lambda_{n,i} - \lambda_i|\|J_r^i U_{i-1}x - U_{i-1}x\| \\
&\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)\|J_r^1 x - x\| \\
&\quad + \textstyle\sum_{i=1}^N |\lambda_{n,i} - \lambda_i|\|J_r^i U_{i-1}x - U_{i-1}x\|.
\end{aligned}$$

Since $r_n \to r$ and $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$, then $\lim_{n\to\infty} \|W_n x - Wx\| = 0$. This completes the proof. □

Next, we will show that the sequences $\{x_n\}$ which is defined by the iterative algorithm (1.5) is a convergent sequence.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space X which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A_i : C \to X$ $(i = 1, 2, \ldots, N)$ be a finite family of accretive operators such that $\bigcap_{i=1}^N N(A_i) \neq \emptyset$, satisfying the range condition:*

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i) \text{ for } i = 1, 2, \ldots, N.$$

Let $f : C \to C$ be an α-contraction mapping with a constant $\alpha \in (0, 1)$. Let $\{\lambda_{n,i}\}$ $(i = 1, 2, \ldots, N)$ be a sequence in $[a, b]$ with $0 < a \leq b < 1$ and $\{r_n\}$ be *a sequence in* $(0,\infty)$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$*. Assume that the following conditions hold:*

(C1) $\alpha_n+\beta_n+\gamma_n=1$;
(C2) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C3) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;
(C4) $\liminf_{n\to\infty}r_n>0$, $\lim_{n\to\infty}\dfrac{r_{n+1}}{r_n}=1$;
(C5) $\lim_{n\to\infty}|\lambda_{n+1,i}-\lambda_{n,i}|=0$ *for all* $i=1,2,\ldots,N$.

For all $n\in\mathbb{N}$*, let* W_n *be a* W*-mapping generated by* $J_r^1,J_r^2,\ldots,J_r^N$ *and* $\lambda_{n,1},\lambda_{n,2},$
$\ldots,\lambda_{n,N}$*. For given* $x_1\in C$*, if* $\{x_n\}$ *is the sequence defined by*

$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n W_n x_n,\ \forall n\geq 1. \tag{3.2}$$

Then $\{x_n\}$ *strongly converges to a common solution of the equations* $A_i x=0$ *for all* $i=1,2,\ldots,N$*.*

Proof. First, we will show that $\{x_n\}$ is bounded. Taking $p\in\bigcap_{i=1}^N N(A_i)$, we have

$$\begin{aligned}
&\|x_{n+1}-p\|\\
&\quad = \|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n(W_nx_n-p)\|\\
&\quad \leq \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\beta_n\|x_n-p\|+\gamma_n\|W_nx_n-p\|\\
&\quad \leq \alpha_n\alpha\|x_n-p\|+\alpha_n\|f(p)-p\|+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\|\\
&\quad = (1-\alpha_n(1-\alpha))\|x_n-p\|+\alpha_n\|f(p)-p\|\\
&\quad \leq \max\Big\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\alpha}\Big\}.
\end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n-p\|\leq\max\Big\{\|x_1-p\|,\frac{\|f(p)-p\|}{1-\alpha}\Big\},\ \forall n\geq 1.$$

Hence, $\{x_n\}$ is bounded and $\{f(x_n)\}$ and $\{W_nx_n\}$ are also.

Next, we show that $\lim_{n\to\infty}\|W_{n+1}\omega_n-W_n\omega_n\|=0$, where $\{\omega_n\}$ is a bounded sequence in C. Let $i=2,3,\ldots,N-2$. From Lemma 3.2, we observe that

$$\begin{aligned}
&\|U_{n+1,N-i}\omega_n-U_{n,N-i}\omega_n\|\\
&\quad =\|\lambda_{n+1,N-i}J_{r_{n+1}}^{N-i}U_{n+1,N-i-1}\omega_n+(1-\lambda_{n+1,N-i})\omega_n\\
&\qquad -\lambda_{n,N-i}J_{r_n}^{N-i}U_{n,N-i-1}\omega_n-(1-\lambda_{n,N-i})\omega_n\|\\
&\quad \leq\lambda_{n+1,N-i}\|J_{r_{n+1}}^{N-i}U_{n+1,N-i-1}\omega_n-J_{r_{n+1}}^{N-i}U_{n,N-i-1}\omega_n\|\\
&\qquad +\lambda_{n+1,N-i}\|J_{r_{n+1}}^{N-i}U_{n,N-i-1}\omega_n-J_{r_n}^{N-i}U_{n,N-i-1}\omega_n\|\\
&\qquad +|\lambda_{n+1,N-i}-\lambda_{n,N-i}|\|J_{r_n}^{N-i}U_{n,N-i-1}\omega_n-\omega_n\|\\
&\quad \leq\|U_{n+1,N-i-1}\omega_n-U_{n,N-i-1}\omega_n\|+\Big|1-\frac{r_{n+1}}{r_n}\Big|\|J_{r_n}^{N-i}U_{n,N-i-1}\omega_n-U_{n,N-i-1}\omega_n\|\\
&\qquad +|\lambda_{n+1,N-i}-\lambda_{n,N-i}|\|J_{r_n}^{N-i-1}U_{n,N-i-1}\omega_n-\omega_n\|\\
&\quad \leq \|U_{n+1,N-i-1}\omega_n-U_{n,N-i-1}\omega_n\|+\bigg(\Big|1-\frac{r_{n+1}}{r_n}\Big|+|\lambda_{n+1,N-i}-\lambda_{n,N-i}|\bigg)M_2,
\end{aligned}$$

where $M_2 > 0$ is a constant such that $M_2 = \sup_{n\geq 1}\big\{\|J^1_{r_n}\omega_n - \omega_n\| + \sum_{i=1}^N \|J^i_{r_n}U_{n,i-1}\omega_n - U_{n,i-1}\omega_n\|\big\}$.

It follows that

$$\begin{aligned}
&\|W_{n+1}\omega_n - W_n\omega_n\|\\
&= \|U_{n+1,N}\omega_n - U_{n,N}\omega_n\|\\
&\leq \|U_{n+1,1}\omega_n - U_{n,1}\omega_n\| + \sum_{i=2}^N \left(\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,i}-\lambda_{n,i}|\right)M_2\\
&\leq\left(\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,1}-\lambda_{n,1}|\right)\|J^1_r\omega_n - \omega_n\| + \sum_{i=2}^N\left(\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,i}-\lambda_{n,i}|\right)M_2\\
&\leq \sum_{i=1}^N \left(\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,i}-\lambda_{n,i}|\right)M_2.
\end{aligned}$$

From the conditions $(C4)$ and $(C5)$, we obtain that

$$\lim_{n\to\infty}\|W_{n+1}\omega_n - W_n\omega_n\| = 0. \tag{3.3}$$

Next, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Let $x_{n+1} = \beta_n x_n + (1-\beta_n)l_n$ for all $n \geq 1$. Then, we have

$$\begin{aligned}
l_{n+1} - l_n =& \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\
=& \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}W_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) - \gamma_n W_n x_n}{1-\beta_n}\\
=& \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})W_{n+1}x_{n+1}}{1-\beta_{n+1}}\\
&-\frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)W_n}{1-\beta_n}\\
=& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - W_{n+1}x_{n+1}) - \frac{\alpha_n}{1-\beta_n}(f(x_n)-W_n x_n)\\
&+W_{n+1}x_{n+1} - W_{n+1}x_n + W_{n+1}x_n - W_n x_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\| \leq& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - W_{n+1}x_{n+1}\|\\
&+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-W_n x_n\| + \|W_{n+1}x_n - W_n x_n\|.
\end{aligned}$$

From the conditions $(C1)$, $(C2)$ and (3.3), we have

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Hence, by Lemma 2.7, we obtain that

$$\lim_{n\to\infty}\|l_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n-x_n\| = 0. \tag{3.4}$$

Note that

$$\begin{aligned}\|x_{n+1}-W_nx_n\| &= \|\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)W_nx_n-W_nx_n\|\\ &\leq \alpha_n\|f(x_n)-W_nx_n\|+\beta_n\|x_n-W_nx_n\|\\ &\leq \alpha_n\|f(x_n)-W_nx_n\|+\beta_n\big(\|x_n-x_{n+1}\|+\|x_{n+1}-W_nx_n\|\big),\end{aligned}$$

which implies that

$$\|x_{n+1}-W_nx_n\| \leq \frac{\alpha_n}{1-\beta_n}\|f(x_n)-W_nx_n\|+\frac{\beta_n}{1-\beta_n}\|x_n-x_{n+1}\|.$$

From the conditions $(C2)$, $(C3)$ and (3.4), we have

$$\lim_{n\to\infty}\|x_{n+1}-W_nx_n\| = 0,$$

and hence

$$\begin{aligned}\|x_n-W_nx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-W_nx_n\|\\ &\to 0 \text{ as } n\to\infty. \end{aligned}\tag{3.5}$$

Note that

$$\begin{aligned}\|x_n-Wx_n\| &\leq \|x_n-W_nx_n\|+\|W_nx_n-Wx_n\|\\ &\leq \|x_n-W_nx_n\|+\sup_{x\in C}\|W_nx-Wx\|.\end{aligned}$$

In view of Lemma 3.3 and (3.5), we obtain that

$$\lim_{n\to\infty}\|x_n-Wx_n\| = 0. \tag{3.6}$$

Let $u_m=\alpha_m f(u_m)+(1-\alpha_m)Wu_m$, where $\{\alpha_m\}$ is satisfies the condition of Lemma 2.9. It follows from Lemma 2.9, then we have $x^*=\lim_{m\to\infty}u_m$. We note that

$$\begin{aligned}&\|u_m-x_n\|\varphi(\|u_m-x_n\|)\\ &\quad=\alpha_n\langle f(u_m)-x_n, j_\varphi(u_m-x_n)\rangle+(1-\alpha_m)\langle Wu_m-x_n, j_\varphi(u_m-x_n)\rangle\\ &\quad=\alpha_m\langle f(u_m)-f(x^*)-u_m+x^*, j_\varphi(u_m-x_n)\rangle\\ &\qquad+\alpha_m\langle f(x^*)-x^*, j_\varphi(u_m-x_n)\rangle+\alpha_m\langle u_m-x_n, j_\varphi(u_m-x_n)\rangle\\ &\qquad+(1-\alpha_m)\langle Wu_m-Wx_n, j_\varphi(u_m-x_n)\rangle\\ &\qquad+(1-\alpha_m)\langle Wx_n-x_n, j_\varphi(u_m-x_n)\rangle\\ &\quad\leq \|u_m-x_n\|\varphi(\|u_m-x_n\|)+\|Wx_n-x_n\|\varphi(\|u_m-x_n\|)\\ &\qquad+\alpha_m(1+\alpha)\varphi(\|u_m-x_n\|)\|u_m-x^*\|.\end{aligned}$$

This implies that

$$\langle f(x^*)-x^*, j_\varphi(x_n-u_m)\rangle \leq \frac{\|Wx_n-x_n\|}{\alpha_m}M_3+(1+\alpha)\|u_m-x^*\|M_3, \tag{3.7}$$

where $M_3>0$ is a constant such that $M_3=\sup_{n\geq}\{\varphi(\|u_m-x_n\|)\}$. Now, taking the upper limit as $n\to\infty$ and $m\to\infty$, respectively, we obtain that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle f(x^*)-x^*, j_\varphi(x_n-u_m)\rangle \leq 0. \tag{3.8}$$

Since j_φ is norm-weak* uniformly continuous on bounded sets, as $m \to \infty$, then

$$\langle f(x^*) - x^*, j_\varphi(x_n - u_m)\rangle \to \langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle.$$

Hence, for each $\epsilon > 0$, there exists $N \geq 1$ such that if $m > N$, for all $n \geq 1$, then

$$\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle < \langle f(x^*) - x^*, j_\varphi(x_n - u_m)\rangle + \epsilon. \tag{3.9}$$

Thus taking upper limit as $n \to \infty$ and $m \to \infty$ in both sides of (3.9), we get from (3.8) that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then we obtain that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j_\varphi(x_n - x^*)\rangle \leq 0. \tag{3.10}$$

Since $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, $\forall t \geq 0$ and $\varphi : [0,\infty) \to [0,\infty)$ is the gauge function, then for $1 \geq k \geq 0$, $\varphi(ky) \leq \varphi(y)$, we get

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(ky)dy \leq k\int_0^t \varphi(y)dy = k\Phi(t).$$

Finally, we will show that $x_n \to x^*$ as $n \to \infty$. From Lemma 2.3, we note that

$$\begin{aligned}
& \Phi(\|x_{n+1} - x^*\|) \\
= \;& \Phi\big(\|\alpha_n(f(x_n) - f(x^*)) + \alpha_n(f(x^*) - x^*) + \beta_n(x_n - x^*) \\
& +\gamma_n(W_nx_n - x^*)\|\big) \\
\leq \;& \Phi\big(\|\alpha_n(f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + \gamma_n(W_nx_n - x^*)\|\big) \\
& +\alpha_n\langle f(x^*) - x^*, j_\varphi(x_{n+1} - x^*)\rangle \\
\leq \;& \Phi\big((1 - \alpha_n(1 - \alpha))\|x_n - x^*\|\big) + \alpha_n\langle f(x^*) - x^*, j_\varphi(x_{n+1} - x^*)\rangle \\
\leq \;& (1 - \alpha_n(1 - \alpha))\Phi(\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, j_\varphi(x_{n+1} - x^*)\rangle.
\end{aligned}$$

From the condition $(C2)$, we see that $\sum_{n=1}^{\infty}(1-\alpha)\alpha_n = \infty$ and by using (3.10), we get

$$\limsup_{n\to\infty} \frac{1}{1-\alpha}\langle f(x^*) - x^*, j_\varphi(x_{n+1} - x^*)\rangle \leq 0.$$

Thus, by Lemma 2.8, we have $\Phi(\|x_n - x^*\|) \to 0$ as $n \to \infty$. From the property of Φ, implies that $x_n \to x^*$. Therefore, we conclude that $x_n \to x^*$. This completes the proof. □

Remark 3.5. Theorems 3.4 improves and extends [17, Theorem 3.1] in the following ways:

(i) The fixed element $u \in C$ is replaced by more general $f(x_n)$, where $f : C \to C$ is a contraction mapping, we obtain that is called viscosity approximation method.

(ii) The normalized duality mapping $J : X \to X^*$ is extended to more general duality mapping $J_\varphi : X \to X^*$ with a gauge function $\varphi : [0, \infty) \to [0, \infty)$ which covers the case $\varphi(t) = t$.

(iii) The algorithm (3.2) likes completely the Hu and Liu's algorithm (1.4), but the mappings S_{r_n} and W_n are very different.

Remark 3.6. The mapping W_n defined by (1.4) is very different from the mapping K_n in [20] because the mapping K_n is contains a finite family of nonexpansive mappings while the mapping W_n is contains a finite family of accretive operators.

If we take $f(x) = u \in C$ in Theorem 3.4, then we have the following result:

Corollary 3.7. *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space X which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A_i : C \to X$ $(i = 1, 2, \dots, N)$ be a finite family of accretive operators such that $\bigcap_{i=1}^N N(A_i) \neq \emptyset$, satisfying the range condition:*

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i) \text{ for } i = 1, 2, \dots, N.$$

Let $\{\lambda_{n,i}\}$ $(i = 1, 2, \dots, N)$ be a sequence in $[a, b]$ with $0 < a \leq b < 1$ and $\{r_n\}$ be a sequence in $(0, \infty)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$. Assume that the conditions $(C1)-(C5)$ of Theorem 3.4 hold. For given $u, x_1 \in C$, if $\{x_n\}$ is the sequence defined by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n W_n x_n, \ \forall n \geq 1. \tag{3.11}$$

Then $\{x_n\}$ strongly converges to a common solution of the equations $A_i x = 0$ for $i = 1, 2, \dots, N$.

If we take $N = 1$ in Theorem 3.4, then we have the following result for a single mapping:

Corollary 3.8. *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space X which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A : C \to X$ be a finite family of accretive operators such that $N(A) \neq \emptyset$, satisfying the range condition:*

$$\overline{D(A)} \subseteq C \subset R(I + rA).$$

Let $f : C \to C$ be an α-contraction mapping with a constant $\alpha \in (0, 1)$. Let $\{r_n\}$ be a sequence in $(0, \infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$. Assume that the conditions $(C1)-(C4)$ of Theorem 3.4 hold. For given $x_1 \in C$, if $\{x_n\}$ is the sequence defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n} x_n, \ \forall n \geq 1. \tag{3.12}$$

Then $\{x_n\}$ strongly converges to a common solution of the equation $Ax = 0$.

4. **Some applications**

In this section, as applications, we will utilize Theorem 3.4 to deduced several results. As a direct consequence of Theorem 3.4, we have the following results:

4.1. **Application to a finite family of continuous pseudocontractive mappings.** Let C be a nonempty closed convex subset of a real Banach space X which admits the duality mapping j_φ. A mapping $T : C \to C$ is said to be *pseudocontractive* if there exists $j_\varphi(x-y) \in J_\varphi(x-y)$ such that

$$\langle Tx - Ty, j_\varphi(x-y)\rangle \leq \|x-y\|\varphi(\|x-y\|),\ \forall x, y \in C.$$

It is well known that the class of pseudocontractive mapping is more general than the class of nonexpansive mapping. Moreover, the class of accretive mappings is the class of pseudo-contractive mappings. A mapping $A : C \to X$ is said to be pseudocontractive if $T := I - A$ is accretive. From Theorem 3.4, we observe that x^* is a zero of the accretive mapping A if and only if it is a fixed point of the pseudocontractive mapping $T := I - A$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space X which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $T_i : C \to X$ ($i = 1, 2, \ldots, N$) be a finite family of continuous pseudocontractive mappings such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. For each $r > 0$, let $J_r^i := (I + r(I - T_i))^{-1} = (2I - T_i)^{-1}$ ($i = 1, 2, \ldots, N$) and $f : C \to C$ be an α-contraction mapping with a constant $\alpha \in (0, 1)$. Let $\{\lambda_{n,i}\}$ ($i = 1, 2, \ldots, N$) be a sequence in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ be a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences $(0, 1)$ which satisfy the conditions $(C1) - (C5)$ in Theorem 3.4. For all $n \in \mathbb{N}$, let W_n be a W-mapping generated by $J_r^1, J_r^2, \ldots, J_r^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. For given $x_1 \in C$, if $\{x_n\}$ is the sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n x_n,\ \forall n \geq 1. \tag{4.1}$$

Then $\{x_n\}$ strongly converges to a common fixed point $x^* \in \bigcap_{i=1}^N Fix(T_i)$.

Proof. For each $i = 1, 2, \ldots, N$, we set $A_i := I - T_i$ into Theorem 3.4. Then, $Fix(T_i) = N(A_i)$ for all $i = 1, 2, .., N$ and hence $\bigcap_{i=1}^N Fix(T_i) = \bigcap_{i=1}^N N(A_i)$. Furthermore, each A_i for $i = 1, 2, \ldots, N$ is m-accretive. Therefore, the proof is complete from Theorem 3.4. □

4.2. **Application to a viscosity approximation with weak contraction.** Let C be a nonempty closed convex subset of a real Banach space X. A mapping $g : C \to C$ is said to be *weakly contractive* (see [1]) if

$$\|g(x) - g(y)\| \leq \|x - y\| - \psi(\|x - y\|),\ \forall x, y \in C,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function such that $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. As a special case, if $\psi(t) = (1 - \alpha)t$ for

$t \in [0, \infty)$ and $\alpha \in (0,1)$, then the weakly contractive mapping g is contraction with a constant α.

In 2001, Rhoades [29] proved a Banach contraction principle for the weakly contractive mapping as follows:

Lemma 4.2. ([29]) *Let (X, d) be a complete metric space, and g be a weakly contractive mapping on X. Then g has a unique fixed point in X.*

Lemma 4.3. ([2]) *Assume that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real number and $\{\alpha_n\}$ is a sequence of a positive real number satisfying the conditions: $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \dfrac{b_n}{\alpha_n} = 0$. Let the recursive inequality*

$$a_{n+1} \leq a_n - \alpha_n \Psi(a_n) + b_n, \ \forall n \geq 0,$$

hold, where $\Psi(a)$ is a continuous and strict increasing function for all $a \geq 0$ with $\Psi(0) = 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. Let $A_i : C \to X$ $(i = 1, 2, \dots, N)$ be a finite family of accretive operators such that $\mathcal{F} := \bigcap_{i=1}^{N} N(A_i) \neq \emptyset$. For all $r > 0$, let $J_r^i := (I + rA_i)^{-1}$ $(i = 1, 2, \dots, N)$ and $g : C \to C$ be a weakly contractive mapping with the function ψ. Let $\{\lambda_{n,i}\}$ $(i = 1, 2, \dots, N)$ be a sequence in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ be a sequence in $(0, \infty)$, $\{\alpha_n\}$ $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences $(0, 1)$ which satisfy the conditions $(C1) - (C5)$ in Theorem 3.4. For all $n \in \mathbb{N}$, let W_n be a W-mapping generated by $J_r^1, J_r^2, \dots, J_r^N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. For given $x_1 \in C$, if $\{x_n\}$ is the sequence defined by*

$$x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \gamma_n W_n x_n, \ \forall n \geq 1. \tag{4.2}$$

Then $\{x_n\}$ strongly converges to a common solution of the equations $A_i x = 0$ for all $i = 1, 2, \dots, N$.

Proof. Since a Banach space X is smooth, then by Lemma 2.5, there exists a sunny nonexpansive retraction Q from C onto $\mathcal{F}$. Moreover, $Q \circ g$ is a weakly contractive mapping. Indeed, for all $x, y \in C$, we see that

$$\begin{aligned} \|(Q \circ g)x - (Q \circ g)y\| &= \|Q\big(g(x)\big) - Q\big(g(y)\big)\| \\ &\leq \|g(x) - g(y)\| \leq \|x - y\| - \psi(\|x - y\|). \end{aligned}$$

By Lemma 4.2, it guarantees that $Q \circ g$ has a unique fixed point $x^* \in C$ such that $Q\big(g(x^*)\big) = x^*$. Now, we let $y_1 \in C$ and define a sequence $\{y_n\}$ by

$$y_{n+1} = \alpha_n g(x^*) + \beta_n y_n + \gamma_n W_n y_n, \ \forall n \geq 1.$$

Then by Corollary 3.7 with $u = g(x^*)$, we have $\lim_{n\to\infty} y_n = Q\big(g(x^*)\big) = x^* \in \mathcal{F}$.

Next, we will show that $x_n \to x^*$ as $n \to \infty$. Note that

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&= \|\alpha_n(g(x_n) - g(x^*)) + \beta_n(x_n - y_n) + \gamma_n(W_n x_n - W_n y_n)\| \\
&\leq \alpha_n\|g(x_n) - g(y_n)\| + \alpha_n\|g(y_n) - g(x^*)\| + \beta_n\|x_n - y_n\|| + \gamma_n\|W_n x_n - W_n y_n\| \\
&\leq \alpha_n\|x_n - y_n\| - \alpha_n\psi(\|x_n - y_n\|) + \alpha_n\|y_n - x^*\| - \alpha_n\psi(\|y_n - x^*\|) \\
&\quad +(1-\alpha_n)\|x_n - y_n\| \\
&\leq \|x_n - y_n\| - \alpha_n\psi(\|x_n - y_n\|) + \alpha_n\|y_n - x^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|y_n - x^*\| = 0$, then by the condition $(C1)$ and Lemma 4.3, we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore, we conclude that $x_n \to x^*$. This completes the proof. □

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