# ON THE STRUCTURE OF SOME DEFICIENCY ZERO GROUPS 

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#### Abstract

Using the notion of the generalized external semidirect product of two groups we investigate the structure of some known deficiency zero groups and also construct some new deficiency zero groups having specified properties. In particular, for any integers $i, j$ with $1 \leq i \leq j$, we construct a 2-generator 2-relation group of order $3^{3 i+j+1}$ and nilpotency class $2 i+1$.


## 1. Introduction

Let $F_{n}$ be the free group on $n$ free generators $a_{1}, \ldots, a_{n}$, and let $\theta$ denote the automorphism defined by setting $a_{i}^{\theta}=a_{i+1}$, where the subscripts are reduced modulo $n$. For any cyclically reduced word

[^0]$w=w\left(a_{1}, \ldots, a_{n}\right)$ in $F_{n}$, the cyclically presented group $K_{n}(w)$ is defined by
$$
\left\langle a_{1}, \ldots, a_{n} \mid w^{\theta^{k-1}}=1, k=1, \ldots, n\right\rangle
$$

Examples of such groups are discussed in [7]. It is clear that $\theta$ induces an automorphism of $K_{n}(w)$ and hence one may take a semi-direct product of $K_{n}(w)$ with a cyclic group of order $n$ with the action induced by $\theta$. It is often easier and more instructive to consider the resulting group rather than $K_{n}(w)$. For instance, see [1], [5] and [13].
In this paper, for a given integer $l$, we consider a factor group $K_{n, l}(w)$ of $K_{n}(w)$ defined by

$$
\left\langle a_{1}, \ldots, a_{n} \mid a_{1}^{l}=\cdots=a_{n}^{l}, w^{\theta^{k-1}}=1, k=1, \ldots, n\right\rangle,
$$

and form the generalized external semi-direct product of $K_{n, l}(w)$ with a cyclic group of order $n$. This allows us to determine the structure of some known deficiency zero groups as well as to construct some new deficiency zero groups having specified properties. In particular, for any integers $i, j$ with $1 \leq i \leq j$, we construct the group

$$
G_{i, j}=\left\langle x, y \mid x^{3^{i}}=y^{3},\left[x, x^{y}\right]=x^{3^{j}}\right\rangle,
$$

and show that $G_{i, j}$ having order $3^{3 i+j+1}$ is of class $2 i+1$. This group is a generalization of the group $G=G_{i, i}$ introduced by Wiegold in [15], wherein the claim about the class is incorrect and the exact order of $G$ is unspecified.

## 2. Preliminary results

In this section we first introduce the notion of generalized semidirect product.

Let $H$ and $K$ be groups. A factor pair of $H$ by $K$ is a pair $(f, \varphi)$ of maps $f: H \times H \longrightarrow K$ and $\varphi: H \longrightarrow \operatorname{Aut}(K)$, denoted by $h \mapsto \varphi_{h}$, such that
(i) $f(x, 1)=f(1, x)=1$ for every $x \in H$ and $\varphi_{1}$ is the identity in $\operatorname{Aut}(K)$.
(ii) $\varphi_{x} \varphi_{y}=\varphi_{x y} \psi_{f(x, y)}$ for $x, y \in H$, where $\psi_{k}$ is the inner automorphism of $K$ induced by $k \in K$.
(iii) $f(x, y z) f(y, z)=f(x y, z) f(x, y)^{\varphi_{z}}$ for $x, y, z \in H$.

Let $(f, \varphi)$ be a factor pair of $H$ by $K$, and define

$$
\left(h_{1}, k_{1}\right)\left(h_{2}, k_{2}\right)=\left(h_{1} h_{2}, f\left(h_{1}, h_{2}\right) k_{1}^{\varphi_{2}} k_{2}\right),
$$

for $\left(h_{1}, k_{1}\right),\left(h_{2}, k_{2}\right) \in H \times K$. This gives a group structure on $H \times K$, called the generalized semi-direct product of $H$ by $K$ corresponding to $(f, \varphi)$. We denote this group by $H \times_{(f, \varphi)} K$ or simply by $H \dot{\ltimes} K$.

We now return to our group $K_{n, l}(w)$ and let $K=K_{n, l}(w), H=$ $\left\langle b \mid b^{n}=1\right\rangle$. We define $f: H \times H \longrightarrow K$ by setting $f(x, 1)=$ $f(1, x)=1$, for $x \in H$ and

$$
f\left(b^{i}, b^{j}\right)= \begin{cases}a_{1}^{-l} & (i=j=n-1) \\ a_{1}^{l} & (i+j=n) \text { or }(i+j>n \text { and } i, j \neq n-1) \\ 1 & (\text { otherwise })\end{cases}
$$

where $1 \leq i, j \leq n-1$. Next on taking $\varphi_{b}=\theta$, we observe that $(f, \varphi)$ is a factor pair of $H$ by $K$. So we may consider the group $E_{n, l}(w)=H \times_{(f, \varphi)} K$.

Using [7, Theorem 1, §20] we are able to write down a presentation for $E_{n, l}(w)$ as follows:

$$
\begin{aligned}
&\left\langle a_{1}, \ldots, a_{n}, b\right| a_{1}^{l}=\cdots=a_{n}^{l}, w^{\theta^{k-1}}=1 \quad(k=1, \ldots, n) \\
&\left.a_{1}^{l}=b^{n}, \quad a_{i}^{b}=a_{i+1} \quad(1 \leq i \leq n)\right\rangle
\end{aligned}
$$

Eliminating the redundant generators $a_{2}, \ldots, a_{n}$ gives

$$
E_{n, l}(w) \cong\left\langle a, b \mid a^{l}=b^{n}, w\left(a, a^{b}, \ldots, a^{b^{n-1}}\right)=1\right\rangle
$$

## 3. Applications and examples

In this section we shall apply our construction to study some known deficiency zero groups as well as to exhibit some infinite classes of deficiency zero finite groups. The method may be used for several known examples.

## (a) The group $G_{7}$

Our first example is the group $G_{7}$ which appeared in [4]. This group was constructed in response to a problem posed by Johnson and Robertson [8], for finite soluble groups of deficiency zero with increasing derived length. The group $G_{7}$, which is defined by

$$
\left\langle x, y \mid x^{4}=y^{3}, x^{-2} y^{-1} x^{-1} y^{-1}(x y)^{2} x y^{-1} x y=1\right\rangle
$$

is of order $2^{10} \cdot 3^{9}$ and has derived length seven.
Here we let $n=3, l=4$ and $w=a_{3} a_{2} a_{1} a_{2} a_{1}^{-2} a_{2}^{-1}$, namely we consider the group $K_{3,4}(w)$ defined by

$$
\begin{array}{r}
\left\langle a_{1}, a_{2}, a_{3}\right| a_{1}^{4}=a_{2}^{4}=a_{3}^{4}, a_{3} a_{2} a_{1}=a_{2} a_{1}^{2} a_{2}^{-1}, a_{1} a_{3} a_{2}=a_{3} a_{2}^{2} a_{3}^{-1} \\
\left.a_{2} a_{1} a_{3}=a_{1} a_{3}^{2} a_{1}^{-1}\right\rangle .
\end{array}
$$

Then $G_{7} \cong \mathbb{Z}_{3} \dot{\propto} K_{3,4}(w)$. Using GAP [12], we observe that $K_{3,4}(w)$
is of order $2^{10} \cdot 3^{8}$ and has derived length 6 . As $K_{3,4}(w) \leq G_{7}^{\prime}$, it is readily seen that $G_{7}$ is of derived length 7 .

## b) The group $G(n)$

We next consider the class $G(n)$ of groups

$$
\left\langle x, z \mid z^{n} x z^{n} x^{-1}=z^{n+2} x^{2} z^{2} x^{2}=1\right\rangle \quad(n=2,3, \ldots),
$$

introduced by Niemenmaa and Rosenberger in [10]. The group $G(n)$ has been investigated by Thomas in [14], and he proved that $G(n)$ is torsion-free whenever $n \geq 3$, by introducing a torsion-free subgroup $K$ of index 4 in $G(n)$. The subgroup $K$ has the following presentation:

$$
\left\langle x, y, z, u \mid x^{n}=y^{n}=z^{n}=u^{n}, x^{2 n+2} y^{2} z^{2} u^{2}=1\right\rangle .
$$

We take $w=w\left(a_{1}, a_{2}, a_{3}, a_{4}\right)=a_{1}^{2 n+2} a_{2}^{2} a_{3}^{2} a_{4}^{2}$ in $F_{4}$. Clearly $K=$ $K_{4, n}(w)$ and $E=E_{4, n}(w)$ is given by

$$
\left\langle a, b \mid a^{n}=b^{4}, b^{12}\left(a^{2} b^{-1}\right)^{4}=1\right\rangle
$$

If we take $u=b^{2}, v=b a b^{-1}$, and $L=\langle a, u, v\rangle$, then we find that $K \leq L$ and $L$ is of index 2 in $E$ with presentation

$$
\left\langle a, u, v \mid a^{n}=u^{2}=v^{n},\left(v^{2} a^{2} u\right)^{2}=u^{-2}\right\rangle
$$

Incidentally, $G(n)$ also has $L=L(n)$ as a subgroup of index 2 with $K \leq L$. Thomas in [14] shows that $G(n)$ is torsion-free by proving that $L$ is a torsion-free group. By the same argument given in [14], we deduce that $E_{4, n}(w)$ is a torsion-free group. It is worth noting that $E_{4,2}(w)$ has a subgroup of index 24 isomorphic to $F_{2}$. The proof goes exactly like that of [14] by showing that $L(2)$ is a split extension of $F_{2}$ by a cyclic group of order 12 .

## c) Fibonacci groups

Here we construct an infinite class of finite metabelian groups using the Fibonacci group $F(r, n)$ defined by

$$
F(r, n)=K_{n}\left(a_{1} \ldots a_{r} a_{r+1}^{-1}\right)
$$

where $r \geq 2$ and subscripts are reduced modulo $n$. It was shown in [3] that if $r \equiv 1(\bmod n)$, then $F(r, n)$ is a metacyclic group of order $r^{n}-1$. It is easy to check that each $a_{i}$ has order $n(r-1)$ in $F(r, n)$ and that $a_{1}^{n}=\cdots=a_{n}^{n}$. So that $E_{n, n}\left(a_{1} \ldots a_{r} a_{r+1}^{-1}\right)$ has a 2-generator, 2-relation presentation of the form

$$
\left\langle a, b \mid a^{n} b^{n}=1,(a b)^{r}=b^{r} a\right\rangle
$$

which is metabelian of order $n\left(r^{n}-1\right)$, where $r \equiv 1(\bmod n)$.

## d) A finite insoluble group

We let $n=10, l=2$ and $w=w\left(a_{1}, \ldots, a_{10}\right)=\left(a_{1} a_{2}\right)^{5} a_{10} a_{2} a_{3} a_{2}$. Then $E_{10,2}(w)$ has a presentation of the form

$$
\left\langle a, b \mid a^{2}=b^{10}, a b^{-1} a b\left(a b a b^{-1}\right)^{5} a b^{2} a b^{-2}\right\rangle .
$$

The group $E_{10,2}(w)$ has been introduced in [2] by Campbell et.al., and they have shown $E_{10,2}(w)$ is a finite insoluble group involving $\operatorname{PSU}(3,4)$. This group was then shown to have order 229,934,420 ,352,000 by Newman and O'Brien [9].

## 4. The group $G_{i, j}$

In what follows we aim to construct an infinite family of finite non-metacyclic 3 -groups of deficiency zero having high nilpotency classes. Examples of such $p$-groups, for $p=2,3$, have been
constructed in $[6,11,15]$. Here we consider the Mennicke group $K_{3}\left(\left[a_{1}, a_{2}\right] a_{1}^{-n}\right)$, where $n$ is a given positive integer; for a brief description of the Mennicke group, see [8]. On setting $K=$ $K_{3, m}\left(\left[a_{1}, a_{2}\right] a_{1}^{-n}\right), m=3^{i}$ and $n=3^{j}$ with $1 \leq i \leq j$, we observe that the group $G_{i, j}=E_{3, m}\left(\left[a_{1}, a_{2}\right] a_{1}^{-n}\right)$ has a deficiency zero presentation of the form

$$
\left\langle x, y \mid x^{3^{i}}=y^{3},\left[x, x^{y}\right]=x^{3^{j}}\right\rangle
$$

We now proceed to determine the order and the nilpotency class of $G_{i, j}$.

Lemma 4.1. For any positive integers $m, n$ with $m$ dividing $n$, let

$$
K=\left\langle a, b, c \mid a^{m}=b^{m}=c^{m}, a^{b}=a^{n+1}, b^{c}=b^{n+1}, c^{a}=c^{n+1}\right\rangle .
$$

Then $K$ is of order $m^{3} n$.

Proof. Let $k=n / m$. Adding the new generator $z=c^{m}$ to those of $K$ yields

$$
K=\left\langle a, b, c, z \mid z=c^{m}, z=a^{m}=b^{m}, a^{b}=a z^{k}, b^{c}=b z^{k}, c^{a}=c z^{k}\right\rangle .
$$

Let $H=\langle a, b, z\rangle$. A straightforward application of the modified Todd-Coxeter algorithm can be used to find a presentation for $H$ on the generators of $H$. We suppress the details and merely observe that $|K: H|=m$ and

$$
H=\left\langle a, b, z \mid z^{m k}=1, z=a^{m}=b^{m}, a^{b}=a z^{k}\right\rangle .
$$

Eliminating $z$ gives

$$
H=\left\langle a, b \mid a^{m^{2} k}=1, a^{m}=b^{m}, a^{b}=a^{m k+1}\right\rangle .
$$

Now the subgroup $L=\left\langle b, a^{m}\right\rangle$ of $H$ is of index $m$ in $H$ and a presentation for $L$ is obtained as follows:

$$
L=\left\langle u, v \mid v=u^{m}, v^{m k}=1\right\rangle
$$

where $u=b$, and $v=a^{m}$. Obviously $L$ is a cyclic group of order $m^{2} k$. Consequently $|K|=m^{4} k=m^{3} n$, as required.

Lemma 4.2. Let $G$ be the group

$$
\left\langle x, y \mid x^{m}=y^{3},\left[x, x^{y}\right]=x^{n}\right\rangle
$$

where $m, n$ are positive integers with $m$ dividing $n$. Assume that $a=x, b=x^{y}$, and $c=x^{y^{-1}}$. Then $a, b$, and $c$ satisfy the relations of $K$, the group given in Lemma 4.1. Furthermore
(i) $a^{y}=b, b^{y}=c$, and $c^{y}=a$;
(ii) $a^{r} b^{s}=b^{s} a^{r} z^{r s}$, $b^{r} c^{s}=c^{s} b^{r} z^{r s}$, and $c^{r} a^{s}=a^{s} c^{r} z^{r s}$, for all positive integers $r, s$, where $z=a^{n}$.

Proof. Indeed, the elements $a, b$, and $c$ of $G$ generate the subgroup $K$ of $G$ introduced in Lemma 4.1. Now as $y^{3}$ is a central element of $G, b^{y}=y^{-2} x y^{2}=y x y^{-1}=c$, which proves (i). The second assertion (ii) follows at once by the fact that $a^{n}$ lies in the center of $G$.

Lemma 4.3. With the notation of Lemma 4.2, let

$$
[x, \underbrace{y, \ldots, y}_{k}]=z^{q_{k}} a^{r_{k}} b^{s_{k}} c^{t_{k}}
$$

Then $q_{k}=r_{k} t_{k}$, and

$$
\begin{aligned}
r_{k+2}+3\left(r_{k+1}+r_{k}\right)=0 & \left(r_{1}=-1, r_{2}=1\right), \\
s_{k+2}+3\left(s_{k+1}+s_{k}\right)=0 & \left(s_{1}=1, s_{2}=-2\right), \\
t_{k+2}+3\left(t_{k+1}+t_{k}\right)=0 & \left(t_{1}=0, t_{2}=1\right) .
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
{[x, \underbrace{y, \ldots, y}_{k+1}] } & =\left[a^{r_{k}} b^{s_{k}} c^{t_{k}}, y\right] \\
& =c^{-t_{k}} b^{-s_{k}} a^{-r_{k}}\left(a^{r_{k}} b^{s_{k}} c^{t_{k}}\right)^{y} \\
& =c^{-t_{k}} b^{-s_{k}} a^{-r_{k}} b^{r_{k}} c^{s_{k}} a^{t_{k}} \\
& =z^{\left(t_{k}-r_{k}\right)\left(s_{k}-t_{k}\right)} a^{t_{k}-r_{k}} b^{r_{k}-s_{k}} c^{s_{k}-t_{k}},
\end{aligned}
$$

by Lemma 4.2. Hence $r_{k+1}=t_{k}-r_{k}, s_{k+1}=r_{k}-s_{k}$ and $t_{k+1}=$ $s_{k}-t_{k}$, from which the result follows, as $[x, y]=a^{-1} b$ and $[x, y, y]=$ $z a b^{-2} c$.

Theorem 4.4. Let

$$
G_{i, j}=\left\langle x, y \mid x^{3^{i}}=y^{3},\left[x, x^{y}\right]=x^{3^{j}}\right\rangle \quad(1 \leq i \leq j) .
$$

Then $G_{i, j}$ having order $3^{3 i+j+1}$ is of nilpotency class $2 i+1$.
Proof. On putting $m=3^{i}$ and $n=3^{j}$, we have $\left|G_{i, j}\right|=3|K|=$ $3 m^{3} n$ by Lemma 4.1. To determine the class of $G_{i, j}$ we observe that

$$
G_{i, j} /\left\langle y^{3}\right\rangle \cong\left\langle x, y \mid x^{3^{i}}=y^{3}=\left[x, x^{y}\right]=1\right\rangle,
$$

which is $\mathbb{Z}_{3^{i}} \backslash \mathbb{Z}_{3}$, the standard wreath product of $\mathbb{Z}_{3^{i}}$ and $\mathbb{Z}_{3}$. As this group is of class $2 i+1$, the class of $G_{i, j}$ is either $2 i+1$ or $2 i+2$. We shall see that the latter case cannot occur by showing that the
term $\Gamma_{2 i+2}\left(G_{i, j}\right)$ of the lower central series of $G_{i, j}$ is in fact trivial. To see this we first note that if $k \geq 2$ then $\Gamma_{k}\left(G_{i, j}\right) \leq G_{i, j}^{\prime} \leq K$, whence $\left[\Gamma_{k}\left(G_{i, j}\right), x\right]$ is contained in the center of $G_{i, j}$, by Lemma 4.2(ii). Therefore $\Gamma_{k}\left(G_{i, j}\right)$ is generated by the conjugates in $G_{i, j}$ of the higher commutator $[x, y, \cdots, y]$ of weight $k$. Working with the difference equations stated in Lemma 4.3 leads to

$$
[x, \underbrace{y, \cdots, y}_{2 k+1}]=\left\{\begin{array}{lll}
a^{3^{k} \varepsilon_{k}} b^{-3^{k^{k}} \varepsilon_{k}} & (k \equiv 0 & (\bmod 3)) \\
b^{-3^{k} \varepsilon_{k}} c^{3^{k^{k}} \varepsilon_{k}} & (k \equiv 1 & (\bmod 3)) \\
c^{3^{k} \varepsilon_{k}} a^{-3^{k_{\varepsilon_{k}}}} & (k \equiv 2 & (\bmod 3))
\end{array}\right.
$$

where $\varepsilon_{k}=(-1)^{3 k+1}$. Now it is clear that $\Gamma_{2 i+2}\left(G_{i, j}\right)=1$, as required.

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