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Author(s):

S. Suantai, B. Panyanak and W. Phuengrattana

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A NEW ONE-STEP ITERATIVE PROCESS FOR APPROXIMATING COMMON FIXED POINTS OF A COUNTABLE FAMILY OF QUASI-NONEXPANSIVE MULTI-VALUED MAPPINGS IN CAT(0) SPACES

S. SUANTAI, B. PANYANAK AND W. PHUENGRATTANA*

(Communicated by Ali Ghaffari)

Abstract. In this paper, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in a CAT(0) space. We also prove strong and *Delta*-convergence theorems of the proposed iterative process under some control conditions. Our main results extend and generalize many results in the literature.
Keywords: Fixed point, quasi-nonexpansive multi-valued mappings, CAT(0) spaces.
MSC(2010): Primary: 47H09; Secondary: 47H10; 47J25.

1. Introduction

Let (X, d) be a metric space. A *geodesic* joining x to y (where $x, y \in X$) is a map γ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. Thus γ is an isometry and $d(x, y) = l$. The image of γ is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We write $\alpha x \oplus (1-\alpha)y$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(y, z) = \alpha d(x, y)$ for $\alpha \in [0, 1]$. The space X is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points.

Following [3], a metric space X is said to be a CAT(0) *space* if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane $\mathbb{E}^2$. It is well known that any complete,

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*Corresponding author.

simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces [3], R-trees [18], the complex Hilbert ball with a hyperbolic metric [15], and many others. It follows from [3] that CAT(0) spaces are uniquely geodesic metric spaces. The fixed point theory in CAT(0) spaces was first studied by Kirk [16, 17]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, there have been many researches concerning the existence and the convergence of fixed points for single-valued and multi-valued mappings in such spaces (e.g., see [2, 5, 7, 8, 18–20]).

The study of fixed points for nonexpansive multi-valued mappings using the Pompeiu-Hausdorff metric was initiated by Markin [21]. Different iterative processes have been used to approximate fixed points of nonexpansive and quasi-nonexpansive multi-valued mappings; in particular, Sastry and Babu [24] considered Mann and Ishikawa iterative processes for a multi-valued mapping T with a fixed point p and proved that these iterative processes converge to a fixed point q of T under certain conditions in Hilbert spaces. Moreover, they illustrated that fixed point q may be different from p. Later, in 2007, Panyanak [22] generalized the results of Sastry and Babu [24] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterative processes for a mapping defined on a noncompact domain. Since then, the strong convergence of the Mann and Ishikawa iterative processes for multi-valued mappings has been rapidly developed, and many papers have appeared (e.g., see [6,12,25,28]). Among other things, Shahzad and Zegeye [26] defined two types of Ishikawa iterative processes and proved strong convergence theorems for such iterative processes involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces. Recently, Abkar and Eslamian [1] established strong and *Delta*-convergence theorems for the multi-step iterative process for a finite family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

In this paper, motivated by the above results, we propose a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces and prove strong and *Delta*-convergence theorems for the proposed iterative process in CAT(0) spaces. We finally provide an example to support our main result.

2. **Preliminaries**

For a nonempty set X, we let $\mathcal{P}(X)$ be the power set of X and $2^X = \mathcal{P}(X) - \{\varnothing\}$. For a metric space (X, d), $x \in X$, and $A, B \in 2^X$, let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, $\operatorname{dist}(x, B) = \inf\{d(x, y) : y \in B\}$, and $h(A, B) = \sup\{\operatorname{dist}(x, B) : x \in A\}$.

We now recall some definitions of continuity for multi-valued mappings (see [4,14] for more details). Let (X, d) and (Y, d) be metric spaces. A multi-valued mapping $T : X \to 2^Y$ is said to be

- *Hausdorff upper semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Ty, Tx) < \varepsilon$ for each $y \in B(x, \delta)$;
- *Hausdorff lower semi-continuous* at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that $h(Tx, Ty) < \varepsilon$ for each $y \in B(x, \delta)$;
- *continuous* at x if T is Hausdorff upper and lower semi-continuous at x.

We say that the multi-valued mapping T is continuous if it is continuous at each point in X.

Let D be a nonempty subset of a metric space X. Let $CB(D)$ and $KC(D)$ denote the families of nonempty closed bounded subsets and nonempty compact convex subsets of D, respectively. The *Pompeiu-Hausdorff distance* [23] on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A)\right\} \text{ for } A, B \in CB(D),$$

where $\operatorname{dist}(x, D) = \inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D.

Note that a continuous multi-valued mapping behaves like a continuous single-valued mapping [14], that is, if a multi-valued mapping $T : D \to CB(D)$ is continuous then for every sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} H(Tx_n, Tx) = 0$.

The set of fixed points of a multi-valued mapping $T : D \to CB(D)$ will be denoted by $F(T) = \{x \in D : x \in Tx\}$.

Definition 2.1. A multi-valued mapping $T : D \to CB(D)$ is said to be

(i) *nonexpansive* [21] if $H(Tx, Ty) \le d(x, y)$, for all $x, y \in D$,
(ii) *quasi-nonexpansive* [24] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \le d(x, p)$, for all $x \in D$ and $p \in F(T)$,
(iii) *hemicompact* if for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = p \in D$.

Definition 2.2. A multi-valued mapping $T : D \to CB(D)$ is said to satisfy *condition* (E_μ) provided that

$$\operatorname{dist}(x, Ty) \le \mu \operatorname{dist}(x, Tx) + d(x, y)$$

for each $x, y \in D$. We say that T satisfies *condition* (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Remark 2.3. From the above definitions, it is clear that:

(i) if T is nonexpansive, then T satisfies the condition (E_1);

(ii) if D is compact, then T is hemicompact.

Although the condition (E) implies the quasi-nonexpansiveness for single-valued mappings [13], but it is not true for multi-valued mappings as the following example.

Example 2.4 ([27, Example 1])**.** Let $D = [0, \infty)$ and $T : D \to CB(D)$ be defined by

$$Tx = [x, 2x] \text{ for all } x \in D.$$

Then T satisfies condition (E) and is not quasi-nonexpansive.

Notice also that the classes of (multi-valued) quasi-nonexpansive mappings, continuous mappings and mappings satisfying condition (E) are different (see Examples 2.5-2.7).

Example 2.5 ([13, Example 2])**.** Let $D = [-1, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{x}{1+|x|} \sin(\frac{1}{x}) \right\} & \text{if } \ x \neq 0; \\ \{0\} & \text{if } \ x = 0. \end{cases}$$

Then T is quasi-nonexpansive and does not satisfy condition (E).

Example 2.6 ([5, p. 984])**.** Let $D = [0, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \{x^2\} & \text{if } \ 0 \leq x < 1; \\ \{0\} & \text{if } \ x = 1. \end{cases}$$

Then T is quasi-nonexpansive and is not continuous. Notice also that the mapping $Tx = \{x^2\}$ on $[0, 1]$ is continuous but is not quasi-nonexpansive nor satisfies condition (E).

Example 2.7 ([13, Example 3])**.** Let $D = [-2, 1]$ and $T : D \to CB(D)$ be defined by

$$Tx = \begin{cases} \left\{ \frac{|x|}{2} \right\} & \text{if } \ -2 \leq x < 1; \\ \left\{ -\frac{1}{2} \right\} & \text{if } \ x = 1. \end{cases}$$

Then T satisfies condition (E) and is not continuous.

The notion of the asymptotic center can be introduced in the general setting of a CAT(0) space X as follows: Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we define a mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known by [10] that in a CAT(0) space, the asymptotic center $A(\{x_n\})$ consists of exactly one point.

We now give the definition and collect some basic properties of the Δ-convergence which will be used in the sequel.

Definition 2.8 ([19])**.** A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2.9 ([19])**.** *Every bounded sequence in a CAT(0) space has a Δ-convergent subsequence.*

Lemma 2.10 ([9])**.** *If D is a nonempty closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.11 ([11])**.** *Let $\{x_n\}$ be a sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.12 ([3])**.** *Let X be a geodesic metric space. The following are equivalent:*

(i) *X is a CAT(0) space.*

(ii) *X satisfies the (CN) inequality: If $x, y \in X$ and $\frac{x\oplus y}{2}$ is the midpoint of x and y, then*

$$d\left(z, \frac{x\oplus y}{2}\right)^2 \le \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2, \ \textit{for all } z \in X.$$

The following lemma is a generalization of the (CN) inequality which can be found in [11].

Lemma 2.13. *Let X be a CAT(0) space. Then*

$$d(z, \lambda x \oplus (1-\lambda)y)^2 \le \lambda d(z,x)^2 + (1-\lambda)d(z,y)^2 - \lambda(1-\lambda)d(x,y)^2,$$

for any $\lambda \in [0,1]$ and $x, y, z \in X$.

In 2012, Dhompongsa et al. [8] introduced the following notation in CAT(0) spaces: Let $x_1, \ldots, x_n$ be points in a CAT(0) space X and $\lambda_1, \ldots, \lambda_n \in (0,1)$ with $\sum_{i=1}^n \lambda_i = 1$, we write

(2.1)

$$\bigoplus_{i=1}^n \lambda_i x_i := (1-\lambda_n)\left(\frac{\lambda_1}{1-\lambda_n}x_1 \oplus \frac{\lambda_2}{1-\lambda_n}x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n}x_{n-1}\right) \oplus \lambda_n x_n.$$

The definition of $\bigoplus$ is an ordered one in the sense that it depends on the order of points $x_1, \dots, x_n$. Under (2.1) we obtain that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i x_i, y\right) \leq \sum_{i=1}^{n} \lambda_i d(x_i, y) \text{ for each } y \in X.$$

3. Main results

In this section, we first introduce a new one-step iterative process for a countable family of quasi-nonexpansive multi-valued mappings in CAT(0) spaces. Let D be a nonempty closed convex subset of a CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\} \subset (0,1)$ satisfying $\sum_{i=0}^{n} \lambda_n^{(i)} = 1$.

Note that, if we put

$$W_n^{(m)} = \bigoplus_{i=0}^{m} \delta_n^{(i,m)} y_n^{(i)},$$

where $\delta_n^{(i,m)} = \frac{\lambda_n^{(i)}}{\sum_{j=0}^{m} \lambda_n^{(j)}}$ for $i = 0, 1, \dots, m$, then we get

$$\begin{aligned}
&W_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\delta_n^{(0,m)}}{1 - \delta_n^{(m,m)}} x_n \oplus \frac{\delta_n^{(1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(1)} \oplus \cdots \oplus \frac{\delta_n^{(m-1,m)}}{1 - \delta_n^{(m,m)}} y_n^{(m-1)} \right) \\
&\quad \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\delta_n^{(0,m-1)} x_n \oplus \delta_n^{(1,m-1)} y_n^{(1)} \oplus \cdots \oplus \delta_n^{(m-1,m-1)} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) \left(\frac{\lambda_n^{(0)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} x_n \oplus \frac{\lambda_n^{(1)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(1)} \oplus \cdots \oplus \frac{\lambda_n^{(m-1)}}{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}} y_n^{(m-1)} \right) \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \left(1 - \delta_n^{(m,m)}\right) W_n^{(m-1)} \oplus \delta_n^{(m,m)} y_n^{(m)} \\
&= \frac{\sum\limits_{j=0}^{m-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{m} \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum\limits_{j=0}^{m} \lambda_n^{(j)}} y_n^{(m)}.
\end{aligned}$$

Therefore, the following result holds:

$$W_n^{(m)} = \frac{\sum_{j=0}^{m-1} \lambda_n^{(j)}}{\sum_{j=0}^{m} \lambda_n^{(j)}} W_n^{(m-1)} \oplus \frac{\lambda_n^{(m)}}{\sum_{j=0}^{m} \lambda_n^{(j)}} y_n^{(m)}. \tag{3.2}$$

The following two lemmas are useful and crucial for our main theorems.

Lemma 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, consider the sequence $\{x_n\}$ generated by (3.1). Then, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$.*

Proof. For $p \in \cap_{i=1}^{\infty} F(T_i)$, we have by (3.1) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, p \right) \\ &\leq \sum_{i=0}^{n} \lambda_n^{(i)} d(y_n^{(i)}, p) \\ &= \sum_{i=0}^{n} \lambda_n^{(i)} \operatorname{dist}(y_n^{(i)}, T_i p) \\ &\leq \sum_{i=0}^{n} \lambda_n^{(i)} H(T_i x_n, T_i p) \\ &\leq \sum_{i=0}^{n} \lambda_n^{(i)} d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

This implies that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$. □

Lemma 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. For $x_1 \in D$, consider the sequence $\{x_n\}$ generated by (3.1). If $\lim_{n\to\infty} \lambda_n^{(i)}$ exists for all $i \in \mathbb{N} \cup \{0\}$ and lies in $(0, 1)$, then $\lim_{n\to\infty} dist(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.*

Proof. For each $p \in \cap_{i=1}^{\infty} F(T_i)$, we obtain by (3.1) that

$$d(x_{n+1}, p) = d\left(\bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, p\right) = d\left(\bigoplus_{i=0}^{n} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} y_n^{(i)}, p\right) = d(W_n^{(n)}, p).$$

It follows by Lemma 2.13 and (3.2) that

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d\left(\frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} W_n^{(n-1)} \oplus \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} y_n^{(n)}, p\right)^2 \\
&\leq \frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(W_n^{(n-1)}, p)^2 + \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(y_n^{(n)}, p)^2 \\
&\quad - \frac{\lambda_n^{(n)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} \frac{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n} \lambda_n^{(j)}} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, p)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d\left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} W_n^{(n-2)} \oplus \frac{\lambda_n^{(n-1)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} y_n^{(n-1)}, p\right)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2 \\
&\leq \sum_{j=0}^{n-1} \lambda_n^{(j)} \left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(W_n^{(n-2)}, p)^2 + \frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1} \lambda_n^{(j)}} d(y_n^{(n-1)}, p)^2\right) \\
&= \sum_{j=0}^{n-1} \lambda_n^{(j)} d\left(\frac{\sum\limits_{j=0}^{n-2} \lambda_n^{(j)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} W_n^{(n-2)} \oplus \frac{\lambda_n^{(n-1)}}{\sum\limits_{j=0}^{n-1} \lambda_n^{(j)}} y_n^{(n-1)}, p\right)^2 + \lambda_n^{(n)} d(y_n^{(n)}, p)^2 \\
&\quad - \lambda_n^{(n)} \sum_{j=0}^{n-1} \lambda_n^{(j)} d(W_n^{(n-1)}, y_n^{(n)})^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{n-1}\lambda_n^{(j)}\left(\frac{\sum_{j=0}^{n-2}\lambda_n^{(j)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}d(W_n^{(n-2)},p)^2+\frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}d(y_n^{(n-1)},p)^2\right.\\
&\quad\left.-\frac{\sum_{j=0}^{n-2}\lambda_n^{(j)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}\frac{\lambda_n^{(n-1)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}d(W_n^{(n-2)},y_n^{(n-1)})^2\right)+\lambda_n^{(n)}d(y_n^{(n)},p)^2\\
&\quad-\lambda_n^{(n)}\sum_{j=0}^{n-1}\lambda_n^{(j)}d(W_n^{(n-1)},y_n^{(n)})^2\\
&= \sum_{j=0}^{n-2}\lambda_n^{(j)}d(W_n^{(n-2)},p)^2+\lambda_n^{(n-1)}d(y_n^{(n-1)},p)^2+\lambda_n^{(n)}d(y_n^{(n)},p)^2\\
&\quad-\frac{\lambda_n^{(n-1)}\sum_{j=0}^{n-2}\lambda_n^{(j)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}d(W_n^{(n-2)},y_n^{(n-1)})^2-\lambda_n^{(n)}\sum_{j=0}^{n-1}\lambda_n^{(j)}d(W_n^{(n-1)},y_n^{(n)})^2\\
&\leq \sum_{j=0}^{n-3}\lambda_n^{(j)}d(W_n^{(n-3)},p)^2+\lambda_n^{(n-2)}d(y_n^{(n-2)},p)^2+\lambda_n^{(n-1)}d(y_n^{(n-1)},p)^2\\
&\quad+\lambda_n^{(n)}d(y_n^{(n)},p)^2-\frac{\lambda_n^{(n-2)}\sum_{j=0}^{n-3}\lambda_n^{(j)}}{\sum_{j=0}^{n-2}\lambda_n^{(j)}}d(W_n^{(n-3)},y_n^{(n-2)})^2\\
&\quad-\frac{\lambda_n^{(n-1)}\sum_{j=0}^{n-2}\lambda_n^{(j)}}{\sum_{j=0}^{n-1}\lambda_n^{(j)}}d(W_n^{(n-2)},y_n^{(n-1)})^2-\lambda_n^{(N)}\sum_{j=0}^{n-1}\lambda_n^{(j)}d(W_n^{(n-1)},y_n^{(n)})^2\\
&\ \vdots\\
&\leq \lambda_n^{(0)}d(W_n^{(0)},p)^2+\sum_{k=1}^{n}\lambda_n^{(k)}d(y_n^{(k)},p)^2-\sum_{k=1}^{n}\frac{\lambda_n^{(k)}\sum_{j=0}^{k-1}\lambda_n^{(j)}}{\sum_{j=0}^{k}\lambda_n^{(j)}}d(W_n^{(k-1)},y_n^{(k)})^2\\
&\leq \sum_{k=0}^{n}\lambda_n^{(k)}d(x_n,p)^2-\sum_{k=1}^{n}\frac{\lambda_n^{(k)}\sum_{j=0}^{k-1}\lambda_n^{(j)}}{\sum_{j=0}^{k}\lambda_n^{(j)}}d(W_n^{(k-1)},y_n^{(k)})^2\\
&= d(x_n,p)^2-\sum_{k=1}^{n}\frac{\lambda_n^{(k)}\sum_{j=0}^{k-1}\lambda_n^{(j)}}{\sum_{j=0}^{k}\lambda_n^{(j)}}d(W_n^{(k-1)},y_n^{(k)})^2.
\end{aligned}
$$

This implies that

$$\sum_{k=1}^{n} \frac{\lambda_n^{(k)} \sum\limits_{j=0}^{k-1} \lambda_n^{(j)}}{\sum\limits_{j=0}^{k} \lambda_n^{(j)}} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{3.3}$$

Since $0 < \lambda_n^{(0)} \leq \sum_{j=0}^{k} \lambda_n^{(j)} \leq 1$ for all $k = 1, 2, \ldots, n$, we have $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \lambda_n^{(k)} \sum_{j=0}^{k} \lambda_n^{(j)}$. So, $0 < \lambda_n^{(0)} \lambda_n^{(k)} \leq \frac{\lambda_n^{(k)} \sum_{j=0}^{k-1} \lambda_n^{(j)}}{\sum_{j=0}^{k} \lambda_n^{(j)}}$ for all $k = 1, 2, \ldots, n$. Then (3.3) becomes

$$\sum_{k=1}^{n} \lambda_n^{(0)} \lambda_n^{(k)} d(W_n^{(k-1)}, y_n^{(k)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2. \tag{3.4}$$

By Lemma 3.1 and the condition $\lim_{n\to\infty} \lambda_n^{(i)}$ exists for all $i \in \mathbb{N} \cup \{0\}$ and lies in $(0, 1)$, we get that

$$\lim_{n\to\infty} d(x_n, y_n^{(1)}) = 0 \text{ and } \lim_{n\to\infty} d(W_n^{(k-1)}, y_n^{(k)}) = 0 \text{ for all } k \geq 2. \tag{3.5}$$

Then, for $k \geq 2$, we have

$$\begin{aligned} d(x_n, y_n^{(k)}) &\leq d(x_n, W_n^{(k-1)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &= d\left(x_n, \bigoplus_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} y_n^{(i)} \right) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &\leq \sum_{i=0}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}) \\ &= \sum_{i=1}^{k-1} \frac{\lambda_n^{(i)}}{\sum\limits_{j=0}^{k-1} \lambda_n^{(j)}} d(x_n, y_n^{(i)}) + d(W_n^{(k-1)}, y_n^{(k)}). \end{aligned}$$

This implies by (3.5) that $\lim_{n\to\infty} d(x_n, y_n^{(k)}) = 0$ for all $k \geq 1$. Since $\operatorname{dist}(x_n, T_i x_n) \leq d(x_n, y_n^{(i)})$ for all $i \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. □

In what follows we get a Δ-convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of quasi-nonexpansive multi-valued* *mappings of* D *into* $KC(D)$ *satisfying the condition (E). Assume that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and* $T_i p = \{p\}$ *for all* $i \in \mathbb{N}$ *and* $p \in \cap_{i=1}^{\infty} F(T_i)$. *Suppose that* $\lim_{n\to\infty} \lambda_n^{(i)}$ *exists for all* $i \in \mathbb{N} \cup \{0\}$ *and lies in* $(0,1)$. *Then, the sequence* $\{x_n\}$ *generated by* (3.1) Δ*-converges to a common fixed point of* $\{T_i\}$.

Proof. By Lemmas 3.1 and 3.2, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \cap_{i=1}^{\infty} F(T_i)$ and $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. Thus the sequence $\{x_n\}$ is bounded. We put $\omega_\Delta(x_n) := \bigcup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $u \in \omega_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.9, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{j\to\infty} u_{n_j} = z \in D$. We will show that $z \in T_1 z$. Since $T_1 z$ is compact, for all $j \in \mathbb{N}$, we can choose $y_{n_j} \in T_1 z$ such that $d(u_{n_j}, y_{n_j}) = \operatorname{dist}(u_{n_j}, T_1 z)$ and $\{y_{n_j}\}$ has a convergent subsequence $\{y_{n_k}\}$ with $\lim_{k\to\infty} y_{n_k} = q \in T_1 z$. By condition (E), we have

$$\operatorname{dist}(u_{n_k}, T_1 z) \le \mu \operatorname{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z).$$

Then we have

$$\begin{aligned} d(u_{n_k}, q) &\le d(u_{n_k}, y_{n_k}) + d(y_{n_k}, q) \\ &= \operatorname{dist}(u_{n_k}, T_1 z) + d(y_{n_k}, q) \\ &\le \mu \operatorname{dist}(u_{n_k}, T_1 u_{n_k}) + d(u_{n_k}, z) + d(y_{n_k}, q). \end{aligned}$$

This implies that

$$\limsup_{k\to\infty} d(u_{n_k}, q) \le \limsup_{k\to\infty} d(u_{n_k}, z).$$

By the uniqueness of asymptotic centers, we have $z = q \in T_1 z$. Similarly, it can be shown that $z \in T_i z$ for all $i = 2, \ldots, N$. Then, $z \in \cap_{i=1}^{\infty} F(T_i)$ and so $\lim_{n\to\infty} d(x_n, z)$ exists. Suppose that $u \neq z$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{j\to\infty} d(u_{n_j}, z) &< \limsup_{j\to\infty} d(u_{n_j}, u) \\ &\le \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, z) \\ &= \limsup_{n\to\infty} d(x_n, z) \\ &= \limsup_{j\to\infty} d(u_{n_j}, z). \end{aligned}$$

This is a contradiction, hence $u = z \in \cap_{i=1}^{\infty} F(T_i)$. This shows that $\omega_\Delta(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$.

Next, we show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{p\}$ and let $A(\{x_n\}) = \{q\}$. Since $p \in \omega_\Delta(x_n) \subset \cap_{i=1}^{\infty} F(T_i)$, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists. By Lemma

2.11, we obtain that $p = q$. Hence, the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i\}$. $\square$

The following result is a strong convergence theorem for a countable family of quasi-nonexpansive multi-valued mappings in complete CAT(0) spaces.

Theorem 3.4. *Let D be a nonempty closed convex subset of a complete CAT(0) space X and let $\{T_i\}$ be a countable family of continuous and quasi-nonexpansive multi-valued mappings of D into $CB(D)$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $i \in \mathbb{N}$ and $p \in \cap_{i=1}^{\infty} F(T_i)$. Let the sequence $\{x_n\}$ generated by (3.1) with $\lim_{n\to\infty} \lambda_n^{(i)}$ exist for all $i \in \mathbb{N} \cup \{0\}$ and lie in $(0, 1)$. Assume that one member of the family $\{T_i\}$ is hemicompact. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n)$ for all $i \in \mathbb{N}$. Without loss of generality, we assume that T_1 is hemicompact. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = p \in D$. By continuity of T_i, we have $\lim_{j\to\infty} \operatorname{dist}(x_{n_j}, T_i x_{n_j}) = \operatorname{dist}(p, T_i p)$ for all $i \in \mathbb{N}$. This implies that $\operatorname{dist}(p, T_i p) = 0$ for all $i \in \mathbb{N}$ and hence $p \in \cap_{i=1}^{\infty} F(T_i)$. It follows by Lemma 3.1 that $\{x_n\}$ converges strongly to p. $\square$

Remark 3.5. Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see [3]), all our results immediately apply to any CAT(κ) space with $\kappa \leq 0$.

Finally, we give a numerical example supporting Theorems 3.3 and 3.4.

Example 3.6. Let X be a real line with the Euclidean norm and $D = [0, 1]$. For $x \in D$, $i = 1, 2, \ldots$, we define mappings T_i on D as follows:

$$T_i x = \left[0, \frac{x}{i}\right] \text{ for all } i \in \mathbb{N}.$$

Let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \bigoplus_{i=0}^{n} \lambda_n^{(i)} y_n^{(i)}, \text{ for all } n \in \mathbb{N}, \tag{3.6}$$

where $y_n^{(0)} = x_n$, $y_n^{(i)} \in T_i x_n$ and the sequences $\{\lambda_n^{(i)}\}$ defined by

$$\lambda_n^{(i)} = \begin{cases} \frac{1}{2^{i+1}} \left(\frac{n}{n+1}\right), & n \geq i+1 \\ 1 - \frac{n}{n+1} \left(\sum_{k=1}^{n} \frac{1}{2^k}\right), & n = i \\ 0, & n < i. \end{cases}$$

Obviously, T_i is quasi-nonexpansive and satisfies condition (E) for all $i \in \mathbb{N}$ and $T_i(0) = \{0\}$ such that $\cap_{i=1}^{\infty} F(T_i) = \{0\}$. It can be observed that all the assumptions of Theorems 3.3 and 3.4 are satisfied.

For any arbitrary $x_1 \in D = [0,1]$, we put $y_n^{(i)} = \frac{x_n}{5i}$ for all $i \in \mathbb{N}$. Then, we rewrite the algorithm (3.6) as follows:

$$x_{n+1} = \lambda_n^{(0)} x_n + \frac{\lambda_n^{(1)} x_n}{5} + \frac{\lambda_n^{(2)} x_n}{10} + \cdots + \frac{\lambda_n^{(n)} x_n}{5n}, \quad \text{for all } n \in \mathbb{N},$$

where

$$\left(\lambda_n^{(i)}\right) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{11}{32} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{n}{2(n+1)} & \frac{n}{4(n+1)} & \frac{n}{8(n+1)} & \frac{n}{16(n+1)} & \frac{n}{32(n+1)} & \frac{n}{64(n+1)} & \cdots & \frac{n}{2^i(n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

The values of the sequence $\{x_n\}$ with different n are reported in Table 1.

TABLE 1. The values of the sequence $\{x_n\}$ in Example 3.6.

	$x_1 = 0.11$	$x_1 = 0.95$
n	x_n	x_n
1	0.1100000	0.9500000
2	0.0440000	0.3800000
3	0.0183333	0.1583333
4	0.0081545	0.0704253
5	0.0037986	0.0328065
6	0.0018280	0.0157875
7	0.0009008	0.0077801
8	0.0004520	0.0039036
9	0.0002300	0.0019863
10	0.0001184	0.0010222
$\vdots$	$\vdots$	$\vdots$
17	0.0000014	0.0000118
18	0.0000007	0.0000064
19	0.0000004	0.0000034
20	0.0000002	0.0000019

From Table 1, it is clear that $\{x_n\}$ converges to 0, where $\{0\} = \cap_{i=1}^{\infty} F(T_i)$.

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(Suthep Suantai) Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail address: suthep.s@cmu.ac.th

(Bancha Panyanak) Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand.
E-mail address: bancha.p@cmu.ac.th

(Withun Phuengrattana) Department of Mathematics, Faculty of Science and Technology, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand, and Research Center for Pure and Applied Mathematics, Research and Development Institute, Nakhon Pathom Rajabhat University, Nakhon Pathom 73000, Thailand.
E-mail address: withun_ph@yahoo.com