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A NEW CHARACTERIZATION OF $L_2(q)$ BY THE LARGEST ELEMENT ORDERS

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(Communicated by Ali Reza Ashrafi)

Abstract. We characterize the finite simple groups $L_2(q)$ by the group orders and the largest element orders, where q is a prime or $q = 2^a$, with $2^a + 1$ or $2^a - 1$ a prime.
Keywords: Finite groups, group orders, largest element orders, characterization.
MSC(2010): Primary: 20D60; Secondary: 20D06.

1. Introduction

Throughout this paper, all groups considered are finite and G denotes a group. We denote by $\pi(x)$ the set of prime divisors of a positive integer x and by $\pi(G)$ the set $\pi(|G|)$. $\pi_e(G)$ and $k(G)$ denote the set of element orders of G and the largest one in $\pi_e(G)$, respectively. G is called a simple K_n-group if G is simple with $|\pi(G)| = n$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertices are the primes in $\pi(G)$ and two primes p and q in $\pi(G)$ are connected by an edge if there exists in G an element of order pq. The connected components of $\Gamma(G)$ are denoted by $\pi_i, 1 \leq i \leq t(G)$, where $t(G)$ is the number of connected components of $\Gamma(G)$. In particular, we denote by π_1 the component containing the prime 2 for a group of even order. For the simple groups the notation is standard and readers may refer to [3].

In 1987, the third author of this paper posed the following conjecture:

Conjecture Let G be a group and M a simple group. Then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$.

It is worth to mention that this conjecture has been completely proved by Mazurov et al in [9]. Thus some authors tried to characterize simple groups by using less conditions. For instance, in [6], L.G. He and G.Y. Chen gave a new characterization of linear simple groups $L_2(q)$ with $q = p^n < 125$ by group

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order, the largest, the second largest and the third largest element orders. Later, they also characterized in [7] the simple K_3-groups by using the group orders and the largest and the second largest element orders. On the other hand, the third and the fourth authors of this paper characterized in [12] all simple K_3-groups and some linear groups $L_2(p)$, where p is a prime with $p = 8n \pm 3 > 3$, by using the group order and the largest element order. In this paper, our goal is to show that each simple linear group $L_2(q)$, where either q is a prime or $q = 2^a$, for $a \in \mathbb{N}$, $a \geq 2$ such that $2^a + 1$ or $2^a - 1$ is a prime, can be characterized by the group order and the largest element order. Our main results are the following:

Theorem 1.1 (Theorem A)**.** *Let G be a group and $a \geq 2$ an integer. If either $2^a + 1$ or $2^a - 1$ is a prime, then $G \cong L_2(2^a)$ if and only if $|G| = |L_2(2^a)|$ and $k(G) = k(L_2(2^a))$.*

Theorem 1.2 (Theorem B)**.** *Let G be a group and $p \geq 5$ a prime. If $|G| = |L_2(p)|$ and $k(G) = k(L_2(p))$, then either $G \cong L_2(p)$ or G is a 2-Frobenius solvable group of order 168. In the latter case, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ with $H \in Syl_2(G)$ elementary abelian of order $p+1 = 8$, and $G/K \cong C_3$. Moreover, $6 \in \pi_e(G)$.*

Remark 1.3. Let G be the group of the library of the small groups of size 168 with position 43 in GAP ([5]). Then $\pi_e(G) = \{1, 2, 3, 6, 7\}$ and G has the normal series: $1 \lhd H \trianglelefteq M \trianglelefteq G$ such that $H \cong C_2 \times C_2 \times C_2$ and $M \cong (C_2 \times C_2 \times C_2) \rtimes C_7$. Both M and G/M are Frobenius groups. This shows that the latter case of Theorem B can occur.

Remark 1.4. Theorem A relies on the Classification of Simple Groups and Theorem B is a generalization of [12, Theorem 1.2].

2. **Preliminaries**

Before taking up the problem, we resume some useful known results.

Recall that G is a 2-Frobenius group if G has a normal series $1 \lhd H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with K/H and H as Frobenius kernels respectively. We call such series $1 \lhd H \trianglelefteq K \trianglelefteq G$ a 2-Frobenius series.

Lemma 2.1 ([10, Theorem])**.** *Let G be a group such that $t(G) \geq 2$. Then G has one of the following structures:*

(a) *G is a Frobenius group or a 2-Frobenius group.*
(b) *G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that $\pi(N) \cup \pi(G/M) \subseteq \pi_1$ and M/N is a non-abelian simple group.*

Lemma 2.2 ([2, Theorem 2])**.** *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \lhd H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, $|G/K|$ divides $|Aut(K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$ and G is solvable.*

Lemma 2.3 ([1, Theorem 1])**.** *Let G be a non-abelian simple group and p a prime. If $p \mid |G|$ and $p > |G|^{1/3}$, then $p \geq 5$ and G is isomorphic either to $L_2(p)$ or to $L_2(p-1)$ and p is a Fermat prime.*

3. Proof of Theorem A

Proof. It is obvious that the necessity holds. We prove the sufficiency. Assume that $|G| = 2^a(2^a-1)(2^a+1) = |L_2(2^a)|$ and $k(G) = k(L_2(2^a))$, where either 2^a+1 or 2^a-1 is a prime. Then $k(G) = 2^a+1$ by [8, II, Satz 8.5]. We divide the proof into two cases.

Case 1. 2^a+1 is a prime.

Write $p := 2^a+1$. Then $p \geq 5$ as $a \geq 2$. Observe that $|G| = p(p-1)(p-2)$ and $k(G) = p$. So $\{p\}$ is a component of $\Gamma(G)$, which implies that $t(G) \geq 2$ and Lemma 2.1 applies.

Suppose first that $G = F \rtimes H$ is a Frobenius group with Frobenius kernel F and Frobenius complement H. If $p \mid |F|$, then $|F| = p$ as $\{p\}$ is a component of $\Gamma(G)$ and F is nilpotent, which yields that $|H| = (p-1)(p-2)$. Moreover, $|H| \mid |F|-1$ leads to $(p-1)(p-2) \mid p-1$. Thus $p = 3$, against $p \geq 5$. Hence $p \mid |H|$. Let $r \in \pi(F)$ and F_r be a Sylow r-subgroup of F. It is clear that $F_r \rtimes H$ is also a Frobenius group with Frobenius kernel F_r and complement H. Note that $|F| \mid (p-1)(p-2)$. We obtain that $|F_r|$ either divides $p-1$ or $p-2$ since $(p-1, p-2) = 1$. Thus $|F_r| - 1 \leq p-2$. On the other hand, $p \mid |H|$ and $|H| \mid |F_r|-1$, this is a contradiction.

Assume then that G is a 2-Frobenius group. It follows by Lemma 2.2 that G has a 2-Frobenius series $1 \lhd H \unlhd K \unlhd G$ such that $|K/H| = p$ and $|G/K| \mid p-1$, implying $p-2 \mid |H|$. Write $K = H \rtimes A$, where A is a cyclic group of order p. Let H_1 be a subgroup of H of order $p-2$, which exists according to the nilpotency of H. Then $H_1 \rtimes A$ is also a Frobenius group with Frobenius kernel H_1 and complement A. Consequently, $|A| \mid |H_1|-1$. That is, $p \mid p-3$, also a contradiction.

Therefore, it follows by Lemma 2.1 that G has a normal series $1 \unlhd N \unlhd M \unlhd G$ such that $\pi(N) \cup \pi(G/M) \subseteq \pi_1$ and M/N is a non-abelian simple group. Further, $p \mid |M/N|$ and $|M/N| < p^3$, which implies that $M/N \cong L_2(p)$ or $L_2(p-1)$ according to Lemma 2.3. If $M/N \cong L_2(p)$, then $|M/N| = p(p+1)(p-1)/2$, leading to $p(p+1)(p-1) \mid 2p(p-1)(p-2)$. This shows that $p+1 \mid 2(p-2)$. Since $2(p-2) = 2(p+1)-6$, we get $p+1 \mid 6$. By $p \geq 5$, we then get that $p = 5$ and thus $a = 2$. In this case, $M/N \cong L_2(5) \cong L_2(2^2)$. Consequently, $|M/N| = |G|$, yielding $N = 1$ and $M = G \cong L_2(2^2)$. Assume then that $M/N \cong L_2(p-1) \cong L_2(2^a)$. This forces $G \cong L_2(2^a)$, as required.

Case 2. 2^a-1 is a prime.

Write $p := 2^a-1$. Obviously, a is a prime and $p \geq 3$. We obtain that $|G| = p(p+1)(p+2)$ and $k(G) = p+2$ by [8, II, Satz 8.4], which implies that $\{p\}$ is a component of $\Gamma(G)$. Moreover, $t(G) \geq 2$.

Suppose first that $G = F \rtimes H$ is a Frobenius group with Frobenius kernel F and a Frobenius complement H. If $p \mid |F|$, then $|F| = p$ as $\{p\}$ is a component of $\Gamma(G)$ and F is nilpotent, indicating that $|H| = (p+1)(p+2)$. Furthermore, $|H| \mid |F| - 1 = p - 1$, a contradiction. Thus $p \mid |H|$. Let $r \in \pi(F)$ and F_r be a Sylow r-subgroup of F. Then $F_r \rtimes H$ is also a Frobenius group with Frobenius kernel F_r and complement H. Note that $|F| \mid (p+1)(p+2)$. We obtain that $|F_r|$ either divides $p+1$ or $p+2$ since $(p+1, p+2) = 1$, leading that $|F_r| - 1 \leq p+1$. Moreover, $p \mid |H|$ and $|H| \mid |F_r| - 1$, which yields to $|F_r| = p+1$ and thus $r = 2$. As a result, $\{2\}$ is also a component of $\Gamma(G)$. Since $t(G) = 2$ by [4, Lemma 1], the argument above shows that G only has two distinct primes, which is contrary to the fact that $|G| = p(p+1)(p+2)$.

Suppose then that G is a 2-Frobenius group. It follows, by [4, Lemma 2], that $t(G) = 2$ and that G has a 2-Frobenius series $1 \lhd H \unlhd K \unlhd G$ such that $|K/H| = p$ and $|G/K| \mid p-1$. Let $K = H \rtimes A$ with A a cyclic subgroup of order p. Let H_1 be a Hall $\pi(p+2)$-subgroup of H. Then $H_1 \rtimes A$ is also a Frobenius group with Frobenius kernel H_1 and Frobenius complement A, yielding that $|A| \mid |H_1| - 1$. Notice that $|H_1| = (p+2)/t$ for some integer t. This shows that p divides $(p+2)/t - 1$, a contradiction.

Consequently, it follows, by Lemma 2.1, that G has a normal series $1 \unlhd N \unlhd M \unlhd G$ such that $\pi(N) \cup \pi(G/M) \subseteq \pi_1$ and M/N is a non-abelian simple group. In particular, $p \mid |M/N|$. If $p = 3$, then $|G| = 60$ and so $|M/N|$ divides 60; but the only non-abelian simple group of order less or equal to 60 is A_5. Thus $M/N \cong A_5$ and consequently $G \cong A_5 \cong L_2(4)$. Let next $p \geq 5$. We show that $N = 1$. Assume the contrary. Then $|M/N| \leq p(p+1)(p+2)/2 \leq p^3$ and, from Lemma 2.3, recalling that $p \mid |M/N|$, we get that $M/N \cong L_2(p)$, with p a prime or $M/N \cong L_2(p-1)$ with p a Fermat prime. If $M/N \cong L_2(p)$, then $p(p+1)(p-1)/2 \mid p(p+1)(p+2)$, implying $p = 7$ and thus $a = 3$. Note that $|G| = 3|M/N|$ and 9 divides $|G|$. We obtain that $|N| = 3$. Let $P_7 \in \mathrm{Syl}_7(G)$. Then $|P_7| = 7$. On the other hand, $N \rtimes P_7 \leq G$. Since $k(G) = 9$, we conclude that $N \rtimes P_7$ is a Frobenius group. However, in this situation, we have $|P_7| \mid |N| - 1$, which is a contradiction. Thus $M/N \cong L_2(p-1)$, where $p - 1 = 2(2^{a-1} - 1)$ is 2. This implies that $a = 2$ and thus $p = 3$, against our assumption $p \geq 5$. Consequently, $N = 1$.

Suppose that $M \neq G$. Then M is a non-abelian simple group satisfying $|M| \leq |G|/2 < p^3$. It follows by Lemma 2.3 that either $M \cong L_2(p)$ or $M \cong L_2(p-1)$. If the former holds, then $p(p+1)(p-1)/2 \mid p(p+1)(p+2)$, leading to $p = 7$ and thus $a = 3$, indicating that $|G/M| = 3$. However, $G/M \leq \mathrm{Out}(M) = \mathrm{Out}(L_2(7)) \cong C_2$, a contradiction. This shows that $M \cong L_2(p-1)$, where $p-1$ is a power of 2. So $p = 3$ against $p \geq 5$. Hence $G = M$ is a simple group.

Note that $|G| = p(p+1)(p+2)$. By [3], it follows that G is not a sporadic simple group. On the other hand, if $G \cong A_n$, then $p \leq n \leq p+2$ since, otherwise, there is an element of order $3p$, which is contrary to the fact that

$k(G) = p+2$. Assume first that $n = p+2$. Then by considering the group orders, we obtain that $p(p+1)(p+2) = \frac{(p+2)!}{2}$, which is impossible being $p \geq 5$. Assume next that $n = p+1$. Then $p(p+1)(p+2) = \frac{(p+1)!}{2}$, yielding $p+2 = \frac{(p-1)!}{2}$. As a result, $p+2 \geq 2(p-1)$, leading to $p \leq 4$, against $p \geq 5$. Consequently, $n = p$. Then $|G| = |A_p|$ and $2(p+1)(p+2) = (p-1)!$, a contradiction. Therefore, G is a simple group of Lie type. We discuss them by a case by case analysis, showing that we always get a contradiction up to $G \cong L_2(2^a)$. Let q denote a prime power.

1. $G \cong B_n(q)$ with $n \geq 2$, or $C_n(q)$ with $n \geq 3$.

Here $p(p+1)(p+2) = \frac{1}{(2,q-1)} q^{n^2} \prod_{i=1}^{n} (q^{2i}-1)$. If $p \mid q$, then q is a power of p, which is impossible since $n \geq 2$. Hence $p \mid q^{2t}-1$ for some $1 \leq t \leq n$. On the other hand, $q^{n^2} \mid p+1$ or $p+2$ as $(p+1, p+2) = 1$. As a consequence, $q^{n^2} \leq p+2 \leq q^{2t}+1 \leq q^{2n}+1$, implying $n = 2$. Moreover, $q^4 = p+2$ or $q^4+1 = p+2$. If the latter holds, then $p = q^4-1 = (q^2-1)(q^2+1)$, against p a prime. Hence $q^4 = p+2$. By considering the group orders, we see that $(q^4-2)(q^4-1)q^4 = \frac{1}{2} q^4 (q^2-1)(q^4-1)$, which implies that $q^4-2 = \frac{1}{2}(q^2-1)$, a contradiction.

2. $G \cong D_n(q)$ or ${}^2D_n(q)$ with $n \geq 4$.

If $G \cong D_n(q)$, then $p(p+1)(p+2) = \frac{1}{(4,q^n-1)} q^{n(n-1)} (q^n-1) \prod_{i=1}^{n-1} (q^{2i}-1)$. Since the p-part of $|G|$ is p and $n(n-1) > 4$, we get $p \nmid q$. As a result $p \mid q^n-1$ or $q^{2t}-1$ for some $1 \leq t \leq n-1$. Assume that the former holds. Note that $q^{n(n-1)} \mid p+1$ or $p+2$. We see that $q^{n(n-1)} \leq p+2 \leq (q^n+1)$, implying $n = 2$, a contradiction. This forces that $p \mid q^{2t}-1$ for some $1 \leq t \leq n-1$. Further, $q^{n(n-1)} \leq p+2 \leq q^{2t}+1 \leq q^{2(n-1)}+1$, also implies that $n = 2$, again a contradiction. As a result, $G \ncong D_n(q)$. Similarly it is checked that $G \ncong {}^2D_n(q)$.

3. $G \cong {}^2A_n(q)$ with $n \geq 2$.

Here $p(p+1)(p+2) = \frac{1}{(n+1,q+1)} q^{\frac{1}{2}n(n+1)} \prod_{i=1}^{n} (q^{i+1}-(-1)^{i+1})$. Since the p-part of $|G|$ is p and $n \geq 2$, we obtain that $p \mid q^{t+1}-(-1)^{t+1}$ for some $1 \leq t \leq n$. Note that $q^{\frac{1}{2}n(n+1)} \mid p+1$ or $p+2$. Hence $q^{\frac{1}{2}n(n+1)} \leq p+2 \leq q^{t+1}-(-1)^{t+1}+2 \leq q^{n+1}+3$, which implies that $n = 2$ and $q^3 \in \{p-1, p, p+1, p+2\}$. Recall that $q^3 \mid p+1$ or $p+2$. If $q^3 \mid p+1 = 2^a$, then q is even and thus $q = 2^b$, for some b with $3b \leq a$. It follows that $q^3 \in \{p-1, p+1\}$. If $q^3 = p+1$, then $2^{3b} = 2^a$ and so $3b = a$, against a a prime. If $q^3 = p-1$, then $2^b = 2(2^{a-1}-1)$, which gives $a = 2, b = 1$. But then $p = 3$ against the fact that we are dealing with $p \geq 5$. It follows that $q^3 \mid p+2$ and, in particular, q is odd. Thus $q^3 = p+2$ and we have $(q^3-2)(q^3-1)q^3 = \frac{1}{(3,q+1)} q^3 (q^2-1)(q^3+1)$ leading to $(3, q+1)(q^3-2)(q^3-1) = (q^2-1)(q^3+1)$, which is easily checked as impossible.

4. $G \cong E_8(q), E_6(q), E_7(q)$ or $F_4(q)$.

Assume that $G \cong E_8(q)$, then $p(p+1)(p+2) = q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$. Then $p \mid q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$. Since the p-part of $|G|$ is p, we obtain that $p \mid q^t - 1$ for some $t \in \{30, 24, 20, 18, 14, 12, 8, 2\}$. On the other hand, $q^{120} \mid p+1$ or $q^{120} \mid p+2$, indicating that $q^{120} \leq p+2 \leq q^{30}+1$, a contradiction. Similarly it is checked that $G \ncong E_6(q), E_7(q)$ or $F_4(q)$.

5. $G \cong G_2(q)$.

Here $p(p+1)(p+2) = q^6(q^6-1)(q^2-1)$. Moreover, $p \mid (q^6-1)(q^2-1)$ and $(q^6-1)(q^2-1) \mid p(p+1)(p+2)$. Since the p-part of $|G|$ is p, we obtain that $p \mid q^6 - 1$. Note that $q^6 \mid p+1$ or $q^6 \mid p+2$. Then $q^6 \leq p+2 \leq q^6+1$, this forces that $q^6 = p+2$ or $q^6 = p+1 = 2^a$. If the latter case holds, then $6 \mid a$, against a a prime. Hence $q^6 = p+2$. This indicates that $q^6(q^6-1)(q^6-2) = q^6(q^6-1)(q^2-1)$, leading to $q^6 = q^2+1$, a contradiction.

6. $G \cong {}^2E_6(q)$.

Here $p(p+1)(p+2) = \frac{1}{(3,q+1)} q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$, which implies $p \mid (q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$. Since the p-part of $|G|$ is p, we obtain that $p \mid q^t - (-1)^t$ for some $t \in \{12, 9, 8, 6, 5, 2\}$. Note that $q^{36} \mid p+1$ or $q^{36} \mid p+2$. Then $q^{36} \leq p+2 \leq q^t - (-1)^t + 2 \leq q^{12}+3$, a contradiction.

7. $G \cong {}^2B_2(q)$ or ${}^2F_4(q)$, where $q = 2^{2m+1}$ with $m \geq 1$.

Suppose that $G \cong {}^2B_2(q)$, where $q = 2^{2m+1}$ with $m \geq 1$. Then $p(p+1)(p+2) = q^2(q^2+1)(q-1)$, which implies that $2^a(2^a+1)(2^a-1) = 2^{4m+2}(2^{4m+2}+1)(2^{2m+1}-1)$ with $m \geq 1$. Thus $a = 4m+2$, which being a a prime, gives $2m+1 = 1$, contrary to $m \geq 1$. Hence, $G \ncong {}^2B_2(q)$. Similarly, $G \ncong {}^2F_4(q)$.

8. $G \cong {}^2G_2(q)$, where $q = 3^{2n+1}$ with $n \geq 1$.

Here $p(p+1)(p+2) = q^3(q^3+1)(q-1)$, which implies $p \mid q^3+1$ or $p \mid q-1$. Moreover, $q^3 \mid p+2$, since, otherwise, $q^3 \mid p+1 = 2^a$, a contradiction. If $p \mid q^3+1$, then $q^3 \leq p+2 \leq q^3+3$, which, being $p+2$ and q odd, implies that $p+2 \in \{q^3, q^3+2\}$. If $q^3 = p+2$, then we get $(q^3-2)(q^2+q+1) = q^3+1$, which is clearly impossible. On the other hand, $p+2 = q^3+2$ gives $p = q^3$, against p a prime.

9. $G \cong {}^3D_4(q)$.

Here $p(p+1)(p+2) = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. Moreover, $p \mid q^8+q+1$ or $p \mid q^t - 1$ for some $t \in \{6, 2\}$. Note that $q^{12} \mid p+1$ or $p+2$. If $p \mid q^t - 1$ for some $t \in \{6, 2\}$, then $q^{12} \leq p+2 \leq q^t+1 \leq q^6+1$, a contradiction. This shows that $p \mid q^8+q^4+1$. Similarly, $q^{12} \leq p+2 \leq q^8+q^4+3$, again a contradiction.

10. $G \cong L_{n+1}(q)$ with $n \geq 1$.

From $p(p+1)(p+2) = \frac{1}{(n+1,q-1)} q^{\frac{1}{2}n(n+1)} \prod_{i=1}^{n}(q^{i+1}-1)$, we obtain that $p \mid (q^{t+1}-1)$ for some $1 \leq t \leq n$. On the other hand, $q^{\frac{1}{2}n(n+1)} \mid p+1$ or $p+2$, so that $q^{\frac{1}{2}n(n+1)} \leq p+2 \leq q^{n+1}+1$, which implies that $n=2$ and $q^3=p+1$ or $p+2$. If the former holds, then $q^3=2^a$, leading to $q=2$ and $a=3$ since a is a prime. This shows that $G \cong L_3(2)$ and thus $|G| = 2^3 \cdot 3 \cdot 7$. On the other hand, we have $|G| = 2^a(2^a-1)(2^a+1) = 2^3 \cdot 7 \cdot 3^2$, a contradiction. If $q^3 = p+2$, we see that $q^3(q^3-1)(q^3-2) = q^3(q^2-1)(q^3-1)$, implying $q^3-q^2=1$. This contradiction shows $n=1$ and thus $G \cong L_2(q)$.

Assume that q is odd. Then $\frac{1}{2}q(q-1)(q+1) = p(p+1)(p+2)$, implying $q \mid p$ or $q \mid p+2$. If $q \mid p$, then $p=q$. We see then that $\frac{1}{2}p(p^2-1) = p(p+1)(p+2)$. This contradiction shows that $q \mid (p+2)$. On the other hand, it follows that $p \mid \frac{1}{2}q(q-1)(q+1)$, yielding to $p \mid \frac{q-1}{2}$ or $p \mid \frac{q+1}{2}$. It follows that $q \leq p+2 \leq \frac{q+5}{2}$, which leads to $q \leq 5$ and $p=3$, against $p \geq 5$. As a result, q is a power of 2. Let $q=2^s$ for some positive integer s. By comparing the orders of $L_2(2^s)$ and $L_2(2^a)$, we obtain that $s=a$ and $G \cong L_2(2^a)$. □

4. Proof of Theorem B

Proof. Let G be a group such that $|G| = p(p+1)(p-1)/2 = |L_2(p)|$ and $k(G) = k(L_2(p)) = p$, where $p \geq 5$ is a prime. Then $\{p\}$ is a component of $\Gamma(G)$, which implies that $t(G) \geq 2$ and Lemma 2.1 applies.

First we show that G is not a Frobenius group. Let $G = F \rtimes H$ be a Frobenius group with Frobenius kernel F and Frobenius complement H. Then, by the Frobenius partition, we have that $p \mid |F|$ or $p \mid |H|$. If the former holds, then $|F| = p$ since $\{p\}$ is a component of $\Gamma(G)$ and F is nilpotent, yielding to $|H| = (p^2-1)/2$. Since $|H| \mid |F|-1$, this forces $(p+1)(p-1)/2 \mid p$, which is a contradiction. Hence $p \mid |H|$. Let $r \in \pi(F)$ and F_r be a Sylow r-subgroup of F. Since F_r is characteristic in F, we have that $F_r \rtimes H$ is also a Frobenius group with Frobenius kernel F_r and complement H. Note that $|F| \mid (p+1)(p-1)/2$. We obtain that $|F_r|$ either divides $(p+1)/2$ or $(p-1)/2$, because $(p+1)/2$ and $(p-1)/2$ are coprime. Thus $p \leq |H| \leq (p-1)/2$, a contradiction.

We suppose then that G is a 2-Frobenius group. It follows, by Lemma 2.2, that G has a 2-Frobenius series $1 \lhd H \unlhd K \unlhd G$ such that $|K/H| = p$ and $|G/K| \mid p-1$. Write $K = H \rtimes A$, where A is a cyclic group of order p. We show that $\pi(H) = \{2\}$. Assume the contrary and let $q \in \pi(H)$ with $q \neq 2$. Let H_q be a Sylow q-subgroup of H. Since $(p+1, p-1) = 2$, we see that $|H_q|$ either divides $(p+1)/2$ or $(p-1)/2$, indicating that $|H_q| \leq (p+1)/2$. On the other hand, since $H_q \rtimes A$ is also a Frobenius group of Frobenius kernel H_q and complement A, we also have $p \mid |H_q|-1$, so $|H_q| \geq p+1$, a contradiction. Thus we have shown that $|H| = 2^a$, for a suitable $a \in \mathbb{N}$.

Next we show that $2^a = p+1$. Recall that we have $p \mid 2^a - 1$ and thus $2^a \geq p+1$. In particular, being $p \geq 5$, we get that $a \geq 3$ and so $p \geq 7$. Moreover, we have $2^a \mid p^2-1$, so that $p^2-1 = 2^a u$, for some $u \in \mathbb{N}$. By $2^a \equiv 1 \pmod{p}$, we get immediately $-1 \equiv u \pmod{p}$. That is, $p \mid u+1$. In particular,

$u \geq p-1$ and then $p^2-1 = 2^a u \geq 2^a(p-1)$. It follows that $p+1 \geq 2^a$ and so $2^a = p+1$.

We now show that H admits no proper A-invariant subgroup. By contradiction, let $1 < U < H$ be a A-invariant subgroup. Then the group $U \rtimes A$ is a Frobenius group and thus $p \mid |U|-1$. In particular, $p \leq |U|-1$. On the other hand, being $U < H$, we also have $|U| < |H|$, so that $|U| \leq p$, it follows that $p \leq |U|-1 \leq p-1$, a contradiction. Consider now $\Phi(H)$. Since this group is characteristic in H, then it is A-invariant. Moreover, by definition, $\Phi(H) < H$. Thus necessarily, we have $\Phi(H) = 1$ and H is an elementary abelian 2-group of order $2^a = p+1$. In particular, G is solvable.

Moreover, for every $s \in \pi((p-1)/2)$, we have $2s \in \pi_e(G)$. Let $x \in G$ be of order s and note that $s \neq 2, p$. Then $H\langle x\rangle \leq G$. If x acts fixed-point-freely on H, then $H\langle x\rangle \leq G$ is a Frobenius group with kernel H and complement $\langle x\rangle$, so that $s \mid 2^a - 1 = p$, which is a contradiction. Thus there exists $y \in H \setminus \{1\}$ such that $xy = yx$.

By Schur-Zassenhaus theorem, H has a complement L in G. Moreover, $G/H \cong L$ is a Frobenius group with kernel K/H. Let $L = A \rtimes B$ be a Frobenius group with kernel A and complement B, respectively. Then $G = HAB$, where $|A| = p$ and $|B| = \frac{p-1}{2} = 2^{a-1}-1$.

Assume that $C_G(H) > H$ as H is abelian. Write $C_G(H) = H \times T$. Since $C_G(H) \trianglelefteq G$, we have $T \trianglelefteq G$. If $p \mid |T|$, then $2p \in \pi_e(G)$, against $k(G) = p$. This implies that T is a normal $\pi((p-1)/2)$-subgroup of G. Recall that G is solvable and B is a Hall $\pi((p-1)/2)$-subgroup of G. It follows that $T \leq B$. Moreover, $T \times A \leq G$, contrary to the fact that $L = A \rtimes B$ is a Frobenius group.

As a result, $C_G(H) = H$. Further, $G/H \leq \mathrm{Aut}(H)$. This indicates that H has a Frobenius group of automorphisms. By [11, Theorem 1(a)], we obtain that $|H| = |C_H(B)|^{|B|}$. Let $|C_H(B)| = 2^m$ for some positive integer $m \leq a$. Then $2^a = (2^m)^{(2^{a-1}-1)}$, leading to $a = m(2^{a-1}-1)$. We see easily that $a = 3$ and $m = 1$. Consequently, G is a 2-Frobenius group with order 168 with $6 \in \pi_e(G)$, as required.

We finally assume that G has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ such that $\pi(N) \cup \pi(G/M) \subseteq \pi_1$ and M/N is a non-abelian simple group. We easily see that $p \mid |M/N|$. Moreover, since $p \geq 5$, we have $|M/N|$ divides $|G|$ and $|G| < p^3$, which implies that $M/N \cong L_2(p)$ or $L_2(p-1)$ by Lemma 2.3. If $M/N \cong L_2(p-1)$, then $|L_2(p-1)|$ divides $|G|$, leading to $p(p-1)(p-2) \mid p(p-1)(p+1)/2$ and forcing $p = 5$. In this case, $M/N \cong L_2(4) \cong L_2(5)$ implies that $G \cong L_2(5)$, as required. To close, assume that $M/N \cong L_2(p)$. Then, clearly, $M = G \cong L_2(p)$. □

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