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A RADICAL EXTENSION OF THE CATEGORY OF S-SETS

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Abstract. Let **S-Set** be the category of S-sets, that is sets equipped with an action of a semigroup S on them. Also, let **S-Pos** be the category of S-posets, that is posets together with the actions compatible with the orders on them. In this paper we show that the category **S-Pos** is a radical extension of **S-Set**; that is there is a radical on the category **S-Pos**, the order desolator radical, whose torsion-free class is **S-Set**.

To do this, first we give a precise definition of a radical on the category **S-Pos** and construct some functors between the above mentioned categories and finally we show that **S-Pos** is a radical extension of **S-Set**.
Keywords: Radical, S-set, S-poset.
MSC(2010): Primary: 18E40; Secondary: 20M30, 06F05.

1. Introduction

The category of S-sets as well as that of ordered algebraic structures, in particular, the category **S-Pos** of partially ordered S-sets, have always been of interest to mathematicians (see, for example, [2–5, 9, 10]).

The purpose of the present paper is to give a new perspective of the category **S-Pos**. In fact, we use the notion of a radical and demonstrate that **S-Pos** may be intuitively, but reasonably and accurately, characterized as a radical extension of **S-Set**. That is, there exists a radical, namely the *order desolator radical*, on **S-Pos** whose torsion-free class is **S-Set**, as a subclass of **S-Pos** (see Section 4).

Although the radical and the torsion theory for S-sets were introduced and investigated by Verena Guruswami [8], to generalize this notion on **S-Pos**, we follow and generalize the category theoretical view of M.M. Clementino, D. Dikranjan and W. Tholen in [7]. In our investigations, it seems necessary to define the radical in a more general manner than what is given in [7], because

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S-Pos is not necessarily a normal category. Indeed, as a topos, the category of S-sets is Barr-exact and protomodular, but lacks a zero object.

In this paper, we first introduce a preradical and a radical on **S-Pos** and examine the algebraic and the categorical properties of these notions. We then introduce, in the third section, the free, or ordering, functor from **S-Set** to **S-Pos** and the order desolator functor from **S-Pos** to **S-Set** and investigate their relationship. Finally, in the last short section, we demonstrate our claim that **S-Pos** is a radical extension of **S-Set**. Now we briefly recall the necessary concepts needed in the sequel.

Recall that, for a semigroup S, a (*left*) S*-set* is a set A equipped with an action $S \times A \to A$ such that, denoting the image of (s,a) by sa, $(st)a = s(ta)$, for every $a \in A$ and $s,t \in S$. The category of all S-sets with *action-preserving* (or *equivariant*) maps between them ($f : A \to B, f(sa) = sf(a)$) will be denoted by **S-Set**. We refer the readers to [9] for further information on the category of S-sets.

A semigroup S is said to be a *partially ordered semigroup* (or simply, a *posemigroup*) if it is a partially ordered set (poset) whose partial order is compatible with the binary operation; that is for all $s, s_1, s_2 \in S$,

$$s_1 \leq s_2 \Rightarrow ss_1 \leq ss_2 \quad \text{and} \quad s_1 s \leq s_2 s.$$

Throughout this paper S is a posemigroup unless stated otherwise.

A poset $(A, \leq)$ is called a (*left*) S*-poset* if A is a (left) S-set such that the action of S satisfies

$$a \leq b \Rightarrow sa \leq sb \quad \text{and} \quad s \leq t \Rightarrow sa \leq ta,$$

for all $a, b \in A$ and $s, t \in S$. In this paper, S-sets are all left S-sets. The morphisms of S-posets are equivariant as well as order preserving maps and they are called S*-poset morphisms*. The family of S-posets and the morphisms between them form a category which is denoted by **S-Pos**. Definitions and results about the category **S-Pos** can be found in [5].

A poset $(B, \leq_B)$ is said to be a *sub-*S*-poset* of an S-poset $(A, \leq_A)$ if B is a sub-S-set of A and $\leq_B \, = \, \leq_A \cap B^2$.

A *regular* S*-poset epimorphism* is the coequalizer of a pair of S-poset morphisms and dually a *regular* S*-poset monomorphism* is the equalizer of a pair of S-poset morphisms. Similarly to [5], it is proved that an S-poset monomorphism $f : A \to B$ is regular if and only if f is an order-embedding; that is,

$$f(a_1) \leq f(a_2) \Leftrightarrow a_1 \leq a_2.$$

Given an S-set A, an equivalence relation θ on A is called an S*-set congruence* (or briefly a *congruence* on A) if $a\theta a'$ implies $(sa)\theta(sa')$ for every $a, a' \in A$ and $s \in S$. The *diagonal congruence* Δ_A on A is the set $\{(a,a) \mid a \in A\}$ and the *all congruence* $A \times A$ is denoted by ∇_A. For any binary relation θ on an S-poset

A, one can define the relation $\leq_\theta$ on A as follows:

$$a \leq_\theta b \Leftrightarrow \exists a_1, \cdots, a_n, b_1, \cdots, b_n;\ a \leq a_1 \theta b_1 \leq \cdots \leq a_n \theta b_n \leq b.$$

Then, an S-set congruence θ on an S-poset A is an *S-poset congruence* if and only if $a\theta b$ whenever $a \leq_\theta b \leq_\theta a$. Given an S-poset A and an S-poset congruence θ on A, the set A/θ of all the equivalence classes together with the order $a/\theta \leq a'/\theta \Leftrightarrow a \leq_\theta b$ constitutes an S-poset with the natural action $sa/\theta = (sa)/\theta$, for every $s \in S$ and $a/\theta \in A/\theta$.

Throughout the paper our standard reference for category theory is [1] and for universal algebra [6].

2. Algebraic properties of a radical in S-Pos

This section is devoted to introducing a preradical on **S-Pos**. We follow the categorical methods used in [7] to define a preradical on **S-Pos**. In [7], the authors work in a normal category with a zero object. They define a normal preradical of the normal category $\mathcal{C}$ to be a functor assigning to every object $X \in \mathcal{C}$ a normal subobject of X. This definition can be generalized to an arbitrary category, but since we do not need the general version, we formulate it just for the category of S-posets.

We denote the set of all congruences on an S-poset A by $\mathsf{Con}(A)$ and give the following definition.

Definition 2.1. In the category **S-Pos** we define the following notions.

(i) A *preradical* (which may also be called a *normal preradical*) is an assignment $r : A \rightsquigarrow r(A)$, assigning to each $A \in$ **S-Pos** a congruence $r(A) \in \mathsf{Con}(A)$ in such a way that every S-poset morphism $f : A \to B$ induces the S-poset morphism $r(f) : r(A) \to r(B)$, meaning that $(f(a), f(a')) \in r(B)$ if $(a, a') \in r(A)$, for every S-poset morphism $f : A \to B$. Note that $r(A)$ and $r(B)$ are, respectively, sub-S-posets of $A \times A$ and $B \times B$, since $r(A) \in \mathsf{Con}(A), r(B) \in \mathsf{Con}(B)$.

(ii) A preradical r is *homomorphic* whenever $(a, a') \in r(A)$ if and only if $(f(a), f(a')) \in r(B)$, for every S-poset morphism $f : A \to B$.

(iii) Given a subclass $\mathcal{M}$ of regular S-poset monomorphisms , a preradical r is *$\mathcal{M}$-hereditary* if $r(f)^{-1}(r(B)) = r(A)$, for every $\mathcal{M}$-morphism $f : A \to B$. More explicitly, $r(A) = \{(a, a') \in A \times A \mid (f(a), f(a')) \in r(B)\}$.

(iv) Given a subclass of S-poset epimorphisms $\mathcal{E}$, a preradical r is *$\mathcal{E}$-cohereditary* if $r(B) = r(f)(r(A))$, for every $\mathcal{E}$-morphism $f : A \to B$.

(v) Finally, a preradical r is a *radical* if $r(A/r(A)) = \Delta_{A/r(A)}$, for every $A \in$ **S-Pos**.

Remark 2.2. (i) Every epi-cohereditary preradical on **S-Pos** is a radical.

(ii) Every homomorphic preradical on **S-Pos** is a radical.

To see (i), let r be an epi-coheredity preradical. We have to show that $r(A/r(A)) = \Delta_{A/r(A)}$, for every $A \in$ **S-Pos**. To do so, consider the canonical S-poset epimorphism $\pi : A \to A/r(A)$. Then we have $r(A/r(A)) = \pi(r(A))$, by the hypothesis. So, $(\pi(a), \pi(a')) \in r(A/r(A))$ if and only if $(a, a') \in r(A)$. Thus, $r(A/r(A)) = \Delta_{A/r(A)}$.

And to prove (ii), let r be a homomorphic preradical. Then, consider the canonical S-poset epimorphism $\pi : A \to A/r(A)$, for every $A \in$ **S-Pos**. Now, since π is an S-poset epimorphism, we get the result analogously to the first part.

Lemma 2.3. *If r is a radical and $\theta \in \mathsf{Con}(A)$ is included in $r(A)$, for some S-poset A, then $r(A/\theta) = r(A)/\theta$.*

Proof. By the definition of a radical, we obtain the induced S-poset morphism $r(A) \to r(A/\theta)$ from the canonical S-poset epimorphism $A \to A/\theta$. Hence, $(a_1, a_2) \in r(A)$ implies that $(a_1/\theta, a_2/\theta) \in r(A/\theta)$. Now, since $(a_1/\theta, a_2/\theta) \in r(A)/\theta$ means $(a_1, a_2) \in r(A)$, we have $(a_1/\theta, a_2/\theta) \in r(A/\theta)$. That is, $r(A)/\theta \subseteq r(A/\theta)$.

For the converse inclusion, we consider the induced S-poset morphism $r(A/\theta) \to r(A/r(A)) = \Delta_{A/r(A)}$ from the S-poset morphism $A/\theta \to A/r(A)$ mapping every a/θ to $a/r(A)$. Now, if $(a_1/\theta, a_2/\theta) \in r(A/\theta)$, then $(a_1/r(A), a_2/r(A)) \in r(A/r(A))$. But, since r is a radical, $r(A/r(A)) = \Delta_{A/r(A)}$ and hence $(a_1/r(A), a_2/r(A)) \in \Delta_{A/r(A)}$. Namely $a_1/r(A) = a_2/r(A)$. Therefore, $(a_1, a_2) \in r(A)$ and hence $(a_1/\theta, a_2/\theta) \in r(A)/\theta$ and $r(A)/\theta = r(A/\theta)$. □

With every preradical r in a category $\mathcal{A}$, one can associate two classes of objects, namely r-*torsion* objects and r-*torsion-free* objects defined, respectively, by

$$\mathcal{T}_r = \{A \in \mathcal{A} \mid A \neq \emptyset \text{ and } r(A) = \nabla_A\},$$
$$\mathcal{F}_r = \{A \in \mathcal{A} \mid r(A) = \Delta_A\}.$$

Definition 2.4. An S-poset A is said to be a *weak sub-S-poset* of the S-poset B, whenever there exists an S-poset monomorphism (not necessarily, embedding) $f : A \to B$.

Remark 2.5. The class $\mathcal{T}_r$ in **S-Pos** is closed under quotients and homomorphic images, while $\mathcal{F}_r$ in **S-Pos** is closed under sub-S-posets, weak sub-S-posets, and products.

To see this, first we show that $\mathcal{T}_r$ is closed under homomorphic images. To do so, let $f : A \to B$ be an S-poset epimorphism with $A \in \mathcal{T}_r$. Then, for every $b, b' \in B$, there exist $a, a' \in A$ such that $b = f(a)$ and $b' = f(a')$. But, since $r(A) = \nabla_A$, we have $(a, a') \in r(A)$ and hence $(b, b') = (f(a), f(a')) \in r(B)$.

Now the closedness of $\mathcal{T}_r$ under quotients follows from the closedness of $\mathcal{T}_r$ under homomorphic images, because A/θ is the image of the natural S-poset epimorphism $A \to A/\theta$, for every $\theta \in \mathsf{Con}(A)$.

Now, suppose $f : A \to B$ is a (regular) monomorphism and $B \in \mathcal{F}_r$. Then, for each $(a, a') \in r(A)$ we get $(f(a), f(a')) \in r(B) = \Delta_B$. So $f(a) = f(a')$ and, since f is a monomorphism, $a = a'$.

Suppose, finally, that $\{A_i\}_{i\in I}$ is a family of r-torsion-free objects. Then, for each $((a_i)_{i\in I}, (a'_i)_{i\in I}) \in r(\prod_{i\in I} A_i)$, the projection S-poset morphism $\pi_j : \prod_{i\in I} A_i \to A_j$ gives $(a_j, a'_j) \in r(A_j) = \Delta_{A_j}$, for every $j \in I$. Hence, $(a_i)_{i\in I} = (a'_i)_{i\in I}$ and $r(\prod_{i\in I} A_i) = \Delta_{\prod_{i\in I} A_i}$.

Note 2.6. It is worth noting that for every $A \in \mathcal{T}_r$ and $B \in \mathcal{F}_r$ the set $Hom(A, B)$, of S-poset morphisms from A to B, is empty or if B has zero elements, then it consists of the constant S-poset morphisms whose image is a zero element in B. Indeed, if $f : A \to B$ is an S-poset morphism then $(f(a), f(a')) \in r(B)$, for each $(a, a') \in A \times A$, since $A \in \mathcal{T}_r$. Now $r(B) = \Delta_B$ implies that $f(a) = f(a')$, for every $a, a' \in A$. That is, f is a constant S-poset morphism whose image is a zero element.

Definition 2.7. (i) A class $\mathcal{C}$ of S-posets is called *pretorsion-free* if it is closed under weak sub-S-posets and products.

(ii) A class $\mathcal{C}$ of S-posets is called *pretorsion* if it is closed under quotients and coproducts.

Proposition 2.8. *There exists a bijective correspondence between radicals of* **S-Pos** *and pretorsion-free classes of objects of* **S-Pos**.

Proof. Remark 2.5 shows that every radical induces a pretorsion-free class $\mathcal{F}_r$. Suppose, conversely, $\mathcal{C}$ is a pretorsion-free class of S-posets and A is an arbitrary S-poset. Then, consider the set $\mathcal{K}_A = \{\theta \in \mathsf{Con}(A) \mid A/\theta \in \mathcal{C}\}$ and take $r_{\mathcal{C}}$, assigning to every S-poset A the congruence $\bigcap \mathcal{K}_A \in \mathsf{Con}(A)$. We show that $r_{\mathcal{C}}$ is a preradical. To do so, consider an S-poset morphism $f : A \to B$. Then, $\bar{A} = A/\mathsf{ker}f$ is a weak sub-S-poset of B and every $\theta \in \mathcal{K}_B$ can be restricted to $\theta|_{\bar{A}} \in \mathsf{Con}(\bar{A})$. Now, we define the congruence $\tilde{\theta}$ on A to be $\tilde{\theta} = \{(a, a') \mid (f(a), f(a')) \in \theta\}$ and show that $\tilde{\theta} \in \mathcal{K}_A$. We note that $\mathsf{ker}f \subseteq \tilde{\theta}$ and $A/\tilde{\theta} \simeq \bar{A}/\theta|_{\bar{A}}$, which is a weak sub-$S$-poset of $B/\theta \in \mathcal{C}$. So, $A/\tilde{\theta} \in \mathcal{C}$ and hence $\tilde{\theta} \in \mathcal{K}_A$. Therefore, $r_{\mathcal{C}}(A) \subseteq \tilde{\theta}$, for every $\theta \in \mathcal{K}_B$, which implies that if $(a, a') \in r_{\mathcal{C}}(A)$ then $(f(a), f(a')) \in r_{\mathcal{C}}(B)$. It is clear that $r_{\mathcal{C}}$ is a radical. Because, every congruence $\theta \in \mathcal{K}_{A/r_{\mathcal{C}}(A)}$ corresponds to a congruence in $\mathcal{K}_A$, by the Correspondence Theorem.

To complete the proof, we have to show that $\mathcal{F}_{r_{\mathcal{C}}} = \mathcal{C}$, for every pretorsion-free class $\mathcal{C}$ and $r = r_{\mathcal{F}_r}$, for every radical r. The second one is an immediate corollary of the next theorem, so we prove the first one. Indeed, for a given pretorsion-free class $\mathcal{C}$, each $A \in \mathcal{C}$, since $\Delta_A \in \mathcal{K}_A$, we have $r_{\mathcal{C}}(A) = \Delta_A$ and hence $\mathcal{C} \subseteq \mathcal{F}_{r_\mathcal{C}}$. For the converse inclusion, let $A \in \mathcal{F}_{r_\mathcal{C}}$. Then $r_\mathcal{C}(A) = \Delta_A$, which implies that $\Delta_A = \bigcap_{\theta \in \mathcal{K}_A} \theta$. Now, since the canonical S-poset morphism $f : A \to \prod_{\theta \in \mathcal{K}_A} A/\theta$ has the kernel $\Delta_A = \bigcap_{\theta \in \mathcal{K}_A} \theta$ and $\mathcal{C}$ is closed under weak sub-S-posets and products, we get the result. □

Theorem 2.9. *For every preradical r on* **S-Pos**, *$r_{\mathcal{F}_r}$ is the smallest radical greater than r under the inclusion order.*

Proof. Given a preradical r, an S-poset A, and a congruence $\theta \in \mathsf{Con}(A)$ with $r(A/\theta) = \Delta_{A/\theta}$, consider the S-poset morphism $r(A) \to r(A/\theta) = \Delta_{A/\theta}$, induced by the canonical S-poset epimorphism $A \to A/\theta$. Then, for each $(a, a') \in r(A)$, since $r(A/\theta) = \Delta_{A/\theta}$, we have $a/\theta = a'/\theta$. So, $r(A) \subseteq \theta$, for each $\theta \in \mathcal{K}_A$ and hence $r(A) \subseteq \bigcap_{\theta \in \mathcal{K}_A} \theta = r_{\mathcal{F}_r}(A)$. Now, suppose $\hat{r}$ is a radical greater than r and A is an S-poset. Then, $r(A/\theta) \subseteq \hat{r}(A/\theta)$, for each $\theta \in \mathsf{Con}(A)$. In particular, $r(A/\hat{r}(A)) \subseteq \hat{r}(A/\hat{r}(A)) = \Delta_{A/\hat{r}(A)}$, since $\hat{r}$ is a radical. Therefore, $\hat{r}(A) \in \mathcal{K}_A$ and hence $r_{\mathcal{F}_r}(A) = \bigcap_{\theta \in \mathcal{K}_A} \theta \subseteq \hat{r}(A)$. □

3. Relations between the categories S-Set and S-Pos

In this section we are going to study the interrelationship between the categories of S-sets and S-posets. In fact, we construct a left adjoint for the forgetful functor U : **S-Pos** $\to$ **S-Set**, the so-called *ordering functor*, where S is a posemigroup. We then give the *order desolator functor* from **S-Pos** to **S-Set** which plays an important role in the sequel. Although there is no adjoint situation between order desolator and ordering functors, in Theorem 3.9 and Corollary 4.3 we show that there is a good connection between these two functors.

To do so, suppose S is a posemigroup. Then, we define an induced binary relation $\leq_S$ by S on each S-set A as follows:

$$a \leq_S b \iff a = b \text{ or there exist } s_1, \cdots, s_n \in S \text{ and } a_1, \cdots a_n \in A$$

such that

$$s_1 \leq s_2, s_3 \leq s_4, \cdots s_{n-1} \leq s_n, a = s_1 a_1, s_2 a_1 = s_3 a_2, s_4 a_2 = s_5 a_3, \cdots s_n a_n = b.$$

Remark 3.1. One can easily see that the binary relation $\leq_S$ is reflexive and transitive but not necessarily antisymmetric. So, $(A/\sim, \leq_S)$ is a poset, where $a \sim b$ if and only if $a \leq_S b$ and $b \leq_S a$, for all $a, b \in A$, and $a/\sim \leq_S a'/\sim$ if and only if $a \leq_S a'$. Also, by a simple calculation, one can easily see that, for each S-set over a posemigroup S, the poset $(A/\sim, \leq_S)$ together with the natural action $s(a/\sim) := (sa)/\sim$, for each $s \in S$ and $a/\sim \in A/\sim$, constitutes an S-poset.

Now, we have the following theorem.

Theorem 3.2. *The forgetful functor U* : **S-Pos** $\to$ **S-Set** *has a left adjoint.*

Proof. To prove, we show that for every S-set A, the natural arrow $\eta_A : A \to U(A/{\sim}, \leq_S)$ mapping every $a \in A$ to $a/{\sim}$ is a U-universal arrow. Indeed, for every S-set morphism $f : A \to U(B, \leq)$, we define a map $\bar{f} : (A/{\sim}, \leq_S\ ) \to (B, \leq)$ which maps every $a/{\sim}$ to $f(a)$ and show that $\bar{f}$ is a unique S-poset morphism which makes the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A/{\sim} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & B \end{array} \tag{3.1}$$

First, we show that $\bar{f}$ is order preserving, which implies that it is well-defined. Suppose $a \leq_S b$ and $a \neq b$. By the definition of $\leq_S$, there exist $s_1, \cdots, s_n \in S$ and $a_1, \cdots a_n \in A$ such that:

$$s_1 \leq s_2, s_3 \leq s_4, \cdots s_{n-1} \leq s_n, a = s_1a_1, s_2a_1 = s_3a_2, s_4a_2 = s_5a_3, \cdots s_na_n = b.$$

Hence we have,

$$f(a) = s_1f(a_1) \leq s_2f(a_1) = s_3f(a_2) \leq s_4f(a_2) = \cdots s_nf(a_n) \ = f(b).$$

Now, if $a/{\sim} = b/{\sim}$, meaning that $a \leq_S b$ and $b \leq_S a$, then, since $\bar{f}$ is order preserving, we have $f(a) \leq f(b) \leq f(a)$. Therefore, $f(a) = f(b)$ and hence $\bar{f}$ is well-defined.

Also, $\bar{f}$ is equivariant, because $\bar{f}(s(a/{\sim})) = \bar{f}((sa)/{\sim}) = f(sa) = sf(a) = s\bar{f}(a/{\sim})$. So, $\bar{f}$ is an S-poset morphism. The diagram (3.1) is commutative because $\bar{f}\eta_A(a) = \bar{f}(a/{\sim}) = f(a)$, for every $a \in A$. And, finally, the uniqueness of $\bar{f}$ follows from the definition of $\bar{f}$. □

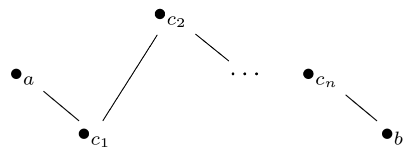
Notation 3.3. We denote the above obtained left adjoint functor of the forgetful functor by F : **S-Set** $\to$ **S-Pos** and call it the *ordering functor*.

Remark 3.4. (i) The natural transformation $\eta : Id \to UF$, assigning to each $A \in$ **S-Set** the S-poset morphism $\eta_A : A \to U(A/{\sim})$, is the unit of the adjoint situation $F \dashv U$.

(ii) The co-unit of the adjoint situation $F \dashv U$ has the identity function as the underlying map.

In the sequel of this section, we consider another important functor, $G :$ **S-Pos** $\to$ **S-Set**, besides the forgetful functor which has a good relation with the ordering functor, meaning that, there is a singleton G-solution set $\{\mu_A : A \to GFA\}$, for each S-set A.

Let $(A, \leq)$ be an S-poset and α_A be the smallest equivalence relation on A containing the order of A. In fact, $a\alpha_Ab$ if and only if a connects to b by finitely many edges with finitely many connectors $c_1, \cdots, c_n \in A$ as nods in the Hasse diagram of the poset A. For instance, $a\alpha_Ab$ in the following.



The following remark is easily derived from the definition of α_A and the definition of S-posets.

Remark 3.5. (i) Given an S-poset A, the equivalence relation α_A is a congruence on A.

(ii) Given an S-poset A, the set A/α_A together with the natural action $s(a/\alpha_A) := (sa)/\alpha_A$, for each $s \in S$ and $a/\alpha_A \in A/\alpha_A$, is an S-set.

Lemma 3.6. *The assignment G : **S-Pos** $\to$ **S-Set**, assigning to every S-poset A the S-set A/α_A, is functorial.*

Proof. Let $f : A \to B$ be an S-poset morphism. Then, $G(f) : A/\alpha_A \to B/\alpha_B$, mapping every a/α_A to $f(a)/\alpha_B$, is an S-set morphism. Indeed, since f is order preserving, f transfers the connectors between a and a' in A to the connectors between $f(a)$ and $f(a')$. Also, $G(f \circ g)$ is clearly $Gf \circ Gg$. ☐

Note 3.7. There is a natural transformation from the forgetful functor to the order desolator functor.

This is because, for every S-poset A, we define $\tau_A : UA \to GA$ to be $\tau_A(a) = a/\alpha_A$. This mapping is equivariant, because $\tau_A(sa) = (sa)/\alpha_A = sa/\alpha_A = s\tau_A(a)$. Also, for every S-poset morphism $f : (A, \leq) \to (B, \leq)$, we have the following commutative diagram.

$$\begin{array}{ccc} U(A,\leq) = A & \xrightarrow{\tau_A} & G(A,\leq) = A/\alpha_A \\ {\scriptstyle Uf}\downarrow & & \downarrow{\scriptstyle Gf} \\ U(B,\leq) = B & \xrightarrow{\tau_B} & G(B,\leq) = B/\alpha_B \end{array}$$

Indeed, $Gf(\tau_A(a)) = Gf(a/\alpha_A) = f(a)/\alpha_B = \tau_B(Uf(a))$, for every $a \in A$.

Definition 3.8. Let us call, the quotient functor of the forgetful functor, G the *order desolator* functor.

Theorem 3.9. *Let S be a posemigroup. Then, for every S-set A, the singleton set containing the morphism $\mu_A = \tau_{(A/\sim)} \circ \eta_A : A \to GFA = \frac{(A/\sim,\leq_S)}{\alpha_{(A/\sim)}}$, mapping every $a \in A$ to $(a/\sim)/\alpha_{(A/\sim)}$, is a G-solution set, meaning that every S-set morphism $f : A \to GB = B/\alpha_B$ factors through μ_A by an S-poset morphism.*

Proof. Let $f : A \to B/\alpha_B$ be an S-set morphism. Then, we define $\bar{f} : (A/\!\sim, \leq_S) \to (B, \leq)$ by $\bar{f}(a/\!\sim) = f(a)$. Completely analogous to the proof of Theorem 3.2, one can see that the map $\bar{f}$ is an S-poset morphism. Also, for every $a \in A$, we have $G(\bar{f}) \circ \mu_A(a) = G(\bar{f})(a/\!\sim\!/\alpha_{A/\sim}) = \bar{f}(a/\!\sim) = f(a)$. That is, f factors through μ_A. □

Note 3.10. Even though for every S-act A there exists a G-solution set, G is not a right adjoint for the ordering functor, because G does not preserve small limits (although it preserves some limits, see Theorem 3.11).

Limits of S-posets are formed on the level of S-sets. Specially, the terminal S-poset (the product of the empty indexed family) is the singleton S-poset and the product of a family of S-posets is their cartesian product, with componentwise action and order. Also, the equalizer of a pair $f, g : A \to B$ of S-poset morphisms is given by $E(f, g) = \{a \in A \mid f(a) = g(a)\}$, with the action and the order inherited from A.

Theorem 3.11. *The order desolator functor preserves finite products.*

Proof. Obviously G preserves the terminal object. So it is enough to show that $G(A\times B) \simeq G(A)\times G(B)$. To do so, define $\varphi : G(A\times B) \to G(A)\times G(B)$, mapping every $((a, b)/\alpha_{A\times B} \in G(A \times B)$ to $(a/\alpha_A, b/\alpha_B) \in G(A) \times G(B)$. First, we show that the map φ is well-defined. Indeed, if $(a, b)/\alpha_{A\times B} = (a', b')/\alpha_{A\times B}$ then there are finitely many connectors, such as $(c_1, d_1), \cdots, (c_n, d_n)$, between (a, b) and (a', b') in $A\times B$. Of course, these connectors give rise to the existence of the connectors $c_1, \cdots, c_n$ between a and a' in A, and $d_1, \cdots d_n$ between b and b' in B. Meaning that

$$\varphi((a, b)/\alpha_{A\times B}) = (a/\alpha_A, b/\alpha_B) = (a'/\alpha_A, b'/\alpha_B) = \varphi((a', b')/\alpha_{A\times B}).$$

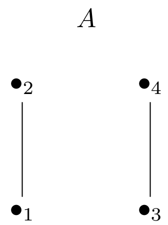
Also, φ is one to one. Because, if

$$\varphi((a, b)/\alpha_{A\times B}) = (a/\alpha_A, b/\alpha_B) = (a'/\alpha_A, b'/\alpha_B) = \varphi((a', b')/\alpha_{A\times B}),$$

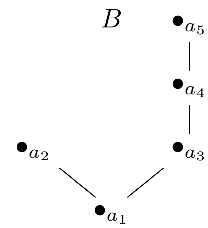
then there are connectors c_i, $i = 1, \ldots, n$, between a and a' in A, and d_j, $j = 1, \ldots, m$, in B. These connectors can be transformed to the connectors (c_k, d_k), $k = 1, \ldots, \max\{m, n\}$, in $A \times B$, by the reflexive property. Obviously, φ is equivariant and onto. So, $G(A \times B) \simeq G(A) \times G(B)$. Preserving the finite product can be proved by induction. □

Although the order desolator functor preserves finite products, it does not preserve equalizers, see the following example.

Example 3.12. Take S to be the one element posemigroup $\{e\}$. Then every poset A can be considered as an S-poset with the trivial action, meaning that $ea = a$, for every $a \in A$. Now consider the S-posets A and B with the trivial action as follows:



and



Also, take the S-posets morphisms $f, g : A \to B$ to be $f(i) = a_i$, for every $i = 1, \ldots, 4$, and $g(1) = a_3, g(2) = g(3) = a_4, g(4) = a_5$. Then, we have $G(E(f, g)) = G(\emptyset) = \emptyset$, while $E(G(f), G(g)) = \{1/\alpha_a, 3/\alpha_A\}$.

4. **S-Pos as a radical extension of S-Set**

In the previous section we introduced two functors, the order desolator and the ordering functors, between the categories **S-Pos** and **S-Set**. In this section, we show that the category **S-Pos** is a radical extension of the category **S-Set**. In fact, the order desolator functor gives rise to a radical in the category **S-Pos** whose class of torsion-free objects is the pretorsion-free class generated from the image of the ordering functor. To do so, we consider the assignment α, assigning to each $A \in$ **S-Pos** the congruence α_A.

In the next theorem we see that α is a radical.

Theorem 4.1. *The assignment α is a radical in* **S-Pos**.

Proof. Clearly α is a preradial. So, to prove the result, it is enough to show that $\alpha(A/\alpha_A) = \Delta_{A/\alpha_A}$, for every S-poset A. Consider the S-poset A/α_A. First, we note that the only order on A/α_A is the inherent order of S, that is $b/\alpha_A \leq b'/\alpha_A$ if there exist $s_1 \leq s_2 \in S$ and $a \in A$ such that $s_1 a/\alpha_A = b/\alpha_A$ and $b'/\alpha_A = s_2 a/\alpha_A$. But, this implies that $s_1 a \leq s_2 a$ in A and hence $s_1 a/\alpha_A = s_2 a/\alpha_A$. Therefore, the order on A/α_A is Δ_{A/α_A}, and therefore $\alpha(A/\alpha_A) = \Delta_{A/\alpha_A}$. □

Here we give another good relationship between the ordering and the order desolator functors. In fact, the torsion-free class associated to the radical α is nothing but the pretorsion-free class $\mathcal{F}$ generated by the image of the ordering functor F. Indeed, the image of the ordering functor F is not necessarily closed under sub-S-posets and products and, by Remark 2.5, every pretorsion-free class is closed under sub-S-posets and products. So, we consider the smallest class $\mathcal{F}$ containing the image of F which is closed under taking sub-S-posets and products, and give the next theorem.

Theorem 4.2. *The radical $r_{\mathcal{F}}$ is precisely the radical α.*

Proof. Applying Proposition 2.8, one can construct the radical $r_{\mathcal{F}}$ mapping each S-poset $(A, \leq)$ to the intersection of those congruences in $\mathsf{Con}(A)$ such that the quotients of A over them belong to $\mathcal{F}$. First, we note that α_A is one of these congruences. Indeed, $A/\alpha_A = F(A/\alpha_A) = ((A/\alpha_A)/\sim, \leq_S)$. The proof of these equalities is completely analogous to the proof of Lemma 3.4.

Now, we show that α_A is the smallest one among these congruences. To do so, let $\theta \in \mathsf{Con}(A)$ be such that $A/\theta \in \mathcal{F}$ and $a, b \in A$ be comparable with the order in $(A, \leq)$. Then, since $A/\theta \in \mathcal{F}$ and S-posets in $\mathcal{F}$ are considered as the S-posets with the order $\leq_S$, the classes a/θ and b/θ can not be comparable with $\leq_\theta$, unless $a/\theta = b/\theta$. Thus, for every two comparable elements $a, b \in A$ we have $a\theta b$. Now, transitivity of θ indicates that $\alpha_A \subseteq \theta$. Therefore $r_{\mathcal{F}} = \alpha$. □

Corollary 4.3. *The torsion-free class $\mathcal{F}_\alpha$ of α is the pretorsion-free class generated by the image of the ordering functor.*

Proof. The result is obviously derived from Theorem 4.2 and Proposition 2.8. □

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