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ON CHARACTERIZATIONS OF HYPERBOLIC HARMONIC BLOCH AND BESOV SPACES

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Abstract. We define hyperbolic harmonic ω-α-Bloch space $\mathcal{B}_{\omega}^{\alpha}$ in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$ and characterize it in terms of

$$\frac{\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}},$$

where $0 \leq \gamma \leq 1$. Similar results are extended to little ω-α-Bloch and Besov spaces. These obtained characterizations generalize the corresponding ones which were obtained by G. Ren and U. Kähler in 2002 and 2005.

Keywords: Hyperbolic harmonic function, Bloch space, Besov space, majorant.
MSC(2010): Primary: 32A18; Secondary: 31B05, 30C20.

1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^n$ with $n \geq 2$, dv the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on the unit sphere $S = \partial\mathbb{B}$. A function $f \in C^2(\mathbb{B})$ is called *hyperbolic harmonic* (briefly, h-harmonic) if it satisfies the *hyperbolic Laplace's equation*

$$\triangle_h u = (1-|x|^2)^2\triangle u + 2(n-2)(1-|x|^2)\langle\nabla u, x\rangle = 0,$$

where $\triangle$ denotes the ordinary *Laplacian operator* and ∇ denotes the gradient. Obviously, when $n = 2$, all h-harmonic functions are harmonic functions. We denote by $H(\mathbb{B})$ the class of all h-harmonic functions on the unit ball $\mathbb{B}$.

For each $\alpha > 0$, the *hyperbolic harmonic α-Bloch space* $\mathcal{B}^{\alpha}$ consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{\alpha} = \sup_{x\in\mathbb{B}}(1-|x|^2)^{\alpha}|\nabla f(x)| < \infty,$$

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and the *little α-Bloch space* $\mathcal{B}_0^\alpha$ consists of the functions $f \in H(\mathbb{B})$ such that

$$\lim_{|x|\to 1^-} \sup_{x\in\mathbb{B}} (1-|x|^2)^\alpha |\nabla f(x)| = 0.$$

The *hyperbolic harmonic Besov space* $\mathcal{B}_p$ is the space of all functions in $H(\mathbb{B})$ for which

$$\int_{\mathbb{B}} (1-|x|^2)^p |\nabla f(x)|^p d\tau(x) < \infty,$$

where $p > n-1$ and $d\tau(x) = (1-|x|^2)^{-n} dv(x)$ is the invariant measure on $\mathbb{B}$ (cf. [4, 10]).

Let $\omega : [0, +\infty) \to [0, +\infty)$ be a continuous increasing function with $\omega(0) = 0$. We call ω *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [5]). Following [3], for each $\alpha > 0$, the *hyperbolic harmonic ω-α-Bloch space* $\mathcal{B}_\omega^\alpha$ consists of all functions $f \in H(B)$ such that

$$\|f\|_{\omega,\alpha} = \sup_{x\in\mathbb{B}} \omega((1-|x|^2)^\alpha) |\nabla f(x)| < \infty,$$

and the *little ω-α-Bloch space* $\mathcal{B}_{\omega,0}^\alpha$ consists of the functions $f \in H(\mathbb{B})$ such that

$$\lim_{|x|\to 1^-} \sup_{x\in\mathbb{B}} \omega((1-|x|^2)^\alpha) |\nabla f(x)| = 0.$$

In particular, when $\omega(t) = t$, the space $\mathcal{B}_\omega^\alpha$ (resp. $\mathcal{B}_{\omega,0}^\alpha$) is $\mathcal{B}^\alpha$ (resp. $\mathcal{B}_0^\alpha$).

Let $\mu, \nu \geq 0$ and f be a continuous function in $\mathbb{B}$. If there exists a constant C such that

$$(1-|x|^2)^\mu (1-|y|^2)^\nu |f(x) - f(y)| \leq C|x-y|,$$

for any $x, y \in \mathbb{B}$, then we say that f is a *weighted Lipschitz function* of indices (μ, ν) (cf. [10, 11]).

In the theory of function spaces, the relationship between Bloch spaces and weighted Lipschitz functions has attracted much attention. In 1986, Holland and Walsh [6] established a classical criterion for analytic Bloch space in terms of weighted Lipschitz functions. Since then, a series of works has been carried out to characterize α-Bloch, little α-Bloch and Besov spaces of holomorphic and harmonic functions along this line. For instance, Li and Wulan [8], Zhao [16] characterized holomorphic α-Bloch space in terms of

$$\frac{(1-|x|^2)^\beta (1-|y|^2)^{\alpha-\beta} |f(x) - f(y)|}{|x-y|}.$$

For the case of harmonic functions, we refer to [11, 14]. In [10], Ren and Kähler extended Holland and Walsh's criterion to the setting of h-harmonic functions. Especially, they proved the following theorems.

Theorem 1.1. *Let $0 < \beta < 1$ and $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}^1$ if and only if*

$$\sup\left\{ (1-|x|^2)^\beta (1-|y|^2)^{1-\beta} \frac{|f(x) - f(y)|}{|x-y|} : x, y \in \mathbb{B}, x \neq y \right\} < \infty.$$

Theorem 1.2. *Let $0 < \beta < 1$ and $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}_0^1$ if and only if*

$$\lim_{|x|\to 1^-} \sup\left\{(1-|x|^2)^\beta(1-|y|^2)^{1-\beta}\frac{|f(x)-f(y)|}{|x-y|} : y \in \mathbb{B}, y \neq x\right\} = 0.$$

Moreover, the same authors characterized the space $\mathcal{B}_p$ as follows.

Theorem 1.3. *Let $f \in H(\mathbb{B})$ and $p \in (2(n-1), \infty)$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}}(1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}}\left|\frac{f(x)-f(y)}{x-y}\right|^p d\tau(x)d\tau(y) < \infty.$$

.

Since the space $\mathcal{B}_\omega^\alpha$ can be viewed as an extension of $\mathcal{B}^\alpha$, the following question arises: can we also characterize it in terms of weighted Lipschitz functions? In [3], Chen et al., obtained the analogue of Theorems 1.1 and 1.2 for the spaces $\mathcal{B}_\omega^\alpha$ and $\mathcal{B}_{\omega,0}^\alpha$ in the unit disc $\mathbb{D}$, respectively.

The main purpose of this paper is to give some characterizations for the spaces $\mathcal{B}_\omega^\alpha, \mathcal{B}_{\omega,0}^\alpha$ and $\mathcal{B}_p$ in the unit ball $\mathbb{B}$ of $\mathbb{R}^n$ along Ren and Kähler's direction. In Section 2, we collect some known results that we need later. Our main results and their proofs are presented in Sections 3 and 4. Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Preliminaries

We shall be using the following notation: we write $x \in \mathbb{R}^n$ in polar coordinate by $x = |x|x'$. For any $a, b \in \mathbb{R}^n$, let

$$[a, b] = \Big||a|b - a'\Big|.$$

Then the symmetry lemma gives that

$$[a, b] = [b, a].$$

For $a \in \mathbb{B}$, we denote by ϕ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\phi_a(0) = a$ and $\phi_a(a) = 0$, which is of the form

$$\phi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x, a]^2}, x \in \mathbb{B}.$$

By simple computations, we have

$$|\phi_a(x)| = \frac{|x-a|}{[x, a]},$$

$$1 - |\phi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{[x, a]^2},$$

and

$$|J\phi_a(x)| = \frac{(1-|a|^2)^n}{[x,a]^{2n}},$$

where $J\phi_a$ denotes the Jacobian of ϕ_a (cf. [1,11]).

For $a \in \mathbb{B}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* $E(a,r)$ with center a and radius r is defined as

$$E(a,r) = \{w \in \mathbb{B} : |\phi_a(w)| < r\}.$$

Clearly, $E(a,r) = \phi_a(\mathbb{B}(0,r))$.

The following result comes from [10, Lemma 2.1].

Lemma 2.1. *Let $r \in (0,1)$ and $y \in E(x,r)$. Then $1-|x|^2 \asymp 1-|y|^2 \asymp [x,y]$.*

As an application of Lemma 2.1, we easily get the following.

Corollary 2.2. *Let $r \in (0,1)$ and $\lambda = \inf_{a\in\mathbb{B},x,y\in E(a,r)} \left(\frac{1-|x|^2}{1-|y|^2}\right)$, then $\lambda \in (0,1)$.*

For $x \in \mathbb{B}$ and $f \in H(\mathbb{B})$, we define $\widetilde{\nabla} f(x)$ of f at x by

$$\widetilde{\nabla} f(x) = \nabla(f \circ \phi_x)(0).$$

We call $|\widetilde{\nabla} f(x)|$ the invariant gradient of f at x, motivated by the following proposition which can be found in [12].

Proposition 2.3. *Let $f \in H(\mathbb{B})$ and $x \in \mathbb{B}$. Then $|\widetilde{\nabla} f(x)| = (1-|x|^2)|\nabla f(x)|$ and*

$$|\widetilde{\nabla}(f \circ \phi)(x)| = |(\widetilde{\nabla} f) \circ \phi(x)|$$

for any Möbius transformation ϕ in $\mathbb{B}$.

Lemma 2.4 ([10])**.** *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}} dv(y) \asymp \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

We end this section with two inequalities which will be used in the sequel.

Lemma 2.5 ([3, Lemma 6])**.** *Let $\omega(t)$ be a majorant and $u \in (0,1]$, $v \in (1,\infty)$. Then for $t \in (0,\infty)$,*

$$\omega(ut) \geq u\omega(t), \quad \omega(vt) \leq v\omega(t).$$

Lemma 2.6. *Let $a, b > 0$, $0 < s < 1$. Then $sa + (1-s)b \geq a^s b^{1-s}$.*

3. Hyperbolic harmonic Bloch space

In this section, we give some characterizations of the spaces $\mathcal{B}_\omega^\alpha$ and $\mathcal{B}_{\omega,0}^\alpha$.

Theorem 3.1. *Let $f \in H(\mathbb{B})$, $0<\beta<1, \beta \le \alpha < 1+\beta$ and $0 \le \gamma \le 1$. Then $f \in \mathcal{B}_\omega^\alpha$ if and only if*

$$L = \sup_{x,y\in\mathbb{B}, x\neq y} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^\gamma|x-y|^{1-\gamma}} < \infty. \tag{3.1}$$

Proof. Assume that (3.1) holds. Fix $r\in(0,1)$ and $x\in\mathbb{B}$, it follows from [10, (3.2)] that

$$|\widetilde{\nabla} f(x)| \le C\int_{E(x,r)} |f(x)-f(y)|d\tau(y).$$

Combing this with Proposition 2.3, we obtain

$$\omega((1-|x|^2)^\alpha)|\nabla f(x)| \le \frac{C\omega((1-|x|^2)^\alpha)}{(1-|x|^2)}\int_{E(x,r)} |f(x)-f(y)|d\tau(y).$$

By Lemmas 2.1, 2.5 and Corollary 2.2, we have

$$\begin{aligned} & \frac{\omega((1-|x|^2)^\alpha)}{(1-|x|^2)}\int_{E(x,r)} |f(x)-f(y)|d\tau(y) \\ \le\ & C\int_{E(x,r)} \omega\big((1-|x|^2)^\beta(\frac{1-|y|^2}{\lambda})^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]}d\tau(y) \\ \le\ & C\lambda^{\beta-\alpha}\int_{E(x,r)} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^\gamma|x-y|^{1-\gamma}}d\tau(y) \\ \le\ & CL\lambda^{\beta-\alpha}\int_{E(x,r)} d\tau \\ =\ & CL\lambda^{\beta-\alpha}\tau(\mathbb{B}(0,r)). \end{aligned}$$

Since $\tau(\mathbb{B}(0,r)) = n\int_0^r t^{n-1}(1-t^2)^{-n}dt$ is a constant, we see that

$$\sup_{x\in\mathbb{B}} \omega((1-|x|^2)^\alpha)|\nabla f(x)| < \infty.$$

Hence $f\in\mathcal{B}_\omega^\alpha$.

Conversely, we assume that $f \in \mathcal{B}^\alpha_\omega$. For $x, y \in \mathbb{B}$,

$$\begin{aligned} |f(x)-f(y)| &= \Big|\int_0^1 \frac{df}{ds}(sx+(1-s)y)ds\Big| \\ &\le \sum_{k=1}^n \Big|(x_k-y_k)\int_0^1 \frac{\partial f}{\partial x_k}(sx+(1-s)y)ds\Big| \\ &\le \sqrt{n}|x-y|\int_0^1 |\nabla f(sx+(1-s)y)|ds \\ &\le C|x-y|\|f\|_{\omega,\alpha}\int_0^1 \frac{ds}{\omega\big((1-|sx+(1-s)y|^2)^\alpha\big)}. \end{aligned}$$

Since for $x, y \in \mathbb{B}$ and $s \in [0,1]$,

$$\begin{aligned} (1-|sx+(1-s)y|^2)^\alpha &\ge (1-|sx+(1-s)y|)^\alpha \\ &\ge (s(1-|x|)+(1-s)(1-|y|))^\alpha \\ &\ge \big(s(\frac{1-|x|^2}{2})+(1-s)(\frac{1-|y|^2}{2})\big)^\alpha \\ &\ge (\frac{s}{2})^\beta(1-|x|^2)^\beta(\frac{1-s}{2})^{\alpha-\beta}(1-|y|^2)^{\alpha-\beta}, \end{aligned}$$

we get

$$\begin{aligned} \frac{|f(x)-f(y)|}{[x,y]^\gamma|x-y|^{1-\gamma}} &\le C\int_0^1 \frac{ds}{\omega\big((1-|sx+(1-s)y|^2)^\alpha\big)} \\ &\le C\int_0^1 \frac{ds}{\omega\big((\frac{s}{2})^\beta(\frac{1-s}{2})^{\alpha-\beta}(1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)} \\ &\le \frac{C}{\omega((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta})}\int_0^1 \frac{ds}{s^\beta(1-s)^{\alpha-\beta}} \\ &\le \frac{C}{\omega((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta})}, \end{aligned}$$

where the last integral converges since $\alpha < 1+\beta$. Thus

$$\omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^\gamma|x-y|^{1-\gamma}} < \infty.$$

This completes the proof. □

Theorem 3.2. *Let* $f \in \mathcal{B}^\alpha_\omega$, $0<\beta<1, \beta \le \alpha < 1+\beta$ *and* $0 \le \gamma \le 1$. *Then* $f \in \mathcal{B}^\alpha_{\omega,0}$ *if and only if*

$$\lim_{|x|\to 1^-} \sup_{x,y\in\mathbb{B},x\ne y} \omega\big((1-|x|^2)^\beta(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^\gamma|x-y|^{1-\gamma}} = 0. \tag{3.2}$$

Proof. Sufficiency. Assume that (3.2) holds. Then for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sup_{x,y\in\mathbb{B},x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} < \epsilon$$

whenever $\delta < |x| < 1$. It follows by an argument similar to that in the proof of Theorem 3.1, we have

$$\omega((1-|x|^2)^{\alpha})|\nabla f(x)| < C \sup_{x,y\in\mathbb{B},x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} < C\epsilon,$$

whenever $\delta < |x| < 1$. Hence

$$\lim_{|x|\to 1^-} \omega((1-|x|^2)^{\alpha})|\nabla f(x)| = 0.$$

Necessity. For $t \in (0,1)$, let $f_t(x) = f(tx)$. By the proof of Theorem 3.1, we have

$$\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|(f-f_t)(x)-(f-f_t)(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} \leq C\|f-f_t\|_{\omega,\alpha}$$

and

$$\begin{aligned}
&\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|f_t(x)-f_t(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}}\\
< \;& \frac{\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)[tx,ty]^{\gamma}}{\omega\big((1-|tx|^2)^{\beta}(1-|ty|^2)^{\alpha-\beta}\big)[x,y]^{\gamma}}\omega((1-|tx|^2)^{\beta}(1-|ty|^2)^{\alpha-\beta})\\
&\times\frac{|f(tx)-f(ty)|}{[tx,ty]^{\gamma}|tx-ty|^{1-\gamma}}\\
\leq \;& \frac{C\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)[tx,ty]^{\gamma}}{\omega\big((1-|tx|^2)^{\beta}(1-|ty|^2)^{\alpha-\beta}\big)[x,y]^{\gamma}}\|f\|_{\omega,\alpha}
\end{aligned}$$

for all $x, y \in \mathbb{B}$. By the triangle inequality, we have

$$\begin{aligned}
&\sup_{x,y\in\mathbb{B},x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}}\\
\leq \;& C\|f-f_t\|_{\omega,\alpha} + \frac{C\omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)[tx,ty]^{\gamma}}{\omega\big((1-|tx|^2)^{\beta}(1-|ty|^2)^{\alpha-\beta}\big)[x,y]^{\gamma}}\|f\|_{\omega,\alpha}.
\end{aligned}$$

In the above inequality, first by letting $|x| \to 1^-$ and then letting $t \to 1^-$, we obtain the desired result. □

By adding the restriction $y \in E(x,r)$ in Theorem 3.1, we characterize $\mathcal{B}^{\alpha}_{\omega}$ in terms of E_f as follows.

Theorem 3.3. *Let $r \in (0,1)$, $f \in H(\mathbb{B})$, $0 \le \beta \le \alpha$ and $0 \le \gamma \le 1$. Then $f \in \mathcal{B}^\alpha_\omega$ if and only if*

$$E_f = \sup_{y \in E(x,r), x \neq y} \omega\big((1-|x|^2)^\beta (1-|y|^2)^{\alpha-\beta}\big) \frac{|f(x)-f(y)|}{[x,y]^\gamma |x-y|^{1-\gamma}} < \infty.$$

Proof. First we prove the sufficiency. Let $f \in H(\mathbb{B})$. For each $x \in \mathbb{B}$, it follows from the proof of Theorem 3.1 that

$$\begin{aligned} |\nabla f(x)| &\le \frac{C}{(1-|x|^2)} \int_{E(x,r)} |f(x)-f(y)| d\tau(y) \\ &\le C \int_{E(x,r)} \frac{|f(x)-f(y)|}{[x,y]^\gamma |x-y|^{1-\gamma}} d\tau(y). \end{aligned}$$

This gives

$$\begin{aligned} |\nabla f(x)| &\le C E_f \int_{E(x,r)} \frac{d\tau(y)}{\omega\big((1-|x|^2)^\beta (1-|y|^2)^{\alpha-\beta}\big)} \\ &\le C \int_{E(x,r)} \frac{d\tau(y)}{\omega\big(\lambda^{\alpha-\beta}(1-|x|^2)^\alpha\big)}. \end{aligned}$$

By Lemma 2.5, we conclude that $f \in \mathcal{B}^\alpha_\omega$.

Conversely, let $f \in \mathcal{B}^\alpha_\omega$ and for any $y \in E(x,r), y \neq x$,

$$\begin{aligned} |f(x)-f(y)| &= \Big| \int_0^1 \frac{df}{ds}(sx+(1-s)y) ds \Big| \\ &\le \sum_{k=1}^n \Big| (x_k - y_k) \int_0^1 \frac{\partial f}{\partial x_k}(sx+(1-s)y) ds \Big| \\ &\le C|x-y| \|f\|_{\omega,\alpha} \int_0^1 \frac{ds}{\omega\big((1-|sx+(1-s)y|^2)^\alpha\big)}. \end{aligned}$$

Since for $s \in [0,1]$, by Lemma 2.6,

$$\begin{aligned} 1-|sx+(1-s)y|^2 &\ge 1-|sx+(1-s)y| \\ &\ge s(1-|x|)+(1-s)(1-|y|) \\ &\ge s\Big(\frac{1-|x|^2}{2}\Big)+(1-s)\Big(\frac{1-|y|^2}{2}\Big) \\ &\ge \frac{1}{2}(1-|x|^2)^s (1-|y|^2)^{1-s} \end{aligned}$$

and $1-|y|^2 \geq \lambda(1-|x|^2)$ for $y \in E(x,r), y \neq x$, we infer that

$$\begin{aligned}\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} &\leq C\int_0^1 \frac{ds}{\omega(\frac{1}{2^{\alpha}}(1-|x|^2)^{\alpha s}(1-|y|^2)^{\alpha-\alpha s})} \\ &\leq C\int_0^1 \frac{ds}{\omega(\frac{1}{2^{\alpha}}(1-|x|^2)^{\alpha}\lambda^{\alpha-\alpha s})} \\ &\leq \frac{C}{\omega((1-|x|^2)^{\alpha})}\int_0^1 \frac{2^{\alpha}ds}{\lambda^{\alpha-\alpha s}} \\ &\leq \frac{C}{\omega((1-|x|^2)^{\alpha})}.\end{aligned}$$

Thus,

$$\sup_{y\in E(x,r),x\neq y} \omega((1-|x|^2)^{\alpha})\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} < \infty.$$

For each $y \in E(x,r)$,

$$(1-|x|^2)^{\alpha} = (1-|x|^2)^{\beta}(1-|x|^2)^{\alpha-\beta} \geq (1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\lambda^{\alpha-\beta}.$$

By Lemma 2.6 again, we deduce that

$$\omega((1-|x|^2)^{\alpha}) \geq \lambda^{\alpha-\beta}\omega((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}),$$

from which we see that $E_f < \infty$. □

Similarly, we can prove the following

Theorem 3.4. *Let $f \in \mathcal{B}^{\alpha}_{\omega}$, $r \in (0,1)$, $0 \leq \beta \leq \alpha$ and $0 \leq \gamma \leq 1$. Then $f \in \mathcal{B}^{\alpha}_{\omega,0}$ if and only if*

$$\lim_{|x|\to 1^-} \sup_{y\in E(x,r),x\neq y} \omega\big((1-|x|^2)^{\beta}(1-|y|^2)^{\alpha-\beta}\big)\frac{|f(x)-f(y)|}{[x,y]^{\gamma}|x-y|^{1-\gamma}} = 0.$$

Remark 3.5. When $\omega(t) = t$, $\gamma = 0$, Li and Wulan [8] obtained the holomorphic version of Theorems 3.3 and 3.4 in the unit ball of $\mathbb{C}^n$.

4. Hyperbolic harmonic Besov space

In this section, we show some characterizations for Besov space $\mathcal{B}_p$ of $H(\mathbb{B})$. Firstly, we generalize Theorem 1.3 into the following form.

Theorem 4.1. *Let $f \in H(\mathbb{B})$, $p \in (2(n-1),\infty)$ and $0 \leq \mu \leq p$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{B}}\int_{\mathbb{B}}(1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}}\frac{|f(x)-f(y)|^p}{[x,y]^{\mu}|x-y|^{p-\mu}}d\tau(x)d\tau(y) < \infty. \tag{4.1}$$

Proof. Assume that $f \in \mathcal{B}_p$. Since $|x-y| \le [x,y]$ for all $x, y \in \mathbb{B}$, it follows from Theorem 1.3 that (4.1) holds.

Conversely, assume that (4.1) holds. From the proof of [10] and Lemma 2.1, we have

$$(1-|x|^2)|\nabla f(x)| \le C \int_{E(x,r)} (1-|x|^2)^{\frac{1}{2}}(1-|y|^2)^{\frac{1}{2}} \frac{|f(x)-f(y)|}{[x,y]} d\tau(y).$$

As an application of Hölder's inequality,

$$\begin{aligned}
(1-|x|^2)^p|\nabla f(x)|^p &\le C \int_{E(x,r)} (1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}} \frac{|f(x)-f(y)|^p}{[x,y]^p} d\tau(y) \\
&\le C \int_{\mathbb{B}} (1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}} \frac{|f(x)-f(y)|^p}{[x,y]^p} d\tau(y) \\
&\le C \int_{\mathbb{B}} (1-|x|^2)^{\frac{p}{2}}(1-|y|^2)^{\frac{p}{2}} \frac{|f(x)-f(y)|^p}{[x,y]^\mu |x-y|^{p-\mu}} d\tau(y),
\end{aligned}$$

and the result follows. □

Secondly, we give a new characterization of $\mathcal{B}_p$ in terms of a double integral of the function $\frac{|f(x)-f(y)|^p}{[x,y]^{2n}}$.

Theorem 4.2. *Let $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}_p$ if and only if*

$$I = \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{[x,y]^{2n}} dv(x) dv(y) < \infty. \tag{4.2}$$

Proof. Assume that $f \in \mathcal{B}_p$. Making the change of variables $y = \phi_x(u)$ we have

$$\begin{aligned}
I &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f\circ\phi_x(0) - f\circ\phi_x(u)|^p}{[x,\phi_x(u)]^{2n}} |J\phi_x(u)| dv(x) dv(u) \\
&= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|f\circ\phi_x(0) - f\circ\phi_x(u)|^p}{(1-|\phi_x(u)|^2)^n} \frac{(1-|u|^2)^n}{[x,u]^{2n}} dv(u) dv(x) \\
&= \int_{\mathbb{B}} \int_{\mathbb{B}} |f\circ\phi_x(0) - f\circ\phi_x(u)|^p dv(u) d\tau(x) \\
&\le C \int_{\mathbb{B}} d\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla}(f\circ\phi_x)(u)|^p dv(u)
\end{aligned}$$

The last inequality follows from the proof of [10, Theorem 4.1].

Since $|\widetilde{\nabla}(f\circ\phi_x)(u)| = |\widetilde{\nabla} f(\phi_x(u))|$, changing variables again leads to

$$\begin{aligned}
I &\le C \int_{\mathbb{B}} d\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla}(f\circ\phi_x)(u)|^p dv(u) \\
&\le C \int_{\mathbb{B}} d\tau(x) \int_{\mathbb{B}} |\widetilde{\nabla} f(w)|^p \frac{(1-|x|^2)^n}{[x,w]^{2n}} dv(w).
\end{aligned}$$

It follows from Fubini's theorem and Lemma 2.4 that

$$\begin{aligned} I &\leq C\int_{\mathbb{B}} |\widetilde{\nabla} f(w)|^p d\tau(w) \\ &= C\int_{\mathbb{B}} (1-|w|^2)^p |\nabla f(w)|^p d\tau(w). \end{aligned}$$

For the converse, we assume that (4.2) holds. Since for $x \in \mathbb{B}$,

$$(1-|x|^2)|\nabla f(x)| \leq C\int_{E(x,r)} |f(x)-f(y)| d\tau(y),$$

by Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned} \int_{\mathbb{B}} (1-|x|^2)^p |\nabla f(x)|^p d\tau(x) &\leq C\int_{\mathbb{B}}\int_{E(x,r)} \frac{|f(x)-f(y)|^p}{[x,y]^{2n}} dv(x)dv(y) \\ &\leq CI, \end{aligned}$$

from which we see that $f \in \mathcal{B}_p$. □

As an application of Theorem 4.2, we end this section with a corollary which can be regarded as an extension of [7, Theorem 1] into the h-harmonic setting.

Corollary 4.3. *Let $f \in H(\mathbb{B})$, $n \leq \alpha, \beta < \infty$. Then $f \in \mathcal{B}_p$ if and only if*

$$J = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{[x,y]^{\alpha+\beta}} (1-|x|^2)^\alpha (1-|y|^2)^\beta d\tau(x)d\tau(y) < \infty. \tag{4.3}$$

Proof. Let (4.3) hold. It follows from the proofs of the above theorems that

$$\begin{aligned} (1-|x|^2)^p|\nabla f(x)|^p &\leq C\int_{E(x,r)} |f(x)-f(y)|^p d\tau(y) \\ &\leq C\int_{E(x,r)} \frac{|f(x)-f(y)|^p}{[x,y]^{\alpha+\beta}} (1-|x|^2)^\alpha (1-|y|^2)^\beta d\tau(y) \\ &\leq C\int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{[x,y]^{\alpha+\beta}} (1-|x|^2)^\alpha (1-|y|^2)^\beta d\tau(y), \end{aligned}$$

from which we see that $f \in \mathcal{B}_p$.

Now, we prove the converse. Suppose that $f \in \mathcal{B}_p$. Then

$$\begin{aligned} J &= \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{[x,y]^{2n}} \frac{(1-|x|^2)^{\alpha-n}(1-|y|^2)^{\beta-n}}{[x,y]^{\alpha+\beta-2n}} dv(x)dv(y) \\ &\leq C\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|f(x)-f(y)|^p}{[x,y]^{2n}} dv(x)dv(y). \end{aligned}$$

Following Theorem 4.2, we conclude that $J < \infty$. □

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