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ON THE CHARACTER SPACE OF BANACH VECTOR-VALUED FUNCTION ALGEBRAS

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Abstract. Given a compact space X and a commutative Banach algebra A, the character spaces of A-valued function algebras on X are investigated. The class of natural A-valued function algebras, those whose characters can be described by means of characters of A and point evaluation homomorphisms, is introduced and studied. For an admissible Banach A-valued function algebra $\mathcal{A}$ on X, conditions under which the character space $\mathfrak{M}(\mathcal{A})$ is homeomorphic to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ are presented, where $\mathfrak{A} = C(X) \cap \mathcal{A}$ is the subalgebra of $\mathcal{A}$ consisting of scalar-valued functions. An illustration of the results is given by some examples.
Keywords: Commutative Banach algebras, Banach function algebras, vector-valued function algebras, vector-valued characters.
MSC(2010): Primary 46J10; Secondary 46J20.

1. Introduction and preliminaries

Let A be a commutative unital Banach algebra over the complex field $\mathbb{C}$. Every nonzero homomorphism $\phi : A \to \mathbb{C}$ is called a *character* of A. Denoted by $\mathfrak{M}(A)$, the set of all characters of A is nonempty and its elements are automatically continuous [13, Lemma 2.1.5]. Consider the Gelfand transform $\hat{A} = \{\hat{a} : a \in A\}$, where $\hat{a} : \mathfrak{M}(A) \to \mathbb{C}$ is defined by $\hat{a}(\phi) = \phi(a)$. The *Gelfand topology* of $\mathfrak{M}(A)$ is the weakest topology with respect to which every $\hat{a} \in \hat{A}$ is continuous. Endowed with the Gelfand topology, $\mathfrak{M}(A)$ is compact and Hausdorff. By [13, Theorem 2.1.8], an ideal M in A is maximal if and only if $M = \ker \phi$, for some $\phi \in \mathfrak{M}(A)$. For this reason, sometimes $\mathfrak{M}(A)$ is called the *maximal ideal space* of A. For more on the theory of commutative Banach algebras see, for example, [5, 6, 13, 19].

1.1. **Function Algebras.** Let X be a compact Hausdorff space and $C(X)$ be the Banach algebra of all continuous functions $f : X \to \mathbb{C}$ equipped with the

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uniform norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A subalgebra $\mathfrak{A}$ of $C(X)$ is called a *function algebra* on X if $\mathfrak{A}$ separates the points of X and contains the constant functions. If $\mathfrak{A}$ is equipped with some complete algebra norm $\|\cdot\|$, then $\mathfrak{A}$ is called a *Banach function algebra*. If the norm $\|\cdot\|$ of $\mathfrak{A}$ is equivalent to the uniform norm $\|\cdot\|_X$, then $\mathfrak{A}$ is called a *uniform algebra*.

Identifying the character space of a Banach function algebra $\mathfrak{A}$ has been always a problem of interest for mathematicians in this field. For every $x \in X$, the evaluation homomorphism $\varepsilon_x : f \mapsto f(x)$ is a character of $\mathfrak{A}$, and the mapping $J : X \to \mathfrak{M}(\mathfrak{A})$, $x \mapsto \varepsilon_x$, imbeds X homeomorphically as a compact subset of $\mathfrak{M}(\mathfrak{A})$. If J is surjective, one calls $\mathfrak{A}$ *natural* [6, Chapter 4]. In this case, the character space $\mathfrak{M}(\mathfrak{A})$ is identical to X. Note that every semisimple commutative Banach algebra A can be considered, through its Gelfand representation, as a natural Banach function algebra on its character space $\mathfrak{M}(A)$.

A relation between the character space $\mathfrak{M}(\mathfrak{A})$ of a Banach function algebra $\mathfrak{A}$ and the character space $\mathfrak{M}(\bar{\mathfrak{A}})$ of its uniform closure $\bar{\mathfrak{A}}$ was revealed in [10] as follows.

Theorem 1.1 (Honary [10]). *The restriction map $\mathfrak{M}(\bar{\mathfrak{A}}) \to \mathfrak{M}(\mathfrak{A})$, $\psi \mapsto \psi|_{\mathfrak{A}}$, is a homeomorphism if and only if $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathfrak{A}$.*

The above result appears to be very useful in identifying the character spaces in a wide class of Banach function algebras. We establish an analogue of this result for vector-valued function algebras in Section 3.

1.2. **Vector-valued function algebras.** Let A be a commutative unital Banach algebra, and let $C(X, A)$ be the space of all A-valued continuous functions on X. Algebraic operations and the uniform norm $\|\cdot\|_X$ on $C(X, A)$ are defined in the obvious way.

Definition 1.2 (c.f. [3, 15]). A subalgebra $\mathcal{A}$ of $C(X, A)$ is called an *A-valued function algebra* on X if (1) $\mathcal{A}$ contains the constant functions $X \to A$, $x \mapsto a$, for all $a \in A$, and (2) $\mathcal{A}$ separates the points of X in the sense that, for every pair $x, y \in X$ with $x \neq y$, and for every maximal ideal M of A, there exists some $f \in A$ such that $f(x) - f(y) \notin M$. If $\mathcal{A}$ is endowed with some algebra norm $|||\cdot|||$ such that the restriction of $|||\cdot|||$ to A is equivalent to the original norm of A and $\|f\|_X \leq |||f|||$, for every $f \in \mathcal{A}$, then $\mathcal{A}$ is called a *normed A-valued function algebra* on X. If the given norm is complete, then $\mathcal{A}$ is called a *Banach A-valued function algebra*. If the given norm is equivalent to the uniform norm $\|\cdot\|_X$, then $\mathcal{A}$ is called an *A-valued uniform algebra*. When no confusion can arise, we use the same notation $\|\cdot\|$ for the norm of $\mathcal{A}$.

Continuing the work of Yood [18], Hausner [9] proved that τ is a character of $C(X, A)$ if, and only if, there exist a point $x \in X$ and a character $\phi \in \mathfrak{M}(A)$ such that $\tau(f) = \phi(f(x))$, for all $f \in C(X, A)$, whence $\mathfrak{M}(C(X, A))$ is homeomorphic to $X \times \mathfrak{M}(A)$. (In this regard, see [1].) We call a Banach A-valued function algebra *natural* if, like $C(X, A)$, its character space is identical to $X \times \mathfrak{M}(A)$. For instance, in Example 4.1, we will see that the A-valued Lipschitz algebra $\operatorname{Lip}(X, A)$ is natural; see also [7, 11]. Natural A-valued function algebras are studied in Section 2.

1.3. **Notations and conventions.** Throughout, X is a compact Hausdorff space, and A is a *semisimple* commutative unital Banach algebra. The unit element of A is denoted by $\mathbf{1}$, and the set of invertible elements of A is denoted by $\operatorname{Inv}(A)$. If $f : X \to \mathbb{C}$ is a function and $a \in A$, we write fa to denote the A-valued function $X \to A$, $x \mapsto f(x)a$. If $\mathfrak{A}$ is a function algebra on X, we let $\mathfrak{A}A$ be the linear span of $\{fa : f \in \mathfrak{A},\ a \in A\}$, so that any element $f \in \mathfrak{A}A$ is of the form $f = f_1a_1 + \cdots + f_na_n$ with $f_j \in \mathfrak{A}$ and $a_j \in A$. Given an element $a \in A$, we use the same notation a for the constant function $X \to A$ given by $a(x) = a$, for all $x \in X$, and consider A as a closed subalgebra of $C(X, A)$. Since A has a unit element $\mathbf{1}$, we identify $\mathbb{C}$ with the closed subalgebra $\mathbb{C}\mathbf{1}$ of A. Whence every continuous function $f : X \to \mathbb{C}$ can be considered as the continuous A-valued function $f\mathbf{1} : x \mapsto f(x)\mathbf{1}$. We drop $\mathbf{1}$ using the same notation f for this A-valued function and adopt the identification $C(X) = C(X)\mathbf{1}$ as a closed subalgebra of $C(X, A)$. Finally, for a family $\mathcal{M}$ of A-valued functions on X, a point $x \in X$, and a character $\phi \in \mathfrak{M}(A)$, we set

$$\mathcal{M}(x) = \{f(x) : f \in \mathcal{M}\}, \quad \phi[\mathcal{M}] = \{\phi \circ f : f \in \mathcal{M}\}.$$

2. Natural vector-valued function algebras

Let $\mathcal{A}$ be an A-valued function algebra on X. Assume that M is a maximal ideal of A, $x_0 \in X$, and set

$$\mathcal{M} = \{f \in \mathcal{A} : f(x_0) \in M\}. \tag{2.1}$$

The fact that $\mathcal{M}$ is an ideal of $\mathcal{A}$ is obvious. We prove that $\mathcal{M}$ is maximal. Take a function $g \in \mathcal{A} \setminus \mathcal{M}$ so that $g(x_0) \notin M$. Since M is maximal in A, there exist $a \in M$ and $b \in A$ such that $\mathbf{1} = a + g(x_0)b$. Consider b as a constant function of X into A and let $f = \mathbf{1} - gb$. Then $f(x_0) = a \in M$ so that $f \in \mathcal{M}$ and $\mathbf{1} = f + gb$ which means that the ideal of $\mathcal{A}$ generated by $\mathcal{M} \cup \{g\}$ is equal to $\mathcal{A}$. Hence $\mathcal{M}$ is maximal in $\mathcal{A}$.

Definition 2.1. An A-valued function algebra $\mathcal{A}$ on X is called *natural* on X, if every maximal ideal $\mathcal{M}$ of $\mathcal{A}$ is of the form (2.1), for some $x_0 \in X$ and $M \in \mathfrak{M}(A)$.

In case $A = \mathbb{C}$, natural A-valued function algebras coincide with natural (complex) function algebras.

Theorem 2.2. *Let $\mathcal{A}$ be an A-valued function algebra on X. If $\mathcal{M}$ is a maximal ideal in $\mathcal{A}$ and $\mathcal{M}(x_0) \neq A$, for some $x_0 \in X$, then*

(1) *$\mathcal{M}(x_0)$ is a maximal ideal of A;*

(2) $\mathcal{M}(x) = A$ *for* $x \neq x_0$*;*
(3) $\mathcal{M} = \{f \in \mathcal{A} : f(x_0) \in M\}$*, where* $M = \mathcal{M}(x_0)$*.*

Proof. It is easily verified that $\mathcal{M}(x_0)$ is an ideal. We show that $\mathcal{M}(x_0)$ is maximal. Assume that $a \notin \mathcal{M}(x_0)$. Then a, as a constant function on X, does not belong to $\mathcal{M}$. Hence, the ideal of $\mathcal{A}$ generated by $\mathcal{M} \cup \{a\}$ is equal to $\mathcal{A}$ meaning that $\mathbf{1} = f + ag$, for some $f \in \mathcal{M}$ and $g \in \mathcal{A}$. In particular, $\mathbf{1} = f(x_0) + ag(x_0)$ which implies that the ideal of A generated by $\mathcal{M}(x_0) \cup \{a\}$ is equal to A. Hence, $\mathcal{M}(x_0)$ is maximal.

Now, assume that $x \neq x_0$. Since $\mathcal{A}$ separates the points of X (Definition 1.2), for the maximal ideal $\mathcal{M}(x_0)$ in A, there is a function $f \in \mathcal{A}$ such that $f(x) - f(x_0) \notin \mathcal{M}(x_0)$. Define $g(s) = f(s) - f(x)$ so that $g(x_0) \notin \mathcal{M}(x_0)$. This implies that $g \notin \mathcal{M}$. Since $\mathcal{M}$ is maximal, there are $h \in \mathcal{M}$ and $k \in \mathcal{A}$ such that $h + kg = \mathbf{1}$. Hence, $\mathbf{1} = h(x) \in \mathcal{M}(x)$ and $\mathcal{M}(x) = A$. □

It is proved in [1] that the algebra $C(X, A)$ satisfies all conditions in Theorem 2.2. Therefore $C(X, A)$ is natural.

Corollary 2.3. *Let* $\mathcal{A}$ *be an* A*-valued function algebra on* X*.*

(1) *The algebra* $\mathcal{A}$ *is natural if, and only if, for every proper ideal* $\mathcal{I}$ *in* $\mathcal{A}$*, there exists some* $x_0 \in X$ *such that* $\mathcal{I}(x_0) \neq A$*.*
(2) *If* $\mathcal{I}$ *is an ideal in* $\mathcal{A}$ *such that* $\mathcal{I}(x_0)$ *and* $\mathcal{I}(x_1)$*, for* $x_0 \neq x_1$*, are proper ideals in* A*, then* $\mathcal{I}$ *cannot be maximal in* $\mathcal{A}$*.*

The next discussion requires a concept of zero sets. The zero set of a function $f : X \to A$ is defined as $Z(f) = \{x : f(x) = \mathbf{0}\}$. This concept of zero set, however, is not useful here in our discussion because, in general, the algebra A may contain nonzero singular elements. Instead, the following slightly modified version of this concept appears to be very useful.

Definition 2.4. For a function $f : X \to A$, the *singular set* of f is defined to be

$$Z_{\mathrm{s}}(f) = \{x \in X : f(x) \notin \mathrm{Inv}(A)\}. \tag{2.2}$$

The following is an analogy of [6, Proposition 4.1.5 (i)].

Theorem 2.5. *Let* $\mathcal{A}$ *be a Banach* A*-valued function algebra on* X*. Then* $\mathcal{A}$ *is natural if, and only if, for each finite set* $\{f_1, \ldots, f_n\}$ *of elements in* $\mathcal{A}$ *with* $\bigcap_{j=1}^n Z_{\mathrm{s}}(f_j) = \emptyset$*, there exist* $g_1, \ldots, g_n \in \mathcal{A}$ *such that*

$$f_1 g_1 + \cdots + f_n g_n = \mathbf{1}.$$

Proof. ($\Rightarrow$) Suppose that $\mathcal{A}$ is natural and, for a finite set $\{f_1, \ldots, f_n\}$ in $\mathcal{A}$, assume that $Z_{\mathrm{s}}(f_1) \cap \cdots \cap Z_{\mathrm{s}}(f_n) = \emptyset$. Let $\mathcal{I}$ be the ideal generated by $\{f_1, \ldots, f_n\}$. If $\mathcal{I} \neq \mathcal{A}$, then, since $\mathcal{A}$ is natural, by Corollary 2.3, there exists a point $x_0 \in X$

such that $\mathcal{I}(x_0) \neq A$. In particular, the elements $f_1(x_0), \dots, f_n(x_0)$ are all singular in A, which means that $x_0 \in Z_s(f_1) \cap \cdots \cap Z_s(f_n)$, a contradiction. Therefore, $\mathcal{I} = \mathcal{A}$ whence there exist $g_1, \dots, g_n \in \mathcal{A}$ such that $f_1 g_1 + \cdots + f_n g_n = \mathbf{1}$.

($\Leftarrow$) To show that $\mathcal{A}$ is natural, we take a maximal ideal $\mathcal{M}$ of $\mathcal{A}$ and, using Corollary 2.3, we show that $\mathcal{M}(x_0) \neq A$, for some $x_0 \in X$. Assume, towards a contradiction, that, for every $x \in X$, there exists a function $f_x \in \mathcal{M}$ such that $f_x(x) = \mathbf{1}$. Set $V_x = f_x^{-1}(\mathrm{Inv}(A))$. Then $\{V_x : x \in X\}$ is an open covering of the compact space X. So there exist finitely many points $x_1, \dots, x_n \in X$ such that $X \subset V_{x_1} \cup \cdots \cup V_{x_n}$. Then $Z_s(f_{x_1}) \cap \cdots \cap Z_s(f_{x_n}) = \emptyset$. By the assumption, there exist functions $g_1, \dots, g_n \in \mathcal{A}$ such that $f_{x_1} g_1 + \cdots + f_{x_n} g_n = \mathbf{1}$. Hence, $\mathbf{1} \in \mathcal{M}$, which is a contradiction. □

Let $f \in \mathcal{A}$ and suppose that $Z_s(f) = \emptyset$ so that $f(X) \subset \mathrm{Inv}(A)$. Since the inverse mapping $a \mapsto a^{-1}$ of $\mathrm{Inv}(A)$ onto itself is continuous, the mapping $x \mapsto f(x)^{-1}$, denoted by $\mathbf{1}/f$, is a continuous A-valued function on X. Hence f is invertible in $C(X, A)$. However, f may not be invertible in $\mathcal{A}$. Let us call $\mathcal{A}$ a *full subalgebra* of $C(X, A)$ if every $f \in \mathcal{A}$ that is invertible in $C(X, A)$ is invertible in $\mathcal{A}$. The following is an analogy of [2, Theorem 2.1].

Theorem 2.6. *Let $\mathcal{A}$ be a Banach A-valued function algebra on X such that $\bar{\mathcal{A}}$, the uniform closure of $\mathcal{A}$, is natural. If $\mathbf{1}/f \in \mathcal{A}$ whenever $f \in \mathcal{A}$ and $Z_s(f) = \emptyset$, then $\mathcal{A}$ is natural.*

Proof. We apply Theorem 2.5 to prove that $\mathcal{A}$ is natural. Let $f_1, \dots, f_n$ be elements in $\mathcal{A}$ such that $Z_s(f_1) \cap \cdots \cap Z_s(f_n) = \emptyset$. We prove the existence of a finite set $\{g_1, \dots, g_n\}$ of elements in $\mathcal{A}$ such that $f_1 g_1 + \cdots + f_n g_n = \mathbf{1}$. Regarding $f_1, \dots, f_n$ as elements of $\bar{\mathcal{A}}$, since $\bar{\mathcal{A}}$ is natural, again by Theorem 2.5, there exist $h_1, \dots, h_n$ in $\bar{\mathcal{A}}$ such that $f_1 h_1 + \cdots + f_n h_n = \mathbf{1}$. For each h_j, there is some $g_j \in \mathcal{A}$ such that $\|h_j - g_j\|_X < \big(\sum_{j=1}^n \|f_j\|_X\big)^{-1}$. Thus

$$\Big\|\mathbf{1} - \sum_{j=1}^n f_j g_j\Big\|_X = \Big\|\sum_{j=1}^n f_j h_j - \sum_{j=1}^n f_j g_j\Big\|_X \leq \sum_{j=1}^n \|f_j\|_X \|h_j - g_j\|_X < 1. \tag{2.3}$$

Hence, for every $x \in X$, $f(x) = \sum f_j(x) g_j(x)$ is an invertible element of A, so that for the function $f = \sum f_j g_j$, which belongs to $\mathcal{A}$, we have $Z_s(f) = \emptyset$. By the assumption, there is a function g in $\mathcal{A}$ such that $\mathbf{1} = fg = \sum f_j (g_j g)$. Now, Theorem 2.5 shows that $\mathcal{A}$ is natural. □

An application of the above theorem is given in Example 4.1.

Let $\mathcal{A}$ be a Banach A-valued function algebra. For every point $x \in X$ and character $\phi \in \mathfrak{M}(A)$ define

$$\varepsilon_x \diamond \phi : \mathcal{A} \to \mathbb{C}, \quad \varepsilon_x \diamond \phi(f) = \varepsilon_x(\phi \circ f) = \phi(f(x)).$$

Then $\varepsilon_x \diamond \phi$ is a character of $\mathcal{A}$ with $\ker(\varepsilon_x \diamond \phi) = \{f \in \mathcal{A} : f(x) \in \ker \phi\}$, which of course is of the form (2.1). Define

$$\mathcal{J} : X \times \mathfrak{M}(A) \to \mathfrak{M}(\mathcal{A}), \quad (x, \phi) \to \varepsilon_x \diamond \phi. \tag{2.4}$$

Theorem 2.7. *The mapping $\mathcal{J}$ is a homeomorphism of $X \times \mathfrak{M}(A)$ onto a compact subset of $\mathfrak{M}(\mathcal{A})$. If $\mathcal{A}$ is natural, then $\mathfrak{M}(\mathcal{A})$ is homeomorphic to $X \times \mathfrak{M}(A)$.*

Proof. Take $x \in X$, $\phi \in \mathfrak{M}(A)$, and set $\tau_0 = \varepsilon_x \diamond \phi$. Let W be a neighbourhood of τ_0 in $\mathfrak{M}(\mathcal{A})$ of the form

$$W = \{\tau \in \mathfrak{M}(\mathcal{A}) : |\tau(f_i) - \tau_0(f_i)| < \varepsilon,\ 1 \le i \le n\},$$

where $f_1, \ldots, f_n \in \mathcal{A}$. Take

$$U = \{y \in X : \|f_i(y) - f_i(x)\| < \varepsilon/2,\ 1 \le i \le n\},$$
$$V = \{\psi \in \mathfrak{M}(A) : |\psi(f_i(x)) - \phi(f_i(x))| < \varepsilon/2,\ 1 \le i \le n\}.$$

Then U is a neighbourhood of x in X and V is a neighbourhood of ϕ in $\mathfrak{M}(A)$, so that $U \times V$ is a neighbourhood of (x, ϕ) in $X \times \mathfrak{M}(A)$. If $(y, \psi) \in U \times V$ then, for every i $(1 \le i \le n)$,

$$\begin{aligned} |\psi(f_i(y)) - \phi(f_i(x))| &\le |\psi(f_i(y)) - \psi(f_i(x))| + |\psi(f_i(x)) - \phi(f_i(x))| \\ &< \|\psi\| \|f_i(y) - f_i(x)\| + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This shows that $\varepsilon_y \diamond \psi \in W$ and thus $\mathcal{J}$ is continuous. Finally, if $\mathcal{A}$ is natural then every maximal ideal of $\mathcal{A}$ is of the form (2.1) which means that every character $\tau \in \mathfrak{M}(\mathcal{A})$ is of the form $\tau = \varepsilon_x \diamond \phi$, for some $x \in X$ and $\phi \in \mathfrak{M}(A)$. Hence, $\mathcal{J}$ is a surjection and thus a homeomorphism. □

3. Characters on vector-valued function algebras

We turn to a more general case where a vector-valued function algebra may not be natural. Let $\mathcal{A}$ be a Banach A-valued function algebra. We show that, under certain conditions, the character space $\mathfrak{M}(\mathcal{A})$ is identical to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$, where $\mathfrak{A} = C(X) \cap \mathcal{A}$ is the subalgebra of $\mathcal{A}$ consisting of scalar-valued functions. To this end, we should restrict ourself to the class of admissible algebras. If $f \in \mathcal{A}$ and $\phi \in \mathfrak{M}(A)$, it is clear that $\phi \circ f \in C(X)$; it is not, however, clear whether the A-valued function $(\phi \circ f)\mathbf{1}$ belongs to $\mathcal{A}$. In fact, [3, Example 2.4] shows that it may very well happen that $(\phi \circ f)\mathbf{1} \notin \mathcal{A}$.

Definition 3.1 ([3])**.** The A-valued function algebra $\mathcal{A}$ is called *admissible* if

$$\{(\phi \circ f)\mathbf{1} : f \in \mathcal{A},\ \phi \in \mathfrak{M}(A)\} \subset \mathcal{A}. \tag{3.1}$$

Note that $\mathcal{A}$ is admissible if, and only if, $\phi[\mathcal{A}]\mathbf{1} \subset \mathcal{A}$, for all $\phi \in \mathfrak{M}(A)$.

Admissible vector-valued function algebras exist around in abundant. Some typical examples are $C(X, A)$, $\mathrm{Lip}(X, A)$, $P(K, A)$, $R(K, A)$, etc. Tensor products of the form $\mathfrak{A} \otimes A$, where $\mathfrak{A}$ is a (Banach) function algebra on X, can be seen as admissible A-valued function algebras. (More details are given in Example 4.4).

During this section, we assume that $\mathcal{A}$ is admissible and set $\mathfrak{A} = \mathcal{A} \cap C(X)$. Then $\mathfrak{A}$ is the subalgebra of $\mathcal{A}$ consisting of all complex functions in $\mathcal{A}$, it forms a complex function algebra by itself, and $\mathfrak{A} = \phi[\mathcal{A}]$, for all $\phi \in \mathfrak{M}(A)$. Our aim is to give a description of maximal ideals in $\mathcal{A}$. To begin, take a character $\phi \in \mathfrak{M}(A)$ and a maximal ideal M of $\mathfrak{A}$, and set

$$\mathcal{M} = \{f \in \mathcal{A} : \phi \circ f \in \mathsf{M}\}. \tag{3.2}$$

Then $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$. One way to see this (though it can be seen directly) is as follows. Take $\psi \in \mathfrak{M}(\mathfrak{A})$ with $\mathsf{M} = \ker \psi$ and define

$$\psi \diamond \phi : \mathcal{A} \to \mathbb{C}, \quad \psi \diamond \phi(f) = \psi(\phi \circ f).$$

Note that $\psi(\phi \circ f)$ is meaningful since $\phi \circ f \in \mathfrak{A}$. The functional $\psi \diamond \phi$ is a character of $\mathcal{A}$ with $\ker(\psi \diamond \phi) = \mathcal{M}$. Hence $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$. The main question is whether any maximal ideal $\mathcal{M}$ of $\mathcal{A}$ is of the form (3.2).

Lemma 3.2. *A maximal ideal $\mathcal{M}$ of $\mathcal{A}$ is of the form (3.2) if and only if $\phi[\mathcal{M}] \neq \mathfrak{A}$ for some $\phi \in \mathfrak{M}(A)$.*

Proof. If $\mathcal{M}$ is of the form (3.2) then clearly $\phi[\mathcal{M}] \neq \mathfrak{A}$. Conversely, assume that $\phi[\mathcal{M}] \neq \mathfrak{A}$ for some $\phi \in \mathfrak{M}(A)$. Then $\phi[\mathcal{M}]$ is an ideal of $\mathfrak{A}$. We show that it is maximal. If $g \notin \phi[\mathcal{M}]$, then $g = g\mathbf{1}$ (as an A-valued function on X) does not belong to $\mathcal{M}$. Since $\mathcal{M}$ is maximal in $\mathcal{A}$, the ideal generated by $\mathcal{M} \cup \{g\}$ is equal to $\mathcal{A}$. This implies that $\mathbf{1} = f + gh$, for some $f \in \mathcal{M}$ and $h \in \mathcal{A}$. Since $\phi \circ g = g$, we get $1 = \phi \circ f + g(\phi \circ h)$. This means that the ideal of $\mathfrak{A}$ generated by $\phi[\mathcal{M}] \cup \{g\}$ is equal to $\mathfrak{A}$. Thus, $\phi[\mathcal{M}]$ is maximal. Set $\mathsf{M} = \phi[\mathcal{M}]$ and $\mathcal{M}_1 = \{f \in \mathcal{A} : \phi \circ f \in \mathsf{M}\}$. Then $\mathcal{M} \subset \mathcal{M}_1$ and both $\mathcal{M}$ and $\mathcal{M}_1$ are maximal ideals. Hence, $\mathcal{M} = \mathcal{M}_1$. ☐

If $\mathcal{M} = \ker \tau$, for some $\tau \in \mathfrak{M}(\mathcal{A})$, then $\mathcal{M}$ is of the form (3.2) if and only if $\tau = \psi \diamond \phi$, for some $\psi \in \mathfrak{M}(\mathfrak{A})$ and $\phi \in \mathfrak{M}(A)$. Let us extend the mapping $\mathcal{J}$ in (2.4) to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ as follows.

$$\mathcal{J} : \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A) \to \mathfrak{M}(\mathcal{A}), \quad \mathcal{J}(\psi, \phi) = \psi \diamond \phi. \tag{3.3}$$

The mapping is injective for if $\psi \diamond \phi = \psi' \diamond \phi'$ then

$$\phi(a) = \psi(\phi(a)) = \psi'(\phi'(a)) = \phi'(a) \quad (a \in A),$$
$$\psi(f) = \psi(\phi(f)) = \psi'(\phi'(f)) = \psi'(f) \quad (f \in \mathfrak{A}),$$

which implies that $\phi = \phi'$ and $\psi = \psi'$. The main question is whether $\mathcal{J}$ is surjective. If $\tau \in \mathfrak{M}(\mathcal{A})$ then $\phi = \tau|_A \in \mathfrak{M}(A)$ and $\psi = \tau|_{\mathfrak{A}} \in \mathfrak{M}(\mathfrak{A})$. The question is whether the equality $\tau = \psi \diamond \phi$ holds true; of course, it does hold if $\phi[\mathcal{M}] \neq \mathfrak{A}$.

Theorem 3.3. *If the mapping $\mathfrak{J}$ in (3.3) is a surjection, then it is a homeomorphism and, therefore, the character space $\mathfrak{M}(\mathcal{A})$ is identical to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$.*

Proof. Suppose that $\mathfrak{J}$ is a surjection (an thus a bijection). Since both the domain and the range are compact Hausdorff spaces, it suffices to prove that $\mathfrak{J}$ is open. Take $\psi_0 \in \mathfrak{M}(\mathfrak{A})$, $\phi_0 \in \mathfrak{M}(A)$ and set $\tau_0 = \mathfrak{J}(\psi_0, \phi_0) = \psi_0 \diamond \phi_0$. Let U and V be neighborhoods of ψ_0 and ϕ_0 of the following form

$$U = \{\psi \in \mathfrak{M}(\mathfrak{A}) : |\psi(f) - \psi_0(f)| < \varepsilon_1 \quad (f \in F_1)\},$$
$$V = \{\phi \in \mathfrak{M}(A) : |\phi(a) - \phi_0(a)| < \varepsilon_2 \quad (a \in F_2)\},$$

where F_1 and F_2 are finite sets in $\mathfrak{A}$ and A, respectively. Take $F = F_1 \cup F_2$ as a finite set in $\mathcal{A}$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and set

$$W = \{\tau \in \mathfrak{M}(\mathcal{A}) : |\tau(f) - \tau_0(f)| < \varepsilon \quad (f \in F)\}.$$

Then W is a neighborhood of τ_0 in $\mathfrak{M}(\mathcal{A})$ and $W \subset \mathfrak{J}(U \times V)$. Hence $\mathfrak{J}$ is open. □

The rest of this section is devoted to investigating conditions under which $\mathfrak{J}$ is surjective.

Theorem 3.4 ($\mathcal{P}$)**.** *For a character $\tau \in \mathfrak{M}(\mathcal{A})$ with $\mathcal{M} = \ker \tau$ and $\phi = \tau|_A$, the following are equivalent.*

(i) $\phi[\mathcal{M}] \neq \mathfrak{A}$.
(ii) $\mathcal{M}$ *is of the form* (3.2) *with* $\mathsf{M} = \phi[\mathcal{M}]$.
(iii) *For every $f \in \mathcal{A}$, if $\phi \circ f = \mathbf{0}$ then $f \in \mathcal{M}$.*
(iv) *For every $f \in \mathcal{A}$, $\tau(\phi \circ f) = \tau(f)$.*
(v) *For every $f \in \mathcal{A}$, if $f(X) \subset \mathcal{M}$ then $f \in \mathcal{M}$.*
(vi) *$\tau = \psi \diamond \phi$, for some $\psi \in \mathfrak{M}(\mathfrak{A})$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is just Lemma 3.2. The implication (ii) $\Rightarrow$ (iii) is clear. To see the implication (iii) $\Rightarrow$ (iv), let $g = f - (\phi \circ f)\mathbf{1}$. Then $\phi \circ g = \mathbf{0}$ and thus $g \in \mathcal{M}$ and $\tau(g) = 0$. Hence, $\tau(\phi \circ f) = \tau(f)$. The implication (iv) $\Rightarrow$ (iii) is clear.

To prove (iii) $\Leftrightarrow$ (v), we note that $f(X) \subset \mathcal{M}$ if and only if $\phi \circ f = \mathbf{0}$. In fact, $f(X) \subset \mathcal{M}$ means that, for every $x \in X$, the element $f(x)$, as a constant function of X into A, belongs to $\mathcal{M}$. This, in turn, means that $\tau(f(x)) = \phi(f(x)) = 0$, for all $x \in X$, which means that $\phi \circ f = \mathbf{0}$.

To prove (iii) $\Rightarrow$ (vi), first note that $\mathcal{A}$ being admissible implies that

$$\mathfrak{A} = \phi[\mathcal{A}] = \{\phi \circ f : f \in \mathcal{A}\}.$$

Define $\psi : \mathfrak{A} \to \mathbb{C}$ by $\psi(\phi \circ f) = \tau(f)$. This is well-defined for if $\phi \circ f = \phi \circ g$ then, by the assumption, $f - g \in \mathcal{M}$ which in turn implies that $\tau(f) = \tau(g)$. Obviously, $\psi \in \mathfrak{M}(\mathfrak{A})$ and $\tau = \psi \diamond \phi$.

Finally, we prove that (vi) $\Rightarrow$ (i). Towards a contradiction, assume that $\phi[\mathcal{M}] = \mathfrak{A}$. Then $\phi \circ f = \mathbf{1}$, for some $f \in \mathcal{M}$. Hence $1 = \psi(\mathbf{1}) = \psi(\phi \circ f) = \tau(f) = 0$ which is absurd. □

Convention. We say that '$\mathcal{A}$ has property $\mathcal{P}$' if every $\mathcal{M} \in \mathfrak{M}(\mathcal{A})$ satisfies one (and hence all) of conditions in Theorem 3.4. Hence $\mathcal{A}$ has $\mathcal{P}$ if and only if the mapping $\mathcal{J}$ in (3.3) is surjective.

Let $\bar{\mathcal{A}}$ denote the uniform closure of $\mathcal{A}$ in $C(X, A)$. The restriction map

$$\mathfrak{M}(\bar{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A}), \quad \bar{\tau} \mapsto \bar{\tau}|_{\mathcal{A}}, \tag{3.4}$$

is one-to-one and continuous with respect to the Gelfand topology [10]. We write $\mathfrak{M}(\bar{\mathcal{A}}) = \mathfrak{M}(\mathcal{A})$ if it is a homeomorphism.

Proposition 3.5. *If $\mathcal{A}$ has $\mathcal{P}$ then $\bar{\mathcal{A}}$ has $\mathcal{P}$. If $\bar{\mathcal{A}}$ has $\mathcal{P}$ and $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathcal{A}$, then $\mathcal{A}$ has $\mathcal{P}$.*

Proof. Suppose that $\mathcal{A}$ has $\mathcal{P}$. Take $\bar{\tau} \in \mathfrak{M}(\bar{\mathcal{A}})$, set $\tau = \bar{\tau}|_{\mathcal{A}}$ and $\phi = \bar{\tau}|_A = \tau|_A$. Since $\mathcal{A}$ has $\mathcal{P}$, by Theorem 3.4 (iv), $\tau(\phi \circ f) = \tau(f)$, for all $f \in \mathcal{A}$. Given $f \in \bar{\mathcal{A}}$, there is a sequence $\{f_n\}$ in $\mathcal{A}$ such that $\|f_n - f\|_X \to 0$. Hence, $\|\phi \circ f_n - \phi \circ f\|_X \to 0$, and thus

$$\bar{\tau}(\phi \circ f) = \lim_{n\to\infty} \bar{\tau}(\phi \circ f_n) = \lim_{n\to\infty} \tau(\phi \circ f_n) = \lim_{n\to\infty} \tau(f_n) = \lim_{n\to\infty} \bar{\tau}(f_n) = \bar{\tau}(f).$$

Again, by Theorem 3.4 (iv), we see that $\bar{\mathcal{A}}$ has $\mathcal{P}$.

Now, assume that $\bar{\mathcal{A}}$ has $\mathcal{P}$, and $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathcal{A}$. Take $\tau \in \mathfrak{M}(\mathcal{A})$ and $\phi = \tau|_A$. Extend τ to a character $\bar{\tau} : \bar{\mathcal{A}} \to \mathbb{C}$ (this is possible since $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathcal{A}$). Note that still we have $\phi = \bar{\tau}|_A$. Since $\bar{\mathcal{A}}$ satisfies $\mathcal{P}$, we have $\bar{\tau}(\phi \circ f) = \bar{\tau}(f)$, for all $f \in \bar{\mathcal{A}}$. This implies that $\tau(\phi \circ f) = \tau(f)$, for all $f \in \mathcal{A}$, and thus $\mathcal{A}$ has $\mathcal{P}$. □

The following is a vector-valued version of Theorem 1.1.

Theorem 3.6. *For an admissible Banach A-valued function algebra $\mathcal{A}$ with $\mathfrak{A} = C(X) \cap \mathcal{A}$, let $\bar{\mathcal{A}}$ and $\bar{\mathfrak{A}}$ be the uniform closures of $\mathcal{A}$ and $\mathfrak{A}$, respectively. Consider the following statements:*

(i) $\mathfrak{M}(\bar{\mathcal{A}}) = \mathfrak{M}(\mathcal{A})$.
(ii) $\|\hat{f}\| \leq \|f\|_X$, *for all* $f \in \mathcal{A}$.
(iii) $\|\hat{f}\| \leq \|f\|_X$, *for all* $f \in \mathfrak{A}$.
(iv) $\mathfrak{M}(\bar{\mathfrak{A}}) = \mathfrak{M}(\mathfrak{A})$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). *If $\mathcal{A}$ satisfies $\mathcal{P}$, then* (iii) $\Rightarrow$ (ii).

Proof. The equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) follow from the main theorem in [10]. The implication (ii) $\Rightarrow$ (iii) is obvious, because $\mathfrak{A} \subset \mathcal{A}$.

Assume that $\mathcal{A}$ satisfies $\mathcal{P}$, and $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathfrak{A}$. Fix a function $f \in \mathcal{A}$ and take an arbitrary character $\tau \in \mathfrak{M}(\mathcal{A})$. Since $\mathcal{A}$ has $\mathcal{P}$, we have $\tau = \psi \diamond \phi$, where $\psi = \tau|_{\mathfrak{A}}$ and $\phi = \tau|_A$. Since $\phi \circ f \in \mathfrak{A}$, we have

$$|\tau(f)| = |\psi(\phi \circ f)| \leq \|\widehat{\phi \circ f}\| \leq \|\phi \circ f\|_X \leq \|f\|_X.$$

Hence $\|\hat{f}\| \leq \|f\|_X$, for all $f \in \mathcal{A}$. □

4. **Examples**

To illustrate the results, we devote this section to some examples.

Example 4.1. Let (X, ρ) be a compact metric space. A function $f : X \to A$ is called an *A-valued Lipschitz function* if

$$L(f) = \sup\left\{ \frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X,\ x \neq y \right\} < \infty. \tag{4.1}$$

The space of A-valued Lipschitz functions on X is denoted by $\mathrm{Lip}(X, A)$. For any $f \in \mathrm{Lip}(X, A)$, the *Lipschitz norm* of f is defined by $\|f\|_L = \|f\|_X + L(f)$. This makes $\mathrm{Lip}(X, A)$ an admissible Banach A-valued function algebra on X with $\mathrm{Lip}(X) = \mathrm{Lip}(X, A) \cap C(X)$, where $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{C})$ is the classical complex Lipschitz algebra on X.

The algebra $\mathrm{Lip}(X)$ satisfies all conditions in the Stone-Weierstrass Theorem and thus it is dense in $C(X)$. On the other hand, by [9, Lemma 1], $C(X)A$ is dense in $C(X, A)$ and thus $\mathrm{Lip}(X)A$ is dense in $C(X, A)$. Since $\mathrm{Lip}(X, A)$ contains $\mathrm{Lip}(X)A$, we see that $\mathrm{Lip}(X, A)$ is dense in $C(X, A)$.

It is easy to verify that if $f \in \mathrm{Lip}(X, A)$ and $Z_s(f) = \emptyset$, then $\mathbf{1}/f \in \mathrm{Lip}(X, A)$. Since $C(X, A)$ is natural, Theorem 2.6 now implies that $\mathrm{Lip}(X, A)$ is natural. By Theorem 2.7, $\mathfrak{M}(\mathrm{Lip}(X, A))$ is homeomorphic to $X \times \mathfrak{M}(A)$. See also [7] and [11].

Example 4.2. Assume that $A = \mathbb{C}^n$, for some positive integer n. Then, for every admissible Banach A-valued function algebra $\mathcal{A}$ on X, the mapping $\mathcal{J}$ in (3.3) is surjective and thus $\mathfrak{M}(\mathcal{A})$ is identical to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(\mathbb{C}^n)$.

To see this, we show that $\mathcal{A}$ satisfies condition (i) of Theorem 3.4. Note that $\mathfrak{M}(\mathbb{C}^n) = \{\pi_1, \ldots, \pi_n\}$, where $\pi_i : \mathbb{C}^n \to \mathbb{C}$ is the projection on i-th component. Assume $\mathcal{M}$ is an ideal in $\mathcal{A}$ and $\mathbf{1} \in \pi_i[\mathcal{M}]$, for all $i = 1, \ldots, n$. Hence, for every i, there is some $f_i \in \mathcal{M}$ such that $\pi_i \circ f_i = \mathbf{1}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. Then e_i, as a constant function of X into A, belongs to $\mathcal{A}$. Since $\mathcal{M}$ is an ideal, we have $\mathbf{1} = e_1 f_1 + \cdots + e_n f_n \in \mathcal{M}$. Hence, $\mathcal{M} = \mathcal{A}$ and $\mathcal{M}$ cannot be maximal.

If $\mathcal{X} = \{1, \ldots, n\}$, then $\mathbb{C}^n = C(\mathcal{X})$. The above example states that, given any admissible Banach $C(\mathcal{X})$-valued function algebra, we have $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(C(\mathcal{X}))$. If $\mathcal{X}$ is an arbitrary compact Hausdorff space, it is unknown whether the result still holds for any admissible Banach $C(\mathcal{X})$-valued function algebra.

But, the following shows that it does hold for admissible $C(\mathcal{X})$-valued uniform algebras.

Example 4.3. Assume that $A = C(\mathcal{X})$, for some compact Hausdorff space $\mathcal{X}$. Then, for every admissible A-valued uniform algebra $\mathcal{A}$ on X, the mapping $\mathfrak{J}$ in (3.3) is surjective, and thus $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(C(\mathcal{X}))$.

To see this, first we show that $\mathcal{A}$ is isometrically isomorphic to $C(\mathcal{X}, \mathfrak{A})$. Take a function $f \in \mathcal{A}$. Then $f(x)$, for every $x \in X$, is a function in $C(\mathcal{X})$. Define $\tilde{f} : \mathcal{X} \to \mathfrak{A}$ by $\tilde{f}(\xi)(x) = f(x)(\xi)$. In fact, $\tilde{f}(\xi) = \phi_\xi \circ f$ where ϕ_ξ is the evaluation character of $A = C(\mathcal{X})$ at ξ, and, since $\mathcal{A}$ is admissible, $\tilde{f}(\xi)$ belongs to $\mathfrak{A}$. Now, define $T : \mathcal{A} \to C(\mathcal{X}, \mathfrak{A})$ by $Tf = \tilde{f}$. It is easily verified that T is an algebra homomorphism, and

$$\|f\|_X = \sup_{x\in X} \|f(x)\| = \sup_{x\in X} \sup_{\xi\in\mathcal{X}} |f(x)(\xi)| = \sup_{\xi\in\mathcal{X}} \|\tilde{f}(\xi)\| = \|\tilde{f}\|_{\mathcal{X}}.$$

Since the range of T contains all elements of the form $g_1h_1+\cdots+g_nh_n$, where $n \in \mathbb{N}$, $g_i \in C(\mathcal{X})$ and $h_i \in \mathfrak{A}$, and these functions are dense in $C(\mathcal{X}, \mathfrak{A})$, we have T surjective. It follows that T is an isometric isomorphism. By [9, Theorem], $\mathfrak{M}(C(\mathcal{X}, \mathfrak{A}))$ is identical to $\mathfrak{M}(\mathfrak{A}) \times \mathcal{X}$, which means that $\mathfrak{M}(\mathcal{A})$ is identical to $\mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$.

Example 4.4 (Tensor Products)**.** Let $\mathfrak{A}$ be a Banach function algebra on X and consider the algebraic tensor product $\mathfrak{A} \otimes A$. There exists, by [5, Theorem 42.6], a linear operator $T : \mathfrak{A} \otimes A \to \mathfrak{A}A$ such that

$$T\Big(\sum_{i=1}^{n} f_i \otimes a_i\Big) = \sum_{i=1}^{n} f_i a_i. \tag{4.2}$$

The operator T is an algebra isomorphism so that $\mathfrak{A} \otimes A$ can be seen as an admissible A-valued function algebra on X. We identify every element $f \in \mathfrak{A} \otimes A$ with its image Tf as an A-valued function on X. Let $\|\cdot\|_\gamma$ be an algebra cross-norm on $\mathfrak{A} \otimes A$ so that the completion $\mathfrak{A} \mathbin{\widehat{\otimes}_\gamma} A$ is a Banach algebra. The mapping T extends to an isometric isomorphism of $\mathfrak{A} \mathbin{\widehat{\otimes}_\gamma} A$ onto a Banach A-valued function algebra on X. For example, if $\|\cdot\|_\epsilon$ is the injective tensor norm, then $\mathfrak{A} \mathbin{\widehat{\otimes}_\epsilon} A$ is isometrically isomorphic to the uniform closure $\overline{\mathfrak{A}A}$ of $\mathfrak{A}A$ and $\|f\|_\epsilon = \|f\|_X$, for all $f \in \mathfrak{A} \otimes A$.

It is proved in [4] that

(1) $\mathfrak{A} \mathbin{\widehat{\otimes}_\gamma} A$ is an admissible Banach A-valued function algebra on X.
(2) If $f \in \mathfrak{A} \mathbin{\widehat{\otimes}_\gamma} A$ and $\phi \in A^*$ then $\phi \circ f \in \mathfrak{A}$ and $\|\phi \circ f\| \le \|\phi\|\|f\|_\gamma$.

We now show that every $\tau \in \mathfrak{M}(\mathfrak{A} \widehat{\otimes}_\gamma A)$ is of the form $\tau = \psi \diamond \phi$, with $\phi = \tau|_A$ and $\psi = \tau|_{\mathfrak{A}}$. Since $\mathfrak{A} \otimes A$ is dense in $\mathfrak{A} \widehat{\otimes}_\gamma A$, it is enough to show that $\tau = \psi \diamond \phi$ on $\mathfrak{A} \otimes A$. First, note that every $f \in \mathfrak{A} \otimes A$ can be seen, through the isomorphism

(4.2), as $f = f_1 a_1 + \cdots + f_n a_n$. Hence, $\phi \circ f = \phi(a_1) f_1 + \cdots + \phi(a_n) f_n$ and

$$\begin{aligned}\tau(f) &= \tau(f_1 a_1 + \cdots + f_n a_n) \\ &= \tau(f_1)\tau(a_1) + \cdots + \tau(f_n)\tau(a_n) \\ &= \psi(f_1)\phi(a_1) + \cdots + \psi(f_n)\phi(a_n) \\ &= \psi(\phi(a_1) f_1 + \cdots + \phi(a_n) f_n) \\ &= \psi(\phi \circ f).\end{aligned}$$

This proves that $\tau = \psi \diamond \phi$ on $\mathfrak{A} \otimes A$ and thus $\tau = \psi \diamond \phi$ on $\mathfrak{A} \widehat{\otimes}_\gamma A$. We conclude that $\mathfrak{M}(\mathfrak{A} \widehat{\otimes}_\gamma A) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$. This result, however, can be derived from the following more general result due to Tomiyama [17].

Theorem 4.5 ([17]). *Suppose that A and B are commutative Banach algebras. If $A \widehat{\otimes}_\gamma B$ is a Banach algebra for a cross-norm γ, then $\mathfrak{M}(A \widehat{\otimes}_\gamma B)$ is homeomorphic to $\mathfrak{M}(A) \times \mathfrak{M}(B)$.*

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References

[1] M. Abel, M. Abtahi, Description of closed maximal ideals in topological algebras of continuous vector-valued functions, *Mediterr. J. Math.* **11** (2014), no. 4, 1185–1193.

[2] M. Abtahi, Normed algebras of infinitely differentiable Lipschitz functions, *Quaest. Math.* **35** (2012), no. 2, 195–202.

[3] M. Abtahi, Vector-valued characters on vector-valued function algebras, *Banach J. Math. Anal.* **10** (2016), no. 3, 608–620.

[4] M. Abtahi and S. Farhangi, Vector-valued spectra of Banach algebra valued continuous functions, *Rev. R. Acad. Cienc. Exacts Fís. Nat. Ser. A Math.* (2016) https://doi.org/10.1007/s13398-016-0367-2.

[5] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[6] H.G. Dales, Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. 24, Clarendon Press, Oxford, New York, 2000.

[7] K. Esmaeili and H. Mahyar, The character spaces and Śilov boundaries of vector-valued Lipschitz function algebras, *Indian J. Pure Appl. Math.* **45** (2014), no. 6, 977–988.

[8] T.W. Gamelin, Uniform Algebras, Prentice Hall Press, 1969.

[9] A. Hausner, Ideals in a certain Banach algebra, *Proc. Amer. Math. Soc.* **8** (1957), 246–249.

[10] T.G. Honary, Relations between Banach function algebras and their uniform closures, *Proc. Amer. Math. Soc.* **109** (1990), no. 2, 337–342.

[11] T.G. Honary, A. Nikou and A.H. Sanatpour, On the character space of vector-valued Lipschitz algebras, *Bull. Iranian Math. Soc.* **40** (2014), no. 6, 1453–1468.

[12] J. Horváth, Topological Vector Spaces and Distributions, Vol. I, Addison-Wesley, Reading, Mass., 1966.

[13] E. Kaniuth, A Course in Commutative Banach Algebras, Grad. Texts in Math. 246, Springer, 2009.

[14] G.M. Leibowitz, Lectures on Complex Function Algebras, Scott Foresman, 1970.
[15] A. Nikou and A.G. O'Farrell, Banach algebras of vector-valued functions, *Glasgow Math. J.* **56** (2014), no. 2, 419–426.
[16] E.L. Stout, The Theory of Uniform Algebras, Bogden & Quigley, 1971.
[17] J. Tomiyama, Tensor products of commutative Banach algebras, *Tôhoku Math. J. (2)* **12** (1960), 147–154.
[18] B. Yood, Banach algebras of continuous functions, *Amer. J. Math.* **73** (1951) 30–42.
[19] W. Żelazko, Banach Algebras, Translated from the Polish by M.E. Kuczma. Elsevier Publishing Co. Amsterdam-London-New York; PWN–Polish Scientific Publishers, Warsaw, 1973.

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