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ON SOME GENERALIZED RECURRENT MANIFOLDS

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Dedicated to the memory of Professor Witold Roter

Abstract. The object of the present paper is to introduce and study a type of non-flat semi-Riemannian manifolds, called, *super generalized recurrent manifolds* which generalizes both the notion of hyper generalized recurrent manifolds [A.A. Shaikh and A. Patra, On a generalized class of recurrent manifolds, *Arch. Math. (Brno)* 46 (2010) 71–78.] and weakly generalized recurrent manifolds [A.A. Shaikh and I. Roy, On weakly generalized recurrent manifolds, *Ann. Univ. Sci. Budapest Rolando Eötvös, Sect. Math.* 54 (2011) 35–45.]. The nature of associated 1-forms of a super generalized recurrent manifold is determined and it is proved that on a Roter type manifold [R. Deszcz, On Roter type manifolds, in: 5^{th} Conference on Geometry and Topology of Manifolds, Krynica, Poland, 2003.] such a notion is equivalent to the notion of generalized Ricci-recurrent manifold [U.C. De, N. Guha and D. Kamilya, On generalized Ricci-recurrent manifolds, *Tensor (N.S.)* 56 (1995), no. 3, 312–317.]. We also obtain a sufficient condition for a super generalized recurrent manifold to be a semisymmetric one and the existence of such notion is ensured by a proper example.
Keywords: Recurrent manifold, hyper generalized recurrent manifold, weakly generalized recurrent manifold, super generalized recurrent manifold, semisymmetric manifold, Roter type manifold.
MSC(2010): Primary: 53B20; Secondary: 53B30, 53C25.

1. Introduction

Let M be a connected semi-Riemannian smooth manifold equipped with a semi-Riemannian metric g. Let ∇, R, S and κ be respectively the Levi-Civita connection, Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature of M. The curvature of a manifold plays the crucial role to determine

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the shape of the manifold. From a given metric one can determine the curvature but the converse is very cumbersome. For the sake of construction of a curvature restricted geometric structure one needs to impose a restriction on the curvature tensor by means of covariant derivatives or otherwise. It is well known that covariant derivative is a generalization of partial derivative and higher order of covariant derivatives imposed on a curvature tensor give rise to different types of curvature restricted geometric structures. For example, Cartan in [1] and [2] introduced the notion of local symmetry and semisymmetry by means of first and second order covariant derivatives respectively. Again, as a generalization of recurrent as well as pseudosymmetric manifold by Chaki [3], Tamássy and Binh [38] introduced the notion of weakly symmetric manifolds. We note that as a generalization of locally symmetric manifold, the notion of recurrent manifold was introduced by Ruse ([25–27], see also [39]) as a kappa space, denoted by K_n, and latter named as recurrent space. Again, in 1979 Dubey [14] introduced the concept of generalized recurrent manifold (briefly, $(GK)_n$). It is noteworthy to mention that $(GK)_n$ does not exist (see, [15, 20–22]). As a generalization of recurrent manifold, recently, Shaikh and his coauthors introduced the notions of quasi generalized recurrent manifold [34] (briefly, $(QGK)_n$), hyper-generalized recurrent manifold [33] (briefly, $(HGK)_n$) and weakly generalized recurrent manifold [35] (briefly, $(WGK)_n$) along with their proper existence by suitable examples (see also [28, 36]). We mention that every recurrent manifold is a 2-recurrent manifold [19]. In [23, Lemma 2] it was stated that every 2-recurrent manifold is semisymmetric [37]. Thus every recurrent manifold is semisymmetric. We also mention that semisymmetric manifolds form a subclass of pseudosymmetric manifolds (see, e.g., [8] and [29]).

The object of the present paper is to introduce and study a generalized class of recurrent manifolds, called, super generalized recurrent manifolds [30] (briefly, $(SGK)_n$). The paper is organized as follows. Section 2 deals with the rudimentary facts of various curvature restricted geometric structures and tensors as preliminaries. Section 3 is concerned with main results. The nature of associated 1-forms of a $(SGK)_n$ is determined and it is shown that if the 1-forms are closed and pairwise codirectional, then such a manifold is semisymmetric. We also obtain a sufficient condition for a $(SGK)_n$ to be K_n, and it is proved that on a Roter type manifold (see [5,6,8,13,29]) a $(SGK)_n$ is equivalent to the notion of generalized Ricci-recurrent manifold [4]. The last section deals with the existence of a $(SGK)_4$ by a proper example with a suitable metric. Finally the conclusion of the whole work is given.

2. **Preliminaries**

We now consider a connected semi-Riemannian smooth manifold (M^n, g), $n \geq 3$ (this condition is to be assumed throughout the paper unless otherwise stated). Let $C^\infty(M)$, $\chi(M)$, $\chi^*(M)$ and $\mathcal{T}^r_k(M)$ be respectively the algebra of all smooth functions, the Lie algebra of all smooth vector fields, the Lie algebra of all smooth 1-forms and the space of all smooth tensor fields of type (r,k) on M.

For $\Pi, \Phi \in \chi^*(M)$, the exterior product $\Pi \wedge \Phi$ is defined as

$$\Pi \wedge \Phi = \frac{1}{2}\left(\Pi \otimes \Phi - \Phi \otimes \Pi\right),$$

where $\otimes$ denotes the tensor product. We note that if $\Pi \wedge \Phi = 0$, then Π and Φ are said to be codirectional. Since ∇ is torsion free, the exterior derivative $d\Pi$ of Π can be expressed as

$$d\Pi(X,Y) = (\nabla_X \Pi)(Y) - (\nabla_Y \Pi)(X)$$

for all $X, Y \in \chi(M)$. We also note that Π is closed if $d\Pi = 0$.

Now for $A, E \in \mathcal{T}^0_2(M)$, the Kulkarni-Nomizu product $A \wedge E$ is defined as (see, e.g., [8,30,31])

$$\begin{aligned}(A \wedge E)(X_1, X_2, Y_1, Y_2) &= A(X_1, Y_2)E(X_2, Y_1) + A(X_2, Y_1)E(X_1, Y_2) \\ &- A(X_1, Y_1)E(X_2, Y_2) - A(X_2, Y_2)E(X_1, Y_1),\end{aligned} \tag{2.1}$$

where $X_1, X_2, Y_1, Y_2 \in \chi(M)$. Throughout the paper we will consider X, Y, $X_i, Y_i \in \chi(M)$, $i = 1, 2, \ldots$, and the same symbol $\wedge$ is used for both Kulkarni-Nomizu product and exterior product.

Again for a symmetric $(0,2)$-tensor A and $X, Y \in \chi(M)$ we can define the $C^\infty(M)$-linear endomorphisms $\mathscr{A}$ and $X \wedge_A Y$ on $\chi(M)$ respectively as

$$g(\mathscr{A}X, Y) = A(X,Y) \quad \text{and} \quad (X \wedge_A Y)X_1 = A(Y, X_1)X - A(X, X_1)Y.$$

The second level (0,2)-tensor A^2 with corresponding endomorphism $\mathscr{A}^2$ for a symmetric $(0,2)$-tensor A is defined as

$$A^2(X,Y) = A(\mathscr{A}X, Y) = g(\mathscr{A}^2 X, Y).$$

In particular, the second level Ricci tensor S^2 is given by

$$S^2(X,Y) = S(\mathscr{S}X, Y),$$

where $\mathscr{S}$ is the Ricci operator such that $S(X,Y) = g(\mathscr{S}X, Y)$.

In terms of the Kulkarni-Nomizu product and $\wedge_A$, the Weyl conformal curvature tensor C (for $n \geq 4$), the projective curvature tensor P, the concircular curvature tensor W and the conharmonic curvature tensor K are respectively given by

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{2(n-1)(n-2)} g \wedge g,$$

$$P = R - \frac{1}{n-1}(\wedge_S),$$

$$W = R - \frac{\kappa}{2n(n-1)} g \wedge g \text{ and}$$

$$K = R - \frac{1}{n-2} g \wedge S.$$

Now for $D \in \mathcal{T}_4^0(M)$ and given $X, Y \in \chi(M)$, the $C^\infty(M)$-linear endomorphism $\mathscr{D}(X,Y)$ is defined as

$$\mathscr{D}(X,Y)X_3 = \mathcal{D}(X,Y)X_3,$$

where $\mathcal{D} \in \mathcal{T}_3^1(M)$ such that $g(\mathcal{D}(X_1,X_2)X_3,X_4) = D(X_1,X_2,X_3,X_4)$. We note that one can easily operate a $C^\infty(M)$-linear endomorphism $\mathscr{L}$ on $T \in \mathcal{T}_k^0(M)$ as

$$\begin{aligned}(\mathscr{L}T)(X_1,X_2,\cdots,X_k) = &- T(\mathscr{L}X_1,X_2,\cdots,X_k) - \cdots \\ &- T(X_1,X_2,\cdots,\mathscr{L}X_k).\end{aligned}$$

In particular, for the endomorphisms $\mathscr{D}(X,Y)$ and $X \wedge_A Y$, the $(0,k+2)$-tensors $D \cdot T$ and $Q(A,T)$ are respectively given as

$$\begin{aligned}&D \cdot T(X_1,X_2,\cdots,X_k,X,Y) = (\mathscr{D}(X,Y)T)(X_1,X_2,\cdots,X_k)\\ &= -T(\mathcal{D}(X,Y)X_1,X_2,\cdots,X_k) - \cdots - T(X_1,X_2,\cdots,\mathcal{D}(X,Y)X_k)\end{aligned}$$

and

$$\begin{aligned}&Q(A,T)(X_1,X_2,\cdots,X_k,X,Y) = ((X \wedge_A Y)T)(X_1,X_2,\cdots,X_k)\\ &= A(X,X_1)T(Y,X_2,\cdots,X_k) + \cdots + A(X,X_k)T(X_1,X_2,\cdots,Y)\\ &- A(Y,X_1)T(X,X_2,\cdots,X_k) - \cdots - A(Y,X_k)T(X_1,X_2,\cdots,X).\end{aligned}$$

Again for $A \in \mathcal{T}_2^0(M)$ and $T \in \mathcal{T}_k^0(M)$, $A \wedge T$ and $\wedge_T$ (see [7,31], and also references therein) are respectively given by

$$\begin{aligned}(A \wedge T)(X_1,X_2,Y_1,Y_2,\cdots,Y_k) =& A(X_1,Y_2)T(X_2,Y_1,\cdots,Y_k)\\ &+ A(X_2,Y_1)T(X_1,Y_2,\cdots,Y_k)\\ &- A(X_1,Y_1)T(X_2,Y_2,\cdots,Y_k)\\ &- A(X_2,Y_2)T(X_1,Y_1,\cdots,Y_k),\end{aligned}$$

$$\begin{aligned}(X \wedge_T Y)(X_1,X_2,\cdots,X_k) =& T(Y,X_1,X_3,\cdots X_k)g(X,X_2)\\ &- T(X,X_1,X_3,\cdots X_k)g(Y,X_2).\end{aligned}$$

From the above expressions we can state the following:

Proposition 2.1. *For $A \in \mathcal{T}_2^0(M)$ and $D \in \mathcal{T}_4^0(M)$, the following conditions hold:*

(i) $\nabla(X \wedge_A Y) = X \wedge_{\nabla A} Y$ *and* $\nabla(g \wedge A) = g \wedge (\nabla A)$,
(ii) $D \cdot (X \wedge_A Y) = X \wedge_{D \cdot A} Y$ *and* $D \cdot (g \wedge A) = g \wedge (D \cdot A)$ *if* $D \cdot g = 0$.

Definition 2.2. A semi-Riemannian manifold M is said to be Einstein (resp. Ein(2)) (see [31], and also references therein) if

$$S = \frac{\kappa}{n} g \quad (\text{resp. } a_1 S^2 + a_2 S + a_3 g = 0),$$

where $a_1, a_2, a_3 \in C^\infty(M)$. Moreover if $a_1 \neq 0$, then an $Ein(2)$ manifold is called proper.

Definition 2.3. A semi-Riemannian manifold M is said to be quasi-Einstein if

$$S = \alpha g + \beta \eta \otimes \eta \tag{2.2}$$

holds for some $\alpha, \beta \in C^\infty(M)$ and $\eta \in \chi^*(M)$. Especially, if α is identically zero, then the manifold is called Ricci simple (see [8] and also references therein).

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally flat spaces, see, [8], and references therein. It is clear that every non-Einstein quasi-Einstein manifold is a proper $Ein(2)$ manifold [17, p. 600].

Definition 2.4. For $T \in \mathcal{T}^0_k(M)$, a semi-Riemannian manifold M is said to be T-recurrent ([25–27]) if

$$\nabla T = \Pi \otimes T \tag{2.3}$$

holds on $\{x \in M : T \neq 0 \text{ at } x\}$ for an 1-form Π, called the associated 1-form. If $T = R$ (resp. S) then the manifold is called recurrent (resp. Ricci recurrent).

Proposition 2.5. *For $A \in \mathcal{T}^0_2(M)$ and $\Pi \in \chi^*(M)$, we have*

(i) $\nabla(X \wedge_A Y) = \Pi \otimes (X \wedge_A Y)$ *if and only if* $\nabla A = \Pi \otimes A$ *and*
(ii) $\nabla(g \wedge A) = \Pi \otimes (g \wedge A)$ *if and only if* $\nabla A = \Pi \otimes A$.

Proposition 2.6. *In a semi-Riemannian manifold M, $\nabla(S - \alpha\kappa g) = \Pi \otimes (S - \alpha\kappa g)$ if and only if $\nabla S = \Pi \otimes S$, where $\Pi \in \chi^*(M)$ and α ($\neq \frac{1}{n}$) is a constant.*

Definition 2.7. For $T \in \mathcal{T}^0_4(M)$, a semi-Riemannian manifold M is said to be T-quasi generalized recurrent with (Π, Ψ, η) [34] (resp. T-hyper generalized recurrent with (Π, Ψ) [33], T-weakly generalized recurrent with (Π, Ψ) [35]) if

$$\nabla T = \Pi \otimes T + \Psi \otimes [g \wedge (g + \eta \otimes \eta)], \tag{2.4}$$

(resp.

$$\nabla T = \Pi \otimes T + \Psi \otimes g \wedge S, \tag{2.5}$$

$$\nabla T = \Pi \otimes T + \Psi \otimes S \wedge S) \tag{2.6}$$

holds on $\{x \in M : T \neq 0 \text{ and } g \wedge (g + \eta \otimes \eta) \neq 0 \text{ at } x\}$ (resp. $\{x \in M : T \neq 0 \text{ and } g \wedge S \neq 0 \text{ at } x\}$, $\{x \in M : T \neq 0 \text{ and } S \wedge S \neq 0 \text{ at } x\}$) for some Π, Ψ, $\eta \in \chi^*(M)$, called the associated 1-forms. If $T = R$ then the manifold is called $(QGK)_n$ (resp. $(HGK)_n$, $(WGK)_n$).

We note that for $\alpha = \beta$, a quasi-Einstein manifold is $(WGK)_n$ if and only if it is $(QGK)_n$, and for $2\alpha = \beta$, a quasi-Einstein manifold is $(HGK)_n$ if and only if it is $(QGK)_n$. We also note that the condition (2.6), in some particular form ($T = R$), was already presented in [24, p. 626]. Precisely, in that paper the following equation was obtained

$$\kappa R_{hijk,l} = -\kappa \Phi_l R_{hijk} + 4\Phi_l (S_{hk} S_{ij} - S_{hj} S_{ik}).$$

Definition 2.8. For $T \in \mathcal{T}^0_4(M)$, a semi-Riemannian manifold M is said to be T-super generalized recurrent manifold [30] if

$$\nabla T = \Pi \otimes T + \Phi \otimes S \wedge S + \Psi \otimes g \wedge S + \Theta \otimes g \wedge g \tag{2.7}$$

holds on $\{x \in M : T \neq 0$ and any one of $S \wedge S, g \wedge S$ is non-zero at $x\}$ for some 1-forms Π, Φ, Ψ and Θ, called the associated 1-forms. Such a manifold is denoted by $TSGK_n$ with $(\Pi, \Phi, \Psi, \Theta)$. Especially, if $T = R$, then the manifold is said to be $(SGK)_n$.

Definition 2.9. For $T \in \mathcal{T}^0_4(M)$, a semi-Riemannian manifold M is said to be T-quasi generalized recurrent with (Π, Φ, Ψ, η) if

$$\nabla T = \Pi \otimes T + \Phi \otimes g \wedge g + \Psi \otimes g \wedge (\eta \otimes \eta)$$

holds on $\{x \in M : T \neq 0$ and $g \wedge (\eta \otimes \eta)$ is non-zero at $x\}$ for some 1-forms Π, Φ, Ψ and η, called the associated 1-forms. Such a manifold is denoted by $TQGK_n$ with (Π, Φ, Ψ).

Definition 2.10. For $Z \in \mathcal{T}^0_2(M)$, a semi-Riemannian manifold M is said to be generalized Z-recurrent manifold [4] (briefly, GZK_n) if

$$\nabla Z = \Pi \otimes Z + \Phi \otimes g \tag{2.8}$$

holds on $\{x \in M : \nabla Z \neq \xi \otimes Z$ at $x\ \forall\ \xi \in \chi^*(M)\} \subset M$ for some Π and $\Phi \in \chi^*(M)$, called the associated 1-forms. In particular, if $Z = S$, the manifold is called generalized Ricci-recurrent [4].

Definition 2.11. For $T \in \mathcal{T}^0_k(M)$, a semi-Riemannian manifold M is said to be T-semisymmetric (briefly, TSS_n) ([2, 37]) if

$$R \cdot T = 0.$$

If we take $T = R$ (resp. S), then the manifold is called semisymmetric (resp. Ricci semisymmetric).

Definition 2.12. A semi-Riemannian manifold M is said to be Roter type (briefly, RT_n) ([5, 6, 8]) if its curvature tensor R can be expressed as

$$R = N_1\ g \wedge g + N_2\ g \wedge S + N_3\ S \wedge S, \tag{2.9}$$

for some N_1, N_2 and $N_3 \in C^\infty(M)$. Moreover it is said to be proper RT_n if $N_3 \neq 0$.

We mention that the notion of generalized Roter type manifold and its warped product is studied in [29, 31, 32]. We note that the condition (2.9), in some particular form, i.e.,

$$R_{hijk} = \frac{2}{\kappa}(S_{hk}S_{ij} - S_{hj}S_{ik})$$

was already presented in [24, p. 625]. Curvature properties of semi-Riemannian manifolds satisfying the condition $R_{hijk} = \Phi(S_{hk}S_{ij} - S_{hj}S_{ik})$ were obtained in [18].

3. Main results

Let M be a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$. Then we have

$$\nabla R = \Pi \otimes R + \Phi \otimes S \wedge S + \Psi \otimes g \wedge S + \Theta \otimes g \wedge g. \tag{3.1}$$

Contraction of (3.1) yields

$$\nabla S = \Pi_1 \otimes S^2 + \Phi_1 \otimes S + \Psi_1 \otimes g, \tag{3.2}$$

where $\Pi_1 = -2\Phi$, $\Phi_1 = \Pi + 2\kappa\Phi + (n-2)\Psi$ and $\Psi_1 = \kappa\Psi + 2(n-1)\Theta$.

Theorem 3.1. *The associated 1-forms of a $(SGK)_n$ are not unique.*

Proof. The second Bianchi identity is given by

$$\begin{aligned}(\nabla_{X_1}R)(X_2, X_3, X_4, X_5)+&(\nabla_{X_2}R)(X_3, X_1, X_4, X_5)\\ &+ (\nabla_{X_3}R)(X_1, X_2, X_4, X_5) = 0.\end{aligned}$$

In view of (3.1), the above identity entails

(3.3)
$$\begin{aligned}&\sum_{X_1,X_2,X_3}\Big[\Pi(X_1)R(X_2, X_3, X_4, X_5) + \Phi(X_1)(S \wedge S)(X_2, X_3, X_4, X_5)\\ &+ \Psi(X_1)(g \wedge S)(X_2, X_3, X_4, X_5) + \Theta(X_1)(g \wedge g)(X_2, X_3, X_4, X_5)\Big] = 0,\end{aligned}$$

where $\sum_{X_1,X_2,X_3}$ denotes the cyclic sum in X_1, X_2 and X_3. Now contracting (3.3) over X_1 and X_5, we get

$$\begin{aligned}&- R(V, X_4, X_2, X_3) + \{\kappa\Psi(X_3) - \Psi(\mathcal{S}(X_3)) + 2(n-2)\Theta(X_3)\}g(X_2, X_4)\\ &+ \{-\kappa\Psi(X_2) + \Psi(\mathcal{S}(X_2)) - 2(n-2)\Theta(X_2)\}g(X_3, X_4)\\ &+ \{\Pi(X_3) + 2\kappa\Phi(X_3) - 2\Phi(\mathcal{S}(X_3)) + (n-3)\Psi(X_3)\}S(X_2, X_4)\\ &+ \{-\Pi(X_2) - 2\kappa\Phi(X_2) + 2\Phi(\mathcal{S}(X_2)) - (n-3)\Psi(X_2)\}S(X_3, X_4)\\ &- 2\Phi(X_3)S^2(X_2, X_4) + 2\Phi(X_2)S^2(X_3, X_4) = 0,\end{aligned}$$

where V is the vector field corresponding to Π, i.e., $g(V, X) = \Pi(X)$ for all $X \in \chi(M)$. Again, contracting over X_3 and X_4, we obtain

$$\begin{aligned}&-\kappa\Pi(X_2) + 2\Big[\Pi(\mathcal{S}(X_2)) + (\kappa^{(2)} - \kappa^2)\Phi(X_2) + 2\kappa\Phi(\mathcal{S}(X_2))\\&-2\Phi(\mathcal{S}^2(X_2)) - (n-2)\{\kappa\Psi(X_2) - \Psi(\mathcal{S}(X_2)) + (n-1)\Theta(X_2)\}\Big] = 0,\end{aligned}$$

where $\kappa^{(2)}$ is the trace of S^2. The result follows from the last relation. □

Remark 3.2. We note that the associated 1-forms of a conformally ($n \geq 4$) (resp. projectively, concircularly, conharmonicly) $(SGK)_n$ are not unique.

Theorem 3.3. *If M is a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$, then its associated 1-forms are linearly dependent with $d\kappa$ such that*

$$d\kappa = \kappa\Pi + 2(\kappa^2 - \kappa^{(2)})\Phi + 2(n-1)[\kappa\Psi + n\Theta].$$

Proof. From (3.1) we have

$$\begin{aligned}(\nabla_X S)(X_1, X_2) =&(\kappa\Psi + 2(n-1)\Theta)(X)g(X_1, X_2) - 2\Phi(X)S^2(X_1, X_2)\\&+ (\Pi + 2\kappa\Phi + (n-2)\Psi)(X)S(X_1, X_2).\end{aligned}$$

Again contracting the above equation over X_1 and X_2, we obtain the result. □

Theorem 3.4. *An Einstein $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ is a K_n and the relation $\frac{\kappa^2}{n^2}\Phi + \frac{\kappa}{n}\Psi + \Theta = 0$ holds.*

Proof. Since the manifold is Einstein, we have $S = \frac{\kappa}{n}g$ and hence (3.1) reduces to

$$\nabla R = \Pi \otimes R + \left[\frac{\kappa^2}{n^2}\Phi + \frac{\kappa}{n}\Psi + \Theta\right] \otimes g \wedge g.$$

Again in [22] Olszak and Olszak showed that for any semi-Riemannian manifold satisfying such curvature condition, the coefficient of $g \wedge g$ is zero. Hence the manifold turns into a K_n and $\frac{\kappa^2}{n^2}\Phi + \frac{\kappa}{n}\Psi + \Theta = 0$ holds. □

Theorem 3.5. *Let M be a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$. If M is quasi-Einstein, then it is a $(QGK)_n$ with $(\Pi, \alpha^2\Phi + \alpha\Psi + \Theta, 2\alpha\beta\Phi + \beta\Psi, \eta)$. Moreover M is $(QGK)_n$ if*

$$\alpha^2\Phi + \alpha\Psi + \Theta = \gamma(2\alpha\beta\Phi + \beta\Psi),$$

where γ is a positive smooth function on M.

Proof. The proof is similar to the proof of the Theorem 3.4. □

We mention that the existence of a quasi-Einstein manifold which is $(QGK)_4$ is given in Example 4.3 of this paper.

Theorem 3.6. *If M is a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ and its associated 1-forms are closed and pairwise codirectional, then it is a semisymmetric manifold.*

Proof. From (3.1) we have

$$\nabla_X R = \Pi(X) \otimes R + \Phi(X) \otimes S \wedge S + \Psi(X) \otimes g \wedge S + \Theta(X) \otimes g \wedge g.$$

Differentiating the above equation covariantly with respect to Y, we get

$$\begin{aligned}
&\nabla_Y(\nabla_X R) = (\nabla_Y \Pi)(X)R + (\nabla_Y \Phi)(X)S \wedge S \\
&+ (\nabla_Y \Psi)(X)g \wedge S + (\nabla_Y \Theta)(X)g \wedge g \\
&+ [\Pi(X)\Psi(Y) + \Psi(X)\Phi_1(Y) + 2\Phi(X)\Psi_1(Y)]\, g \wedge S \\
&+ \Pi(X)\Pi(Y)R + \Psi(X)\Pi_1(Y)g \wedge S^2 + [\Pi(X)\Theta(Y) + \Psi(X)\Psi_1(Y)]\, g \wedge g \\
&+ 2\Phi(X)\Pi_1(Y)S \wedge S^2 + [\Pi(X)\Phi(Y) + 2\Phi(X)\Phi_1(Y)]\, S \wedge S.
\end{aligned}$$

In view of the last relation we get

$$\begin{aligned}
&R(X,Y) \cdot R = \nabla_X(\nabla_Y R) - \nabla_Y(\nabla_X R) \\
&= d\Pi(X,Y)R + 2\,[\Phi(Y)\Psi(X) - \Phi(X)\Psi(Y)]\, g \wedge S^2 \\
&+ \Big[d\Phi(X,Y) + \Phi(Y)(\Pi(X) + 2(n-2)\Psi(X)) \\
&\qquad\qquad - \Phi(X)(\Pi(Y) + 2(n-2)\Psi(Y))\Big] S \wedge S \\
&+ [d\Psi(X,Y) - 4(n-1)\,(\Theta(Y)\Phi(X) - \Theta(X)\Phi(Y))]\, g \wedge S \\
&+ \Big[d\Theta(X,Y) - \Theta(Y)(\Pi(X) + 2(n-1)\Psi(X)) \\
&\qquad\qquad + \Theta(X)(\Pi(Y) + 2(n-1)\Psi(Y))\Big] g \wedge g.
\end{aligned}$$

Thus we have

$$\begin{aligned}
R \cdot R =\,& d\Pi R - 4(\Phi \wedge \Psi)g \wedge S^2 + [d\Phi - 2\Phi \wedge (\Pi - 2(n-2)\Psi)]S \wedge S \\
&+ [d\Psi - 8(n-1)\Phi \wedge \Theta]g \wedge S + [d\Theta - 2\Theta \wedge (\Pi + 2(n-1)\Psi)]g \wedge g
\end{aligned} \tag{3.4}$$

and hence the theorem is proved. □

Proposition 3.7. *A $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ is a generalized Ricci-recurrent manifold if and only if $\Phi = 0$ or it is a proper $Ein(2)$ manifold.*

Proof. The result follows from (3.2). □

Theorem 3.8. *Let M be a proper RT_n. Then the following statements are equivalent:*

(i) *M is a $(SGK)_n$.*
(ii) *M is a generalized Ricci-recurrent manifold.*

Proof. Since M is a proper RT_n, it is a proper $Ein(2)$ manifold [31]. Now if M is a $(SGK)_n$, then by Proposition 3.7, it is a generalized Ricci-recurrent manifold and hence (ii)⇒ (i).

Now differentiating (2.9) covariantly and then using the defining condition of generalized Ricci-recurrent manifold, it is easy to check that M is a $(SGK)_n$. Hence (ii)$\Rightarrow$ (i). $\square$

By using Proposition 2.5, we can easily state the following:

Theorem 3.9. *Let M be a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$. If M is generalized Ricci-recurrent with associated 1-forms $\overline{\Pi}$ and $\overline{\Phi}$, then it is*

(i) *projectively $(SGK)_n$ with $\left(\Pi, \Phi, \Psi, \Theta - \frac{\overline{\Phi}}{2(n-2)}\right)$ for $\Pi = \overline{\Pi}$,*
(ii) *concircularly $(SGK)_n$ with $\left(\Pi, \Phi, \Psi, \Theta - \frac{\kappa\overline{\Pi}+n\overline{\Phi}-\kappa\Pi}{2n(n-1)}\right)$,*
(iii) *conharmonicly $(SGK)_n$ with $\left(\Pi, \Phi, \Psi - \frac{\overline{\Pi}-\Pi}{n-2}, \Theta - \frac{\overline{\Phi}}{n-2}\right)$,*
(iv) *conformally $(SGK)_n$ with $\left(\Pi, \Phi, \Psi - \frac{\overline{\Pi}-\Pi}{n-2}, \Theta - \frac{\overline{\Phi}}{n-2} + \frac{\kappa\overline{\Pi}+n\overline{\Phi}-\kappa\Pi}{2(n-1)(n-2)}\right)$ for $n \geq 4$.*

From definition, we can state the following:

Proposition 3.10. *Let $T \in \mathcal{T}_4^0(M)$. If $\nabla(T-R) = \Pi\otimes(T-R)$, then $(TSGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ and $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ are equivalent. Conversely, if M is a $(TSGK)_n$ and $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$, then $\nabla(T-R) = \Pi\otimes(T-R)$.*

Corollary 3.11. *If M is a $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$, then it is*

(i) *conformally $(SGK)_n$ $(n \geq 4)$ with $(\Pi, \Phi, \Psi, \Theta)$ if and only if it is Ricci recurrent with Π as its 1-form of recurrence.*
(ii) *projectively $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ if and only if it is Ricci recurrent with Π as its 1-form of recurrence.*
(iii) *concircularly $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ if and only if $d\kappa = \kappa\Pi$.*
(iv) *conharmonicly $(SGK)_n$ with $(\Pi, \Phi, \Psi, \Theta)$ if and only if it is Ricci recurrent with Π as its 1-form of recurrence.*

Proof. From Proposition 3.10 and Proposition 2.5, it follows that M is conformally $(SGK)_n$ $(n \geq 4)$ if and only if $(S - \frac{2\kappa}{n-1}g)$ is recurrent with associated 1-form Π. Again from Proposition 2.6, $(S - \frac{2\kappa}{n-1}g)$ is recurrent with associated 1-form Π if and only if M is Ricci recurrent with associated 1-form Π. Hence (i) is proved.

Again from Proposition 3.10, M is projectively $(SGK)_n$ if and only if $\frac{1}{n-2}X \wedge_S Y$ is recurrent, or S is recurrent (by Proposition 2.5) and hence (ii) is proved.

Similarly (iii) and (iv) can be proved. $\square$

4. Examples

Example 4.1. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1, x^2, x^3, x^4 > 0\}$ be an open connected subset of $\mathbb{R}^4$ such that

$$f_1^2 - f_2^2 > 0,$$

where $f_1 = \left(x^3+x^4\right)(x^1)^2(x^2)^2$ and $f_2 = \left(x^1+x^2\right)(x^3)^2(x^4)^2$. We note that if $x^1 > x^3$ and $x^2 > x^4$, then

$$\begin{aligned} f_1 - f_2 &= \left(\frac{x^3+x^4}{(x^3x^4)^2} - \frac{x^1+x^2}{(x^1x^2)^2}\right)\left(x^1x^2x^3x^4\right)^2 \\ &= \left(\frac{1}{x^3x^4}\left(\frac{1}{x^3}+\frac{1}{x^4}\right) - \frac{1}{x^1x^2}\left(\frac{1}{x^1}+\frac{1}{x^2}\right)\right)(det(g))^2 > 0. \end{aligned}$$

Let M be endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^idx^j = x^2(dx^1)^2 + x^1(dx^2)^2 + x^4(dx^3)^2 + x^3(dx^4)^2. \tag{4.1}$$

The non-zero components (up to symmetry) of the Riemann-Christoffel curvature tensor R, the Ricci tensor S and the scalar curvature κ are given by

$$R_{1212} = \frac{1}{4}\left(\frac{1}{x^2}+\frac{1}{x^1}\right),\ R_{3434} = \frac{1}{4}\left(\frac{1}{x^4}+\frac{1}{x^3}\right)\ \text{and}$$

$$S_{11} = \mu_1 g_{11},\quad S_{22} = \mu_1 g_{22},\quad \mu_1 = -\frac{(x^1+x^2)(x^3x^4)^2}{4\left(x^1x^2x^3x^4\right)^2} = -\frac{f_2}{4(det(g))^2},$$

$$S_{33} = \mu_2 g_{33},\quad S_{44} = \mu_2 g_{44},\quad \mu_2 = -\frac{(x^3+x^4)(x^1x^2)^2}{4\left(x^1x^2x^3x^4\right)^2} = -\frac{f_1}{4(det(g))^2},$$

$$\kappa = g^{hk}S_{hk} = 2(\mu_1+\mu_2) = -\frac{f_1+f_2}{2\left(x^1x^2x^3x^4\right)^2} = -\frac{f_1+f_2}{2(det(g))^2}.$$

From the main results of [37] it follows that M is semisymmetric manifold. Evidently, M is a 2-quasi-Einstein manifold. We refer to [9, 10] and [11] for recent results on 2-quasi-Einstein manifolds. Again the non-zero components (up to symmetry) of ∇R and ∇S are given as

$$R_{1212,1} = -\frac{\frac{x^1}{x^2}+2}{4(x^1)^2},\ R_{1212,2} = -\frac{\frac{x^2}{x^1}+2}{4(x^2)^2},$$

$$R_{3434,3} = -\frac{\frac{x^3}{x^4}+2}{4(x^3)^2},\ R_{3434,4} = -\frac{\frac{x^4}{x^3}+2}{4(x^4)^2}\ \text{and}$$

$$S_{11,1} = \frac{x^1+2x^2}{4(x^1)^3x^2},\ S_{11,2} = \frac{2x^1+x^2}{4(x^1)^2(x^2)^2},\ S_{22,1} = \frac{x^1+2x^2}{4(x^1)^2(x^2)^2},$$

$$S_{22,2} = \frac{2x^1+x^2}{4x^1(x^2)^3},\ S_{33,3} = \frac{x^3+2x^4}{4(x^3)^3x^4},$$

$$S_{33,4} = \frac{2x^3+x^4}{4(x^3)^2(x^4)^2},\ S_{44,3} = \frac{x^3+2x^4}{4(x^3)^2(x^4)^2},\ S_{44,4} = \frac{2x^3+x^4}{4x^3(x^4)^3}.$$

Also the non-zero components (up to symmetry) of $g \wedge g$, $g \wedge S$ and $S \wedge S$ are given by

$$(g\wedge g)_{1212} = -2x^1x^2,\ (g\wedge g)_{1313} = -2x^2x^4,\ (g\wedge g)_{1414} = -2x^2x^3,$$

$$(g\wedge g)_{2323}=-2x^1x^4,\ (g\wedge g)_{2424}=-2x^1x^3,\ (g\wedge g)_{3434}=-2x^3x^4;$$
$$(g\wedge S)_{1212}=\frac{1}{2}\left(\frac{1}{x^2}+\frac{1}{x^1}\right),\ (g\wedge S)_{1313}=\frac{f_1+f_2}{4(x^1)^2x^2(x^3)^2x^4},$$
$$(g\wedge S)_{1414}=\frac{f_1+f_2}{4x^1(x^2)^2(x^3)^2x^4},\ (g\wedge S)_{2323}=\frac{f_1+f_2}{4(x^1)^2x^2x^3(x^4)^2},$$
$$(g\wedge S)_{2424}=\frac{f_1+f_2}{4x^1(x^2)^2x^3(x^4)^2},\ (g\wedge S)_{3434}=\frac{1}{2}\left(\frac{1}{x^4}+\frac{1}{x^3}\right)\text{ and}$$
$$(S\wedge S)_{1212}=-\frac{\left(x^1+x^2\right)^2}{8(x^1)^3(x^2)^3},\ (S\wedge S)_{1313}=-\frac{\left(x^1+x^2\right)\left(x^3+x^4\right)}{8(x^1)^2x^2(x^3)^2x^4},$$
$$(S\wedge S)_{1414}=-\frac{\left(x^1+x^2\right)\left(x^3+x^4\right)}{8(x^1)^2x^2x^3(x^4)^2},\ (S\wedge S)_{2323}=-\frac{\left(x^1+x^2\right)\left(x^3+x^4\right)}{8x^1(x^2)^2(x^3)^2x^4},$$
$$(S\wedge S)_{2424}=-\frac{\left(x^1+x^2\right)\left(x^3+x^4\right)}{8x^1(x^2)^2x^3(x^4)^2},\ (S\wedge S)_{3434}=-\frac{\left(x^3+x^4\right)^2}{8(x^3)^3(x^4)^3}.$$

Then it is easy to check that the manifold M is a $(SGK)_4$ with (Π,Φ,Ψ,Θ), where Π, Φ, Ψ and Θ are given by

$$\Pi_i=\begin{cases}\frac{8\Theta_1(f_1-f_2)^2-x^1(x^2)^2\left(x^1+2x^2\right)\left(x^3+x^4\right)^2}{(x^1+x^2)(x^3+x^4)(f_1+f_2)} & \text{for } i=1\\ \frac{8\Theta_2(f_1-f_2)^2-(x^1)^2x^2\left(2x^1+x^2\right)\left(x^3+x^4\right)^2}{(x^1+x^2)(x^3+x^4)(f_1+f_2)} & \text{for } i=2\\ \frac{8\Theta_3(f_1-f_2)^2-\left(x^1+x^2\right)^2x^3(x^4)^2\left(x^3+2x^4\right)}{(x^1+x^2)(x^3+x^4)(f_1+f_2)} & \text{for } i=3\\ \frac{8\Theta_4(f_1-f_2)^2-\left(x^1+x^2\right)^2(x^3)^2x^4\left(2x^3+x^4\right)}{(x^1+x^2)(x^3+x^4)(f_1+f_2)} & \text{for } i=4,\end{cases}\tag{4.2}$$

$$\Phi_i=\begin{cases}\frac{2x^1(x^2)^2(x^3)^2(x^4)^2\left(\frac{8\Theta_1x^1}{x^3+x^4}-\frac{x^1+2x^2}{(f_1-f_2)}\right)}{x^1+x^2} & \text{for } i=1\\ \frac{2(x^1)^2x^2(x^3)^2(x^4)^2\left(\frac{8\Theta_2x^2}{x^3+x^4}-\frac{2x^1+x^2}{(f_1-f_2)}\right)}{x^1+x^2} & \text{for } i=2\\ \frac{2(x^1)^2(x^2)^2x^3(x^4)^2\left(\frac{8\Theta_3x^3}{x^1+x^2}+\frac{x^3+2x^4}{(f_1-f_2)}\right)}{x^3+x^4} & \text{for } i=3\\ \frac{2(x^1)^2(x^2)^2(x^3)^2x^4\left(\frac{8\Theta_4x^4}{x^1+x^2}+\frac{2x^3+x^4}{(f_1-f_2)}\right)}{x^3+x^4} & \text{for } i=4,\end{cases}\tag{4.3}$$

$$\Psi_i=\begin{cases}\frac{x^1(x^2x^3x^4)^2\left[16\Theta_1x^1(f_1-f_2)-\left(x^1+2x^2\right)\left(x^3+x^4\right)\right]}{f_1^2-f_2^2} & \text{for } i=1\\ \frac{x^2(x^1x^3x^4)^2\left[16\Theta_2x^2(f_1-f_2)-\left(2x^1+x^2\right)\left(x^3+x^4\right)\right]}{f_1^2-f_2^2} & \text{for } i=2\\ \frac{x^3(x^1x^2x^4)^2\left[16\Theta_3x^3(f_1-f_2)+\left(x^1+x^2\right)\left(x^3+2x^4\right)\right]}{f_1^2-f_2^2} & \text{for } i=3\\ \frac{x^4(x^1x^2x^3)^2\left[16\Theta_4x^4(f_1-f_2)+\left(x^1+x^2\right)\left(2x^3+x^4\right)\right]}{f_1^2-f_2^2} & \text{for } i=4.\end{cases}\tag{4.4}$$

We note that the associated 1-forms of such $(SGK)_4$ are not unique (hence Theorem 3.1 is verified). We also note that the manifold is neither hyper generalized recurrent nor weakly generalized recurrent. Since the metric g is the Cartesian product of two 2-dimensional metrics, the main results of [37] states that M is a semisymmetric manifold. Moreover, in view of Proposition 3.4 of [16], M is a Roter type manifold. The last conclusion, is also an immediate consequence of the considerations presented in [12] (p. 12 and Theorem 4.1).

Using the local components of R, $g\wedge g$, $g\wedge S$ and $S\wedge S$, we can easily check that the manifold fulfills (2.9) and hence is a Roter type manifold, where

$$\begin{aligned}
N_1 &= -\frac{\left(x^1+x^2\right)\left(x^3+x^4\right)(f_1+f_2)}{8(f_1-f_2)^2} = -\frac{f_1 f_2 (f_1+f_2)}{8(f_1-f_2)^2(det(g))^2},\\
N_2 &= -\frac{2(x^1)^2(x^2)^2\left(x^1+x^2\right)(x^3)^2(x^4)^2\left(x^3+x^4\right)}{(f_1-f_2)^2} = -\frac{2f_1f_2}{8(f_1-f_2)^2},\\
N_3 &= -\frac{2(x^1)^2(x^2)^2(x^3)^2(x^4)^2(f_1+f_2)}{(f_1-f_2)^2} = -\frac{2(f_1+f_2)(det(g))^2}{(f_1-f_2)^2}.
\end{aligned}$$

Consequently in view of Theorem 3.8, the manifold is generalized Ricci-recurrent satisfying

$$\nabla S = \overline{\Pi}\otimes S + \overline{\Phi}\otimes g,$$

where $\overline{\Pi}$ and $\overline{\Phi}$ are given by

$$\overline{\Pi}_i = \begin{cases} \frac{\left(x^1+2x^2\right)(x^3)^2(x^4)^2}{x^1(f_1-f_2)} & \text{for } i=1\\ \frac{\left(2x^1+x^2\right)(x^3)^2(x^4)^2}{x^2(f_1-f_2)} & \text{for } i=2\\ \frac{(x^1)^2(x^2)^2\left(x^3+2x^4\right)}{-x^3(f_1-f_2)} & \text{for } i=3\\ \frac{(x^1)^2(x^2)^2\left(2x^3+x^4\right)}{-x^4(f_1-f_2)} & \text{for } i=4, \end{cases} \tag{4.5}$$

$$\overline{\Phi}_i = \begin{cases} \frac{\left(x^1+2x^2\right)\left(x^3+x^4\right)}{4x^1(f_1-f_2)} & \text{for } i=1\\ \frac{\left(2x^1+x^2\right)\left(x^3+x^4\right)}{4x^2(f_1-f_2)} & \text{for } i=2\\ \frac{\left(x^1+x^2\right)\left(x^3+2x^4\right)}{-4x^3(f_1-f_2)} & \text{for } i=3\\ \frac{\left(x^1+x^2\right)\left(2x^3+x^4\right)}{-4x^4(f_1-f_2)} & \text{for } i=4. \end{cases} \tag{4.6}$$

Also it is easy to check that the manifold under consideration is semisymmetric. Again, M is not Ricci recurrent but generalized Ricci-recurrent and $(SGK)_4$. Hence by Theorem 3.9, it is conformally, concircularly and conharmonicly $(SGK)_4$ with distinct associated 1-forms.

Remark 4.2. If we consider the following metrics

$$ds^2 = x^2(dx^1)^2 + x^1(dx^2)^2 + x^4(dx^3)^2 - x^3(dx^4)^2,$$

$$ds^2 = x^2(dx^1)^2 + x^1(dx^2)^2 - x^4(dx^3)^2 - x^3(dx^4)^2 \text{ and}$$

$$ds^2 = x^2(dx^1)^2 - x^1(dx^2)^2 + x^4(dx^3)^2 - x^3(dx^4)^2$$

on a suitable open connected subset of $\mathbb{R}^4$, then it can be easily seen that all the above results are true.

Example 4.3 ([36]). Let M be an open connected subset of $\mathbb{R}^4$ such that $x^1, x^2, x^3, x^4 > 0$, endowed with the Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1+x^3}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + e^{x^1}(dx^4)^2. \tag{4.7}$$

Then the non-zero components (up to symmetry) of the Riemann-Christoffel curvature tensor R and the Ricci tensor S are given by

$$R_{1313} = -\frac{1}{2}e^{x^1+x^3}, \quad R_{1414} = -\frac{e^{x^1}}{4},$$

$$S_{11} = \frac{1(1+2e^{x^1+x^3})}{4e^{x^1+x^3}}g_{11}.$$

The scalar curvature of this metric $\kappa = 0$. Again the non-zero components (up to symmetry) of ∇R and ∇S are given by:

$$R_{1313,1} = -\frac{e^{x^1+x^3}}{2} = R_{1313,3}, \qquad S_{11,1} = S_{13,3} = -\frac{e^{x^1+x^3}}{2}.$$

Using the local components of R, ∇R, g and S, we can easily check that the manifold is Ricci simple satisfying

$$S = \eta \otimes \eta$$

for

$$\eta_i(x) = \begin{cases} -\frac{1}{2}\sqrt{1+2e^{x^1+x^3}} & \text{for} \quad i=1 \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, any Ricci simple manifold is a quasi-Einstein manifold. Again it is easy to check that the manifold is a $(QGK)_4$ with (Π, Φ, Ψ, η) for $\Phi \equiv 0$,

$$\Pi_i(x) = \begin{cases} -\frac{2e^{x^1+x^3}}{1-2e^{x^1+x^3}} & \text{for} \quad i=1,3 \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

and

$$\Psi_i(x) = \begin{cases} \frac{2e^{x^1+x^3}}{1-4e^{2x^1+2x^3}} & \text{for} \quad i=1,3 \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Conclusion

In the present paper we introduce a generalized class of recurrent manifolds, named, super generalized recurrent manifold and also study the curvature properties of such a manifold. It is shown that its associated 1-forms are not unique, and they are linearly dependent with $d\kappa$. It is also shown that if the associated 1-forms are closed and pairwise codirectional, then a $(SGK)_n$ is semisymmetric. It is proved that an Einstein $(SGK)_n$ is a K_n. Also we obtain a sufficient condition (namely, Roter type condition) for the equivalency of a $(SGK)_n$ and a generalized Ricci-recurrent manifold. Finally the existence of a $(SGK)_4$ is given by a non-trivial example. Also an example of a quasi-Einstein $(QGK)_4$ is given.

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