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ON PM$^+$ AND FINITE CHARACTER BI-AMALGAMATION

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Abstract. Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and $J^{'}$ be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J^{'})$. The bi-amalgamation of A with (B, C) along $(J, J^{'})$ with respect of (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J^{'}) = \{(f(a) + j, g(a) + j^{'})/a \in A, (j, j^{'}) \in J \times J^{'}\}.$$

In this paper, we study the transference of pm^+, pm and finite character ring-properties in the bi-amalgamation.
Keywords: Bi-amalgamated algebras, pm^+ rings, pm rings, rings with finite character.
MSC(2010): Primary: 13A15; Secondary: 13B02.

1. **Introduction**

Throughout, all rings considered are commutative with unity. A ring R is called a pm ring (also called Gefland ring) if each prime ideal is contained in exactly one maximal ideal. This class of rings has been introduced in [10] by G. De Marco and A.Orsatti, and studied in [4, 5, 17]. It contains the class of Von Neumann regular rings, local rings, zero-dimensional rings, rings of functions, etc.... In particular, any ring of the form $C(X)$, the ring of continuous real valued functions on a (completely regular) topological space X is pm ([11, 7.15]). However, this last class of rings have a stronger property than pm; in fact in a ring $C(X)$ where X is a topological space, the prime ideals containing a given prime ideal form a chain. In [2], W.D. Burgess and R. Raphael introduced a pm^+ rings as a rings with this last stronger property. Any local domain is a pm ring but would be pm^+ only if all its prime ideals formed a chain, as, for example, in a valuation domain. It is also proved that a ring R is pm^+ if and only if, for each multiplicative subset set S of R, $S^{-1}R$ a pm ring (see [2] for more details).

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A ring R has finite character if each non-zero ideal of R is contained in at most finitely many maximal ideals of R. The notion of the finite character domain has been introduced by Griffin [12]. While the concept of finite character ring was been extended to the rings with zero-divisors in 1971 by Larsen [14], where he characterized the prüfer finite character rings. A ring R is called $h-$local if it is a pm ring and has finite character. Thus, R is $h-$local if and only if modulo any non-zero prime ideal it is a local ring and modulo any non-zero ideal it is a semi-local ring. In [15, 16], Matlis studied the h-local domains and gave some further characterizations. In [19], Olberding shows how diverse examples of h-local Prüfer domains arise as overrings of Noetherian domains and polynomial rings in finitely many variables.

Let $f : A \to B$ and $g : A \to C$ be two commutative ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') = \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [13] as a natural generalization of duplications [8, 9] and amalgamations [6, 7]. Given a ring homomorphism $f : A \to B$ and an ideal J of B, the bi-amalgamation $A \bowtie^{\mathrm{id}_A, f} (f^{-1}(J), J)$ coincides with the amalgamated algebra introduced in 2009 by D'Anna, Finocchiaro, and Fontana [6, 7] as the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A,\ j \in J\}.$$

When $A = B$ and $f = \mathrm{id}_A$, the amalgamated $A \bowtie^{\mathrm{id}_A} I$ is called amalgamated duplication of a ring A along the ideal I, and denoted $A \bowtie I$ (introduced in 2007 by D'Anna and Fontana, [9]). This construction can be presented as a bi-amalgamation algebra as follows:

$$A \bowtie I = A \bowtie^{\mathrm{id}_A, \mathrm{id}_A} (I, I).$$

In this paper, we investigate the transfer of pm, pm^+, finite character and h-local properties to bi-amalgamation algebras. Our main goal is to provide new classes of rings which satisfied these properties.

2. **Results**

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $I := f^{-1}(J) = g^{-1}(J')$. Throughout this paper, $A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g).

In our study of the transfer of properties defined in the introduction, we need the description of the prime and maximal spectrums of the bi-amalgamation construction. For thus, let's adopt the following notations:

For $L \in \mathrm{Spec}(f(A)+J)$ and $L' \in \mathrm{Spec}(g(A)+J')$, consider the prime ideals of $A \bowtie^{f,g} (J,J')$ given by:

$$\begin{array}{rcl} \overline{L} & := & \big(L \times (g(A)+J')\big) \cap \big(A \bowtie^{f,g} (J,J')\big) \\ & = & \big\{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J', f(a)+j \in L\big\}, \\ \overline{L'} & := & \big((f(A)+J) \times L'\big) \cap \big(A \bowtie^{f,g} (J,J')\big) \\ & = & \big\{(f(a)+j, g(a)+j') \mid a \in A, (j,j') \in J \times J', g(a)+j' \in L'\big\}. \end{array}$$

Using [13, Lemmas 5.1, 5.2 and Propositions 5.3, 5.7], we deduce the following lemma.

Lemma 2.1. *Under the above notation, let P be a prime (resp. maximal) ideal of $A \bowtie^{f,g} (J,J')$. Then*

(1) $J \times J' \subseteq P \Leftrightarrow \exists!\ \mathfrak{p} \supseteq I$ *in* $\mathrm{Spec}(A)$ *(resp.* $\mathrm{Max}(A)$*) such that* $P = \mathfrak{p} \bowtie^{f,g} (J,J')$.
In this case, $\exists\ L \supseteq J$ *in* $\mathrm{Spec}(f(A)+J)$ *(resp.* $\mathrm{Max}(f(A)+J)$*) and* $\exists\ L' \supseteq J'$ *in* $\mathrm{Spec}(g(A)+J')$ *(resp.* $\mathrm{Max}(g(A)+J')$*) such that* $P = \overline{L} = \overline{L'}$.

(2) $J \times J' \nsubseteq P \Leftrightarrow \exists!\ L$ *in* $\mathrm{Spec}(f(A)+J)$ *or in* $\mathrm{Spec}(g(A)+J)$ *(resp. in* $\mathrm{Max}(f(A)+J)$ *or in* $\mathrm{Max}(f(A)+J)$*) such that* $J \nsubseteq L$ *or* $J' \nsubseteq L$ *and* $P = \overline{L}$.

Consequently, we have

$$\mathrm{Spec}\big(A \bowtie^{f,g} (J,J')\big) = \big\{\overline{L} \mid L \in \mathrm{Spec}\big(f(A)+J\big) \cup \mathrm{Spec}\big(g(A)+J'\big)\big\},$$

and

$$\mathrm{Max}\big(A \bowtie^{f,g} (J,J')\big) = \big\{\overline{L} \mid L \in \mathrm{Max}\big(f(A)+J\big) \cup \mathrm{Max}\big(g(A)+J'\big)\big\}.$$

The notations and the facts of the previous lemma will be used in the sequel without explicit reference.

Our first result investigate the transfer of the pm^+ property to the bi-amalgamation construction.

Proposition 2.2. *$A \bowtie^{f,g} (J,J^{'})$ is pm^+ ring if and only if $f(A)+J$ and $g(A)+J^{'}$ are pm^+ rings.*

Proof. ($\Rightarrow$) Using [13, Proposition 4.1], we have the following isomorphisms of rings

$$\frac{A \bowtie^{f,g} (J,J')}{0 \times J'} \cong f(A)+J \quad \text{and} \quad \frac{A \bowtie^{f,g} (J,J')}{J \times 0} \cong g(A)+J'.$$

Thus, following [2, Lemma 3.7], $f(A)+J$ and $g(A)+J'$ are pm^+ rings.
($\Leftarrow$) Let P be a prime ideal of $A \bowtie^{f,g} (J,J')$. We have to show that any two

ideals of $A \bowtie^{f,g} (J, J')$ which contains P are comparable. So, let $P_1, P_2 \in \mathrm{Spec}\left(A \bowtie^{f,g} (J, J')\right)$ containing P.
If $J \times J' \subseteq P$, then $J \times J' \subseteq P_1$ and $J \times J' \subseteq P_2$, and hence there exist $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(A)$ containing I such that $P = \mathfrak{p} \bowtie^{f,g} (J, J')$, $P_1 = \mathfrak{p}_1 \bowtie^{f,g} (J, J')$ and $P_2 = \mathfrak{p}_2 \bowtie^{f,g} (J, J')$. Following [13, Proposition 4.1], we have the following isomorphism of rings:

$$\begin{array}{cccc} \varphi: & \frac{\frac{A\bowtie^{f,g}(J,J')}{J\times J'}}{(f(a)+j,g(a)+j')} & \begin{array}{c}\longrightarrow \\ \longmapsto\end{array} & \begin{array}{c}\frac{A}{I} \\ \overline{a}\end{array} \end{array},$$

Thus, since $P \subseteq P_1$ and $P \subseteq P_2$, we have $\frac{\mathfrak{p}}{I} \subseteq \frac{\mathfrak{p}_1}{I}$ and $\frac{\mathfrak{p}}{I} \subseteq \frac{\mathfrak{p}_2}{I}$. On the other hand, by [13, Proposition 4.1], $\frac{A}{I} \cong \frac{f(A)+J}{J}$. Hence, since $f(A) + J$ is a pm^+ ring and by using [2, Lemma 3.7], $\frac{A}{I}$ is a pm^+ ring. Thus, $\frac{\mathfrak{p}_1}{I}$ and $\frac{\mathfrak{p}_2}{I}$ are comparable, and so are $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Consequently, P_1 and P_2 are comparable.
Now, suppose that $J \times J' \nsubseteq P$. Then, there exist L in $\mathrm{Spec}(f(A) + J)$ or in $\mathrm{Spec}(g(A) + J)$ such that $J \nsubseteq L$ or $J' \nsubseteq L$ and $P = \overline{L}$. In the first case, we have

$$0 \times J' \subseteq \left(L \times (g(A) + J')\right) \cap \left(A \bowtie^{f,g} (J, J')\right) = \overline{L} = P.$$

Thus, $\frac{P_1}{0\times J'}$ and $\frac{P_2}{0\times J'}$ are prime ideals of the pm^+ ring $\frac{A\bowtie^{f,g}(J,J')}{\{0\}\times J'} \cong f(A) + J$ containing the prime ideal $\frac{P}{0\times J'}$. Thus, $\frac{P_1}{0\times J'}$ and $\frac{P_1}{0\times J'}$ are comparable, and so are P_1 and P_2. Similarly, in the second case, we conclude that P_1 and P_2 are comparable. Accordingly, the ring $A \bowtie^{f,g} (J, J')$ is pm^+. □

Corollary 2.3. (1) *$A \bowtie^f J$ is pm^+ if and only if A and $f(A) + J$ are pm^+.*
(2) *$A \bowtie I$ is pm^+ if and only if A is pm^+.*

Example 2.4. For a given prime positive integer p, let $\mathbb{Z}_p$ be ring of p-adic integers. Clearly, $\mathbb{Z}_p$ is a pm^+ ring since it is a valuation domain. Thus, using the previous corollary, for any non zero positive integer n, the ring $\mathbb{Z}_p \bowtie (p^n\mathbb{Z}_p)$ is a pm^+ ring.

The next result study the transfer of the pm property to the bi-amalgamation construction.

Proposition 2.5. *$A \bowtie^{f,g} (J, J')$ is pm ring if and only if $f(A) + J$ and $g(A) + J'$ are pm rings.*

Proof. ($\Rightarrow$). Clearly, any homomorphic image of a pm ring is also pm. Thus, using [13, Proposition 4.1], $f(A) + J$ and $g(A) + J'$ are pm rings.
($\Leftarrow$). Let P be a prime ideal of $A \bowtie^{f,g} (J, J')$ contained in $M_1, M_2 \in \mathrm{Max}\left(A \bowtie^{f,g} (J, J')\right)$. If $J \times J' \subseteq P$, then $J \times J' \subseteq M_1$ and $J \times J' \subseteq M_1$, and hence there exist $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(A)$ containing I such that $P = \mathfrak{p} \bowtie^{f,g} (J, J')$, $M_1 = \mathfrak{m}_1 \bowtie^{f,g} (J, J')$ and $M_2 = \mathfrak{m}_2 \bowtie^{f,g} (J, J')$. As in the proof of Proposition

2.2, we deduce that $\frac{\mathfrak{m}_1}{I}$ and $\frac{\mathfrak{m}_2}{I}$ are maximal ideals of $\frac{A}{I}$ (which is isomorphic to a homomorphic image of $f(A)+J$, by [13, Proposition 4.1], and so it is a *pm* ring), and they contain the prime ideal $\frac{\mathfrak{p}}{I}$. Thus, $\frac{\mathfrak{m}_1}{I} = \frac{\mathfrak{m}_2}{I}$, and so $\mathfrak{m}_1 = \mathfrak{m}_2$. Consequently, $M_1 = M_2$.

Now, suppose that $J \times J^{'} \nsubseteq P$. Then, there exist L in $\mathrm{Spec}(f(A)+J)$ or in $\mathrm{Spec}(g(A)+J)$ such that $J \nsubseteq L$ or $J' \nsubseteq L$ and $P = \overline{L}$. In the first case, as in the proof of Proposition 2.2, we have $0 \times J^{'} \subseteq P$. Thus, $\frac{M_1}{0\times J'}$ and $\frac{M_2}{0\times J'}$ are maximal ideals of the *pm* ring $\frac{A\bowtie^{f,g}(J,J^{'})}{\{0\}\times J'} \cong f(A)+J$ containing the prime ideal $\frac{P}{0\times J'}$. Thus, $\frac{M_1}{0\times J'} = \frac{M_2}{0\times J'}$, and so $M_1 = M_2$. Similarly, in the second case, we conclude that $M_1 = M_2$. Accordingly, the ring $A \bowtie^{f,g} (J, J')$ is *pm*. □

Corollary 2.6. (1) *$A \bowtie^f J$ is pm if and only if A and $f(A)+J$ are pm.*
(2) *$A \bowtie I$ is pm if and only if A is pm.*

Recall that the ring R is called clean if each element in R can be expressed as the sum of a unit and an idempotent. The concept of clean rings was introduced by Nicholson [18]. Examples of clean rings include all commutative Von Neumann regular rings and local rings. A basic property of clean rings is that any homomorphic image of a clean ring is again clean. In [1], D.D. Anderson and V.P. Camillo proved that any clean ring is *pm*, and the equivalence holds when the ring has only finite number of minimal prime ideals. The notion of cleanness of bi-amalgamations is not yet performed. However, using the *pm* property, we can deduce the following result.

Corollary 2.7. (1) *If $A \bowtie^{f,g} (J, J')$ is clean, then so are $f(A)+J$ and $g(A)+J'$.*
(2) *If $f(A)+J$ and $g(A)+J'$ have a finite number of minimal prime ideals, then $A \bowtie^{f,g} (J, J')$ is clean if and only if $f(A)+J$ and $g(A)+J'$ are clean.*

In particular, $A \bowtie^{f,g} (J, J')$ is Noetherian and clean if and only if $f(A)+J$ and $g(A)+J'$ are Noetherian and clean.

Proof. (1). Follows from [13, Proposition 4.1] since every homomorphic image of a clean ring is clean (by [1, Proposition 2]).
(2). From [1, Theroem 5] and Proposition 2.5, it suffices to show that $A \bowtie^{f,g} (J, J')$ has only a finite number of minimal ideals. Using [13, Proposition 4.1], when $J = 0$ (resp. $J' = 0$), we have $A \bowtie^{f,g} (J, J') \cong g(A)+J'$ (resp. $A \bowtie^{f,g} (J, J') \cong g(A)+J'$), and so $A \bowtie^{f,g} (J, J')$ is satisfied the desired property. So, we may assume that $J \neq 0$ and $J' \neq 0$.

Let P be a minimal prime ideal of $A \bowtie^{f,g} (J, J')$. From Lemma 2.1, there exists L in $\mathrm{Spec}(f(A)+J)$ or in $\mathrm{Spec}(g(A)+J')$ such that $P = \overline{L}$. Take, for example, the first case and let $K \in \mathrm{Spec}(f(A)+J)$ such that $K \subseteq L$. Then, $0 \times J' \subseteq \overline{K} \subseteq \overline{L} = P$. Hence, $\overline{K} = \overline{L}$, and so $K = L$. Thus, L is a minimal prime ideal of $f(A)+J$. Consequently, since $f(A)+J$ and $g(A)+J'$ have only finite number of minimal prime ideals, $A \bowtie^{f,g} (J,J')$ has also only finite number of minimal prime ideals.

The last statement follows from [13, Proposition 4.2] and the fact that Noetherian rings have only finite number of minimal prime ideals. $\square$

Example 2.8. For each integer $n>1$, the Krull dimension of the Noetherian ring $\mathbb{Z}/n\mathbb{Z}$ is 0. Thus, by [1, Corollary 11], $\mathbb{Z}/n\mathbb{Z}$ is a clean ring. By using Corollary 2.7, for each integer $1<m<n$, the ring $\mathbb{Z}/n\mathbb{Z} \bowtie (\overline{m})$ is clean.

Recall that a ring R is semi-local if it has a finite number of maximal ideals. Clearly, every semi-local ring has finite character. The converse is not true (take the ring $\mathbb{Z}$ for example). Our next result proves that the finite character property and the semi-local property coincide over the bi-amalgamation construction. This is due to the form of maximal ideals of bi-amalgamation.

Proposition 2.9. *Suppose that J and $J^{'}$ are non-zero ideals of B and C, respectively. The following assertions are equivalent:*

(1) *$A \bowtie^{f,g} (J,J^{'})$ is semi-local.*
(2) *$A \bowtie^{f,g} (J,J^{'})$ has finite character.*
(3) *$f(A)+J$ and $g(A)+J^{'}$ are semi-local.*

Proof. (1) $\Rightarrow$ (2). Clear, since every semi-local ring has finite character.
(2) $\Rightarrow$ (3). Since $J\times\{0\}$ and $\{0\}\times J^{'}$ are non zero ideals of $A \bowtie^{f,g} (J,J^{'})$, there exist a finite number of maximal ideals of $A \bowtie^{f,g} (J,J^{'})$ containing $J\times\{0\}$ and $\{0\}\times J^{'}$, respectively. Therefore, $\frac{A\bowtie^{f,g}(J,J^{'})}{\{0\}\times J^{'}} \cong f(A)+J$ and $\frac{A\bowtie^{f,g}(J,J^{'})}{J\times\{0\}} \cong g(A)+J^{'}$ are semi-local rings.
(3) $\Rightarrow$ (1). Following Lemma 2.1,

$$\mathrm{Max}\big(A \bowtie^{f,g} (J,J')\big) = \big\{\overline{L} \mid L \in \mathrm{Max}\big(f(A)+J\big) \cup \mathrm{Max}\big(g(A)+J'\big)\big\}.$$

Thus, $\mathrm{Max}\big(A \bowtie^{f,g} (J,J')\big)$ is finite since $\mathrm{Max}\big(f(A)+J\big)$ and $\mathrm{Max}\big(g(A)+J'\big)$ are finite, and so $A \bowtie^{f,g} (J,J^{'})$ is semi-local. $\square$

Corollary 2.10. *Suppose that J and $I := f^{-1}(J)$ are a non-zero ideals of B and A, respectively. Then, $A \bowtie^{f} J$ has finite character if and only if A and $f(A)+J$ are semi-local.*

The transfer of h-local property to the bi-amalgamation algebras is easily deduced from Propositions 2.5 and 2.9 as follows:

Corollary 2.11. *Suppose that J and $J^{'}$ are non zero ideals of B and C respectively. Then, $A \bowtie^{f,g} (J,J^{'})$ is h-local if and only if $f(A)+J$ and $g(A)+J^{'}$ are pm and semi-local.*

Corollary 2.12. *Suppose that J and $I := f^{-1}(J)$ are a non-zero ideals of B and A, respectively. Then, $A \bowtie^f J$ is h-local if and only if A and $f(A) + J$ are pm and semi-local.*

Example 2.13. Let R be an Artinian ring (e.g., $\mathbb{Z}/n\mathbb{Z}$) and I a non-zero ideal of R. Then, since R is zero-dimensional (and so *pm*) ring and semi-local and by using the above corollary, $R \bowtie I$ is h-local.

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