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A STUDY ON DIMENSIONS OF MODULES

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Abstract. In this article we study relations between some algebraic operations such as tensor product and localization from one hand and some well-known dimensions such as uniform dimension, hollow dimension and type dimension from the other hand. Some minor applications to the ring $C(X)$ are observed.

Keywords: Type dimension, uniform dimension, hollow dimension.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. **Introduction**

In this article, by R, we mean a commutative ring with identity unless otherwise stated and all modules are assumed to be unitary. By $C(X)$, we always mean the ring of all real valued continuous functions over a completely regular space, with pointwise addition and multiplication. For undefined concepts and terminologies on $C(X)$ see [10]. This article is concerned with uniform dimension, hollow dimension and type dimension. The first two dimensions are quite well known and the reader can find a rich literature about them. See [11] and [16] for undefined terms and concepts on uniform dimension and hollow dimension respectively (infinite uniform dimension has been introduced in [6]).

Type dimension has been independently introduced in [18] and [5] as a type analog of uniform dimension (see [9] for a systematic study of type dimension and all related concepts). A module A is called *atomic* if $A \neq 0$ and for any $x, y \in A \setminus \{0\}$, xR and yR have non-zero isomorphic submodules. It is obvious that every uniform module is atomic but the converse is not true. The abelian group $\oplus^{\aleph_0}\mathbb{Z}_p$, is atomic but has no finite uniform dimension. We should emphasize here that the concept of atomic modules defined in [14] is used for a

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different class of modules. Two modules A and B are called *orthogonal*, written $A \perp B$, if they do not have non-zero isomorphic submodules. Modules A and B are *parallel*, denoted as $A||B$, if for any $(0) \neq V \subseteq A$, there exist $(0) \neq aR \subseteq V$ and $(0) \neq bR \subseteq B$ with $aR \cong bR$, and dually, for any $(0) \neq W \subseteq B$, there exist $(0) \neq aR \subseteq W$ and $bR \subseteq A$ such that $aR \cong bR$. The type dimension of a module M, denoted by $\text{t.dim}M$, is the finite or infinite cardinal number defined by

$$\text{t.dim}M = \sup\{|I| : \exists \oplus_{i\in I} K_i \leq R; \text{ for all } i \neq j, K_i \perp K_j\},$$

where $|I|$ denotes the cardinality of the set I (see [8, p. 15]). If in this definition all orthogonality restrictions are omitted, we obtain the definition of uniform dimension.

2. When is $\text{u.dim}(M \otimes N) = \text{u.dim}M \cdot \text{u.dim}N$?

Let F be a field and V, W two finite dimensional vector spaces. It is well-known that

$$\dim(V \otimes_F W) = \dim V \cdot \dim W. \tag{2.1}$$

Knowing that uniform dimension, hollow dimension and type dimension can be considered as generalizations of dimension of vector spaces, we can substitute these dimensions with dimension of vector spaces and ask whether the relation (2.1) still holds for modules over commutative rings.

In the next section we review the literature around the topic. Hence, some results which seems to be folklore are stated and proved.

2.1. Uniform dimension and tensor product.

As we mentioned in the introduction one of our aims in this work is to study those rings over which for certain modules M and N we have $\text{u.dim}(M \otimes_R N) = \text{u.dim}M \cdot \text{u.dim}N$. Putting $M = R$, then $R \otimes_R N \cong N$, we need R to be a uniform ring. Therefore, from the very beginning we suppose that R is a domain. To ensure that $(M \otimes N) \neq 0$, we restrict ourselves to torsion-free modules over a domain. The next lemma is well-known.

Lemma 2.1. *Let R be a domain, M and N two non-zero torsion free R-modules, then $M \otimes_R N$ is non-zero.*

Infinite uniform dimension of a module M defined as the supremum of all cardinals k such that M contains the direct sum of k nonzero submodules (see [6]). Let R be a domain and M and N be two torsion free modules (with infinite uniform dimension) over R. Is there any relation between $\text{u.dim}(M \otimes_R N)$ and $\text{u.dim}M \cdot \text{u.dim}N$, where by product we mean product of two infinite cardinals. The next proposition gives a positive answer to this question.

Proposition 2.2. *Let R be an integral domain and M, N two (non-zero) torsion free R-modules. Then* $\text{u.dim}(M \otimes_R N) = \text{u.dim}M \cdot \text{u.dim}N$.

Proof. Let K be the quotient field of R. It is easy to verify that u.dim$M = [M \otimes_R K : K]$ (see [11, Chapter 3, p. 94, Exercise 4]). As much as, $(M \otimes_R N) \otimes_R K \cong (M \otimes_R K) \otimes_R (N \otimes_R K)$, we have u.dim$(M \otimes_R N) = [((M \otimes_R N) \otimes_R K) : K] = [(M \otimes_R K) \otimes_R (N \otimes_R K) : K] = [(M \otimes_R K) : K][(N \otimes_R K) : K] =$ u.dim$M \cdot$ u.dim$N$. □

Corollary 2.3. *Let R be a domain and M, N two torsion free R-modules. Then $M \otimes_R N$ is uniform if and only if M and N are uniform.*

The concept of *an attained cardinal* and its relation with *inaccessible cardinals* has been considered in [6]. A cardinal number α is *attained* in M if M contains a direct sum of α nonzero submodules. If α is not a limit cardinal, i.e., if it is of the form $\aleph_\beta + 1$, for some ordinal β, then α is attained in M. An infinite cardinal α is called regular if $\alpha_i < \alpha$ for $i \in I$ with $|I| < \alpha$ implies $\sum \alpha_i < \alpha$; otherwise it is called singular. An uncountable, regular, limit cardinal is said to be *inaccessible*.

Corollary 2.4. *Over an integral domain R, the uniform dimension of every torsion free module is attained (even if it is inaccessible).*

Proof. Let M be a torsion free module and K be the field of fractions of R. We have already observed that u.dim$M =$ u.dim$(M \otimes_R K)$. The right side is the dimension of the K-vector space $M \otimes_R K$, and hence is attained. □

3. $C(X)$ and type dimension

In this section we characterize basic concepts such as orthogonal ideals, parallel ideals and atomic ideals related to type dimension in $C(X)$. We first characterize parallel and orthogonal ideals in $C(X)$. Moreover, we characterize atomic ideals of $C(X)$.

Before characterizing atomic ideals in $C(X)$ we need a lemma. Let I be an ideal in $C(X)$ and $B \subseteq X$. In the sequel we denote $\Delta(I) = \bigcap_{f \in I} Z(f)$ and $M_B = \{f \in C(X) \mid B \subseteq Z(f)\}$, where $Z(f) = f^{-1}\{0\}$.

Lemma 3.1. *Let $f \in C(X)$. Then*

(1) $\mathrm{Ann}(f) = M_{cl(X \setminus Z(f))}$.
(2) $\Delta(\mathrm{Ann}(f)) = \cap_{h \in \mathrm{Ann}(f)} A(h) = \mathrm{cl}_X(X \setminus Z(f))$.

Proof. (1). Let $g \in \mathrm{Ann}(f)$, we have $fg = 0$ and this implies that $Z(f) \cup Z(g) = X$, i.e., $X \setminus Z(f) \subseteq Z(g)$. Since $Z(g)$ is a closed set, we have clint$(X \setminus Z(f)) \subseteq Z(g)$, that is $g \in M_{cl(X \setminus Z(f))}$. Now let $g \in M_{cl(X \setminus Z(f))}$. This means that $\mathrm{cl}(X \setminus Z(f)) \subseteq Z(g)$. That is $X = Z(f) \cup Z(g) = Z(fg)$ or equivalently $fg = 0$. (2). We know that $\Delta(\mathrm{Ann}(f)) = \bigcap_{g \in \mathrm{Ann}(f)} Z(g)$ hence $X \setminus \Delta(I) = \cup_{g \in \mathrm{Ann}(f)}(X \setminus Z(g))$. Now if $f \in \mathrm{Ann}(f)$, then $fI = (0)$. That is for every $g \in I$; $Z(f) \cup Z(g) = X$, i.e., $X \setminus Z(g) \subseteq Z(f)$. This means that $\cup_{g \in \mathrm{Ann}(f)}(X \setminus Z(g)) \subseteq Z(f)$, and this is equal to $X\backslash\bigcap_{g\in \mathrm{Ann}(f)} Z(g) \subseteq Z(f)$. Hence $\mathrm{cl}(X\backslash\Delta(\mathrm{Ann}(f)) \subseteq Z(f)$ or equivalently $f \in M_{cl(X\backslash\Delta(\mathrm{Ann}(f))}$.

□

Corollary 3.2. *In $C(X)$, annihilators are always strongly z-ideals and closed (i.e. intersection of maximal ideals).*

Proposition 3.3. *Let X be a completely regular space and I an ideal in $C(X)$. Then the following conditions are equivalent.*

(1) *I is atomic;*
(2) *for every $0 \neq f, g \in I$, $fg \neq 0$;*
(3) *for every $f, g \in I \setminus \{0\}$, there exists $0 \neq h \in (f) \cap (g)$;*
(4) *I is uniform;*
(5) *I is minimal (simple submodule).*

Proof. (1) $\Longrightarrow$ (2). Let I be atomic and $0 \neq f, g \in I$. On the contrary, suppose that $fg = 0$. This shows that $(f)(g) = (0)$. Hence there are no non-zero ideals $H \subseteq (f)$ and $K \subseteq (g)$ such that $K \cong H$. Thus $\mathrm{Ann}(H) = \mathrm{Ann}(K)$ and $HK \subset (f)(g) = (0)$. Therefore $H^2 = (0)$ and hence $H = (0)$, is a contradiction.
(2) $\Longrightarrow$ (1). Let for every non-zero $f, g \in I$, $fg \neq 0$. This means that $Z(f) \cup Z(g) \subsetneqq X$. Now let $x \in X \setminus (Z(f) \cup Z(g))$. Since X is a completely regular space, there exists non-zero $h \in C(X)$ such that $h(x) = 1$. Hence $Z(f) \cup Z(g) \subseteq \mathrm{int}_X Z(h)$, i.e., $h \neq 0$ and $h \in (f) \cap (g)$. Hence $0 \neq (h) \subseteq (f)$ and $0 \neq (h) \subseteq (g)$.
(2) $\Longrightarrow$ (3). On the contrary, suppose that $(f) \cap (g) = (0)$, this implies that $(f)(g) = (0)$ and hence $fg = 0$, a contradiction.
(3) $\Longrightarrow$ (4). Just follow the definition of uniform module.
(4) $\Longleftrightarrow$ (5). It was proved in [3].
(5) $\Longrightarrow$ (1). The verification is immediate.

□

Corollary 3.4. *Let X be a completely regular space. Then the following conditions are equivalent in $C(X)$.*

(1) *$C(X)$ is atomic;*
(2) *$C(X)$ is a domain;*
(3) *X is a singleton.*

The next proposition deals with orthogonality in $C(X)$.

Proposition 3.5. *Let X be a completely regular space and I, J two ideals in $C(X)$. Then the following conditions are equivalent.*

(1) *$I \perp J$;*
(2) *$IJ = (0)$.*

Proof. (1) $\Longrightarrow$ (2). Let $IJ \neq (0)$. Then there are $f \in I$ and $g \in J$ such that $fg \neq 0$ (or equivalently $Z(f) \cup Z(g) \neq X$). If $(f) \cong (g)$ we have nothing to prove (we get a contradiction). So suppose that $(f) \ncong (g)$ which means that

$\text{int}Z(f) \neq \text{int}Z(g)$. Since $Z(f) \cup Z(g) \subsetneq X$, there exists $h \in C(X)$ such that $Z(f) \cup Z(g) \subseteq \text{int}Z(h)$ and $h \neq 0$. Therefore $h \in (f)$ and $h \in (g)$, which imply that $(h) \subseteq I$ and $(h) \subseteq J$ (In fact we have shown that $I \cap J = (0)$).
(2) $\Longrightarrow$ (1). By contrary, suppose that there are $(0) \neq K \leq I$ and $(0) \neq H \leq J$ such that $K \cong H$. This last term implies that $\text{Ann}(K) = \text{Ann}(H)$, hence $IH = (0)$ and we have $I \subseteq \text{Ann}(K)$. That is $IK = (0)$, hence $K^2 = (0)$ implying that $K = (0)$, a contradiction. □

Remark 3.6. By the same argument as in the proof of part (2) $\Longrightarrow$ (1) above, one can show that, in any reduced ring, $I \cap J = (0)$ if and only if $I \perp J$.

Corollary 3.7. *Let R be a commutative reduced ring. Then*

$$\text{u.dim}R = \text{t.dim}R.$$

Recall that when $I||J$, we have for every $0 \neq K \subseteq I$, K is not perpendicular to J and also for every $0 \neq T \subseteq J$, T is not perpendicular to I. We use this equivalent definition for parallel ideals in the following.

Proposition 3.8. *Let I, J be two ideals in $C(X)$. Then the following conditions are equivalent.*

(1) $I||J$;
(2) $\text{Ann}(I) = \text{Ann}(J)$;
(3) $\text{int}\Delta(I) = \text{int}\Delta(J)$.

Proof. (2) $\Longleftrightarrow$ (3). is well-known.
(1) $\Longrightarrow$ (2). By contrary suppose that there exists a non-zero $K \subseteq I$ such that $K \perp J$, we have $KJ = (0)$. This implies that $\text{Ann}(J) \cap I = (0)$ and hence $\text{Ann}(J) \subseteq \text{Ann}(I)$. Similarly, $\text{Ann}(I) \subseteq \text{Ann}(J)$.
(2) $\Longleftrightarrow$ (1). Now by contrary suppose that $I \nparallel J$, i.e., there exists $0 \neq K \leq I$ such that $K \perp J$. By the above proposition, this implies that $KJ = (0)$. That is $K \subseteq \text{Ann}(J)$ and hence $K \subseteq \text{Ann}(I)$ which means that $KI = (0)$ and therefore $K^2 = (0)$, but in $C(X)$, this means that $K = (0)$, a contradiction.
□

4. When is $\text{h.dim}(M \otimes_R N) = \text{h.dim}M \cdot \text{h.dim}N$?

To answer the aforementioned question, we need to consider some natural restrictions. Again, from $R \otimes_R M \cong M$ and the fact that over R, at least for a good class of modules, one expects to have $\text{h.dim}(M \otimes_R N) = \text{h.dim}M \cdot \text{h.dim}N$, one concludes that $\text{h.dim}R = 1$. Hence it is not surprising, if as one of our assumptions in this section we suppose that R is a local ring.

Our second natural restriction comes from the well-known observation that over a local ring R, for any two non-zero finitely generated modules M and N, $M \otimes_R N = 0$ implies that $M = 0$ or $N = 0$ (an application of Nakayama's lemma). Bearing these facts in mind, we may rephrase our initial question, over a local ring R as follows: when is $\text{h.dim}(M \otimes_R N) = \text{h.dim}M \cdot \text{h.dim}N$, for any two finitely generated R-modules M and N?

Since over a local ring, for finitely generated modules, flatness, projectivity and freeness are all equivalent, we may first observe this fact for finitely generated free R-modules.

Lemma 4.1. *Let R be a local ring and M, N two finitely generated free R-modules. Then* $\text{h.dim}(M \otimes_R N) = \text{h.dim}M \cdot \text{h.dim}N$.

Proof. Without loss of generality we may suppose that $M = R^n$ and $N = R^m$. From [4, 5.4.(1)], we know that $\text{h.dim}R^n = n$ and $\text{h.dim}R^m = m$. On the other hand, $R^n \otimes R^m \cong R^{nm}$ and therefore $\text{h.dim}(R^n \otimes_R R^m) = nm$. This shows that $\text{h.dim}(M \otimes_R N) = \text{h.dim}M \cdot \text{h.dim}N$. □

Proposition 4.2. *Let R be a local ring and M and N be finitely generated R-modules with finite hollow dimension. Then* $\text{h.dim}(M \otimes_R N) \leq \text{h.dim}M \cdot \text{h.dim}N$.

Proof. Without loss of generality, we suppose that $M = \frac{R^n}{A}$ and $N = \frac{R^m}{B}$. It is well-know that $M \otimes N = \frac{R^n}{A} \otimes_R \frac{R^m}{B} \cong \frac{R^n \otimes_R R^m}{C}$, where C is the submodule of $R^n \otimes R^m$ generated by all elements $x' \otimes y$ and $x \otimes y'$ with $x \in R^n$, $x' \in A$, $y \in R^m$, $y' \in B$. Now the hollow dimension of factor modules is less or equal to the hollow dimension of the module itself. Therefore $\text{h.dim}(M \otimes_R N) = \text{h.dim}(\frac{R^n}{A} \otimes_R \frac{R^m}{B} \cong \frac{R^n \otimes_R R^m}{C}) \leq \text{h.dim}(R^n \otimes_R R^m) = \text{h.dim}R^n \cdot \text{h.dim}R^m = nm$. □

Corollary 4.3. *Let R be a local ring and M, N two finitely generated R-modules. If M and N are hollow modules, then $M \otimes_R N$ is hollow module.*

5. Localization and uniform dimension

In this section, our main concern is the relation between localization and uniform dimension. First, we begin with a useful fact.

Lemma 5.1. *Let R be a domain and M be a torsion free R-module and P be a prime ideal in R. Then M_P as R_P-module has finite uniform dimension n if and only if* $\text{u.dim}M = n$.

Proof. First recall that torsion freeness is a local property over integral domains ([2, p. 45, Problem 13]). By [2, Proposition 3.5], we have

$$R_P \otimes_R M \cong M_P$$

Now by Proposition 2.2, we may conclude that M_P has uniform dimension n if and only if M has uniform dimension n. □

For a ring, it is possible that localization at prime ideals are all fields and hence with uniform dimension 1, but the ring itself has no finite uniform dimension. Now we may ask when is the converse true ?

Proposition 5.2. *Let R be a ring and* $\text{u.dim}R = \beta$. *Then for every prime ideal P of R,* $\text{u.dim}R_P \leq \beta$.

Proof. On the contrary, suppose that there exists a prime ideal P such that $\text{u.dim}R_P = \lambda > \beta$. Then λ is the supremum of all the independent families of ideals in R_P. Bringing back these families to R by contraction, one may get a supremum over all the contract families. But this supremum must be larger than β, a contradiction. □

The reader is reminded that R is semilocal if and only if R has finite hollow dimension. The next result and its corollary are generalizations of this fact.

Theorem 5.3. *Let R be a ring,* $\text{h.dim}R = |\text{Max}(R)| = \alpha$ *and* $\text{u.dim}R = \beta$. *If β is attained, then the following statements are equivalent:*

(1) $\beta \leq \alpha$;
(2) $\text{u.dim}R_P \leq \alpha$ *for each prime ideal P of R;*
(3) $\text{u.dim}R_M \leq \alpha$ *for each maximal ideal M of R.*

Proof. (1)$\Longrightarrow$(2). By Proposition 5.2, $\text{u.dim}R_P \leq \beta$ and hence $\text{u.dim}R_P \leq \alpha$.
(2)$\Longrightarrow$(3). is evident.
(3)$\Longrightarrow$(1). On the contrary, suppose that $\beta > \alpha$. Since there exists an infinite direct sum A in R, with β summands and R has only α maximal ideals, by the generalized pigeon hole principal, there is some maximal ideal M containing β summands of A. Now localizing R at M, one gets a contradiction by observing that R_M contains a direct sum of β summands, while $\text{u.dim}R_M \leq \alpha < \beta$. □

Corollary 5.4. *Let R be a semilocal ring such that every localization at a maximal ideal has finite uniform dimension. Then R has finite uniform dimension.*

Remark 5.5. Let λ be the Souslin number of X. It is easy to see that $\lambda \leq |\beta X|$. Now by Theorem 5.3, $\text{u.dim}C(X)_P \leq |\beta X|$ for every prime ideal P.

5.1. **Localization and type dimension.** We begin with a lemma whose proof is straightforward.

Lemma 5.6. *Suppose that by e and c we mean usual extension and contraction. Then we have:*

(1) *If $I \perp J$, then $I^e \perp J^e$;*
(2) *If $A^c \perp B^c$, then $A \perp B$.*

Using this lemma and the same argument as in the proof of Proposition 5.7 we have:

Proposition 5.7. *Let R be a ring and* $\text{t.dim}R = \beta$. *Then for every prime ideal P of R,* $\text{t.dim}R_P \leq \beta$.

5.2. **Partial answers to some questions.** In [1, p. 10, Question 4], it has been asked for the characterization of all the ring extensions $R \subseteq T$ such that u.dim$R =$ u.dim$T$. Then the authors of [1] answered the question in a positive way when T is the total classical ring of quotient of R. In [7], J. Dauns and M. Motamedi proved that if T is either a normalizing extension of R or $T = R[x]$, then u.dim$R =$ u.dim$T$. For finite uniform dimensional case these questions were answered by R. Shock in [17] (for $T = R[x]$) and by C. Lanski in [15] (for a normalizing extension).

Furthermore, in [1, p. 12, Question 1], the following question was asked by the authors: what are the topological spaces X such that u.dim$C(X)_P \leq \lambda$, for all prime ideals P in C(X), where λ is a given cardinal number. Concerning this question one can use Proposition 5.7 to give a partial answer. For those cardinal numbers λ, with u.dim$C(X) = c(X) \leq \lambda$, the above inequality is correct for every topological space. So the question can be reformulated by this restriction that $\lambda < c(X)$. An algebraic interpretation of this new reformulation is that, when do localizations of a ring R, at prime ideals have uniform dimension strictly less than u.dim$R$?

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