ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 5, pp. 1265–1279

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 5, pp. 1265–1279
Online ISSN: 1735-8515

APPLICATION OF FRAMES IN CHEBYSHEV AND CONJUGATE GRADIENT METHODS

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(Communicated by Mohammad Asadzadeh)

Abstract. Given a frame of a separable Hilbert space H, we present some iterative methods for solving an operator equation $Lu = f$, where L is a bounded, invertible and symmetric operator on H. We present some algorithms based on the knowledge of frame bounds, Chebyshev method and conjugate gradient method, in order to give some approximated solutions to the problem. Then we investigate the convergence and optimality of them.

Keywords: Hilbert spaces, dual space, frame, Chebyshev polynomials, iterative method.
MSC(2010): Primary: 65F10; Secondary: 65F08.

1. Introduction

The analysis of numerical schemes for operator equation

$$Lu = f, \tag{1.1}$$

where $L : H \to H$ is a boundedly invertible and self adjoint operator on a separable Hilbert space H is a field of enormous current interest. Linear differential or integral equation in variational form are among such operations. Inverting the operator L can be complicated if the dimension of H is enough large, hence, a suitable option is to use an algorithm to obtain the approximations of the solution. In this among, one of the powerful mathematical tool is to use wavelet spaces that are applicable in many areas such as image and signal processing. Wavelet basis functions on any bounded domain include locally compact support that allows us to have a sparse stiffness matrix. Moreover, they can be used to construct adaptive numerical schemes that are guaranteed to converge with optimal order (see for instance [7]). In [1,6–8] some numerical

Article electronically published on 31 October, 2017.
Received: 31 August 2015, Accepted: 22 May 2016.
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algorithms for solving this system have been developed by using wavelets. Usually, constructing wavelets with specific properties on bonded domain or on a closed manifold is a hard mission and some serious drawbacks such as stability problems cannot be avoided [10, 11]. On the other hand, using wavelet basis functions usually generate a coefficient matrix with relatively high condition number. This necessity to have basis functions for wavelet spaces can generate the above mentioned drawbacks. Therefore, a slightly weaker concept, namely frames, can be applied to get rid of constructing basis functions. Let H be a separable Hilbert space with dual H^* and Λ be a countable set of indices. A family $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a frame for H, if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$,

$$A\|f\|_H^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 \leq B\|f\|_H^2. \tag{1.2}$$

The constants A and B are called lower and upper frame bound, respectively. Those sequences which satisfy only the upper inequality in (1.2) are called Bessel sequences. A frame is called a tight frame, if $A = B$. If $A = B = 1$, it is called a Parseval frame. For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span. Since frames in comparison with wavelets have more freedom in generating the solution space, hence construction of frames is much easier than that of wavelets. Moreover, frames can be generated by functions with smaller compact support that makes sparser the coefficient matrix. On the other hand, since we are working with a weaker concept, one can expect to have a coefficient matrix with relatively low condition number. In [9, 14, 19] we can see some iterative methods for solving the equation (1.1) by using frames. Richardson method is a usual iterative method in solving operator equations with boundary conditions (see [9]). Fornasier and Stevenson used the steepest descent method to solve the operator equations in frame spaces. We present two algorithms in which we use Chebyshev and conjugate gradient methods for solving operator equations. In proving the convergence of both algorithms, we observe that the selection of suitable frame bounds is very efficient for faster convergence of an algorithm.

This paper is organized as follows. In Section 2, we give some preliminaries on frames. Algorithm of Chebyshev and its convergence for solving operator equation is given in Section 3. In Chapter 4, we present conjugate gradient method to approximate the solution of an operator equation in the frame space. Finally, in Section 5, we present some numerical experiments to confirm our theoretical results.

2. **Preliminaries**

For a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$, let $T : \ell_2(\Lambda) \to H$ be the synthesis operator

$$T((c_\lambda)_\lambda) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and $T^* : H \to \ell_2(\Lambda)$ be the analysis operator

$$T^*(f) = (\langle f, \psi_\lambda\rangle)_{\lambda\in\Lambda}.$$

Also, suppose that $S := TT^* : H \to H$ is the frame operator

$$S(f) = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda.$$

Note that T and T^* are surjective and injective operators, respectively, and T^* is the adjoint of T. Because of (1.2), T is bounded and

$$\|T\|_{\ell_2(\Lambda)\to H} = \|T^*\|_{H\to ell_2(\Lambda)} \le \sqrt{B}. \tag{2.1}$$

The operator S is a positive definite invertible operator satisfying

$$AI_H \le S \le BI_H \tag{2.2}$$

and $B^{-1}I_H \le S^{-1} \le A^{-1}I_H$ (see [5]). Moreover, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda},$$

is a frame (called the canonical dual frame) for H with bounds B^{-1}, A^{-1}. In this case every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \widetilde{\psi}\rangle \psi_\lambda = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \widetilde{\psi}. \tag{2.3}$$

In [5], it has been shown that if $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a frame for H with bounds A and B, and if L is a bounded invertible operator on H, then the sequence $\Phi = (L\psi_\lambda)_{\lambda\in\Lambda}$ would be a frame for H with bounds $\frac{A}{\|L^{-1}\|^2_{H\to H}}$ and $B\|L\|^2_{H\to H}$.
Furthermore, if L is self adjoint, S is the frame operator of Ψ and S' is the frame operator of Φ, then for any $f \in H$,

$$S'f = \sum_{\lambda\in\Lambda} \langle f, L\psi_\lambda\rangle L\psi_\lambda = L(\sum_{\lambda\in\Lambda} \langle f, L\psi_\lambda\rangle \psi_\lambda) = L(\sum_{\lambda\in\Lambda} \langle Lf, \psi_\lambda\rangle \psi_\lambda) = LSLf$$

that means $S' = LSL$ (for more information we refer to [3,5]).

3. Chebyshev method

In this section, we verify how to use the frames for solving an operator equation with given boundary conditions by applying the Chebyshev method. Moreover, by using the bounds of the frame, we present a stop criteria.

One way to numerically approach the solution of the equation (1.1) with given boundary conditions is the Richardson iterative method . First of all, rewrite (1.1) as

$$u = (I - L)u + f.$$

Now, for a given $u_0 \in H$ and $k \geq 0$ define:

$$u_{k+1} = (I - L)u_k + f. \tag{3.1}$$

In this case, since $Lu - f = 0$ then
$u_{k+1} - u = (I - L)u_k + f - u - (f - Lu) = (I - L)u_k - u + Lu = (I - L)(u_k - u)$.
Therefore,

$$\|u_{k+1} - u\|_H \leq \|I - L\|_{H \to H} \|u_k - u\|$$

that means u_k converges to u, if

$$\|I - L\|_{H \to H} < 1.$$

In general, the non-stationary Richardson iteration is

$$u_{k+1} = u_k + a_k(f - Lu_k), \quad k = 0, 1, 2, \dots, \tag{3.2}$$

where u_0 is an initial guess and $a_k > 0$ are parameters to be chosen. This equation easily induces that the residual $r_k = f - Lu_k$ and the error vector $u - u_k$ can be written as

$$r_k = Q_k(L)r_0, \quad u - u_k = Q_k(L)(u - u_0),$$

where $Q_k(x) = \prod_{i=0}^{k-1}(1 - a_i x), \quad Q_k(0) = 1$. By a suitable choice of the parameters $\{a_i\}_{i=0}^{k-1}$ in (3.2), it may be possible to improve the rate of convergence of the iteration (3.2). Such process is called polynomial acceleration. The Chebyshev polynomials have the important minimax property that makes them useful for convergence acceleration [18].

The following theorem represents an iterative scheme based on the Richardson iterative method and the knowledge of some frame bounds to give an approximated solution for (1.1).

Theorem 3.1. *Let $\Psi = (\psi_\lambda)_{\lambda \in \Lambda} \subset H$ be a frame for H with frame operator S, and let L be as in (1.1). Suppose that A and B are the frame bounds of the frame $(L\psi_\lambda)_{\lambda \in \Lambda}$. Let $u_k = u_{k-1} + \frac{2}{A+B} LS(f - Lu_{k-1})$ for $k \geq 1$.*
Then

$$\|I - \frac{2}{A+B} LSL\|_{H \to H} \leq \frac{B - A}{B + A}. \tag{3.3}$$

Proof. For every $v \in H$ we have

$$\begin{aligned}
\left\langle \left(I - \tfrac{2}{A+B} LSL\right) v, v \right\rangle &= \|v\|_H^2 - \tfrac{2}{A+B} \langle LSLv, v\rangle \\
&= \|v\|_H^2 - \tfrac{2}{A+B} \langle S'v, v\rangle \\
&= \|v\|_H^2 - \tfrac{2}{A+B} \textstyle\sum_\lambda |\langle v, \phi_\lambda\rangle|^2 \\
&\leq \|v\|_H^2 - \tfrac{2A}{A+B} \|v\|_H^2 \\
&= \left(\tfrac{B-A}{B+A}\right) \|v\|_H^2.
\end{aligned} \tag{3.4}$$

The inequality given in (3.4) is obtained by the frame property of $\Phi = (L\psi_\lambda)_{\lambda\in\Lambda}$. Similarly, we have

$$-\left(\frac{B-A}{B+A}\right) \|v\|_H^2 \leq \left\langle \left(I - \frac{2}{A+B} LSL\right) v, v \right\rangle.$$

So, we conclude that

$$\left\| I - \frac{2}{A+B} LSL \right\|_{H\to H} \leq \frac{B-A}{B+A}. \tag{3.5}$$

□

Note that by definition of u_k, in Theorem 3.1 we obtain

$$\begin{aligned}
u - u_k &= u - u_{k-1} - \frac{2}{A+B} LSL(u - u_{k-1}) \\
&= \left(I - \frac{2}{A+B} LSL\right)(u - u_{k-1}) \\
&= \left(I - \frac{2}{A+B} LSL\right)^2 (u - u_{k-2}) = ... \\
&= \left(I - \frac{2}{A+B} LSL\right)^k (u - u_0).
\end{aligned}$$

Thus

$$\|u - u_k\|_H \leq \|I - \frac{2}{A+B} LSL\|_{H\to H}^k \|u\|_H, \tag{3.6}$$

that means the iterative method proposed in Theorem 3.1 is convergent for each initial guess u_0. Now, let $h_n = \sum_{k=1}^n a_{n_k} u_k$, where $\sum_{k=1}^n a_{n_k} = 1$ and u_k is given as in Theorem 3.1.

The condition $\sum_{k=1}^{n} a_{n_k} = 1$ guarantees that if $u_1 = u_2 = ... = u_n = u$, then $h_n = u$. Therefore,

$$\begin{aligned} u - h_n &= \sum_{k=1}^{n} a_{n_k} u - \sum_{k=1}^{n} a_{n_k} u_k = \sum_{k=1}^{n} a_{n_k} (u - u_k) \\ &= \sum_{k=1}^{n} a_{n_k} \left(I - \tfrac{2}{A+B} LSL \right)^k (u - u_0). \end{aligned}$$

Putting $R = I - \frac{2}{A+B} LSL$ and $Q_n(x) = \sum_{k=1}^{n} a_{n_k} x^k$, we obtain

$$u - h_n = Q_n(R)(u - u_0) \tag{3.7}$$

that is, the error is a polynomial in R applied to the initial error $u - u_0$.

Remark 3.2. Since

$$-\frac{B-A}{B+A}\|f\|_H^2 \leq \langle (I - \frac{2}{A+B} LSL) f, f \rangle \leq \frac{B-A}{B+A}\|f\|_H^2,$$

the spectrum of R is obtained in $[-\alpha_0, \alpha_0]$ with $\alpha_0 = \frac{B-A}{B+A}$.

Since LSL is a positive operator, the spectral theorem applies and yields

$$\|u - h_n\|_H \leq \|Q_n(R)\| \|u - u_0\|_H \leq \max_{|x| \leq \alpha_0} |Q_n(x)| \|u - u_0\|_H. \tag{3.8}$$

Now, in order to minimize this error we try to find

$$\min_{Q_n(1)=1} \max_{|x| \leq \alpha_0} |Q_n(x)|, \tag{3.9}$$

where the min is taken over all polynomials of degree less than or equal to n, with $Q_n(1) = \sum_{k=1}^{n} a_{n_k} = 1$. Having the answer of this, our request can be given in terms of the Chebyshev polynomials that are defined by

$$C_0(x) = 1, C_1(x) = x$$

and for $n \geq 2$,

$$C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x). \tag{3.10}$$

It is readily seen that

(3.11)

$$C_n(x) = \begin{cases} \cos(n \cos^{-1}(x)), & |x| \leq 1 \\ \cosh(n \cosh^{-1}(x)) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n} \right), & |x| \geq 1. \end{cases}$$

The following lemma has been established in [4].

Lemma 3.3. *For given* $a < b < 1$ *let* $P_n(x) = \frac{C_n\left(\frac{2x-a-b}{b-a}\right)}{C_n\left(\frac{2-a-b}{b-a}\right)}$. *Then*

$$\max_{a \leq x \leq b} |P_n(x)| \leq \max_{a \leq x \leq b} |Q_n(x)|,$$

for all polynomials Q_n *of degree* n *satisfying* $Q_n(1) = 1$. *Furthermore,*

$$\max_{a \leq x \leq b} |P_n(x)| = \frac{1}{C_n\left(\frac{2-a-b}{b-a}\right)}.$$

Theorem 3.4. *The polynomial* $\frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$ *minimizes the error* $\|u-h_n\|$ *given in the inequality* (3.8).

Proof. It is enough to apply Lemma 3.3 with $a=-\alpha_0$ and $b=\alpha_0$.
In this case,

$$P_n(x)=\frac{C_n(\frac{2x+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})}{C_n(\frac{2+\alpha_0-\alpha_0}{\alpha_0+\alpha_0})}=\frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}. \tag{3.12}$$

□

Proposition 3.5. *The approximated solution* h_n *with the error* $\|u-h_n\|$ *satisfies the recurrence relation*

$$h_n=\beta_n\left(h_{n-1}-h_{n-2}+\frac{2}{A+B}LSL(u-h_{n-1})\right)+h_{n-2},$$

where $\beta_n=\frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$.

Proof. By Theorem 3.4 we obtain a recurrence relation for

$$P_n(x)=\frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})},$$

or equivalently $C_n(\frac{1}{\alpha_0})P_n(x)=C_n(\frac{x}{\alpha_0})$. Combining this formula with the recurrence relation (3.10) for C_n gives

$$C_n(\frac{1}{\alpha_0})P_n(x)=\frac{2x}{\alpha_0}C_{n-1}(\frac{x}{\alpha_0})-C_{n-2}(\frac{x}{\alpha_0})=\frac{2x}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(x)-C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(x).$$

Now, replacing x by R induces the operator identity

$$C_n(\frac{1}{\alpha_0})P_n(R)=\frac{2R}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(R)-C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(R).$$

Multiplying this operator identity by $u-u_0$ gives

$$C_n(\frac{1}{\alpha_0})P_n(R)(u-u_0)=(\frac{2R}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})P_{n-1}(R)-C_{n-2}(\frac{1}{\alpha_0})P_{n-2}(R))(u-u_0),$$

and by (3.7) we obtain

$$C_n(\frac{1}{\alpha_0})(u-h_n)=\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})R(u-h_{n-1})-C_{n-2}(\frac{1}{\alpha_0})(u-h_{n-2}).$$

Writing $R=I-\frac{2}{A+B}LSL$ induces

$$C_n(\frac{1}{\alpha_0})(u-h_n)=\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})(I-\frac{2}{A+B}LSL)(u-h_{n-1})-C_{n-2}(\frac{1}{\alpha_0})(u-h_{n-2}),$$

or equivalently

$$\begin{aligned} C_n(\frac{1}{\alpha_0})u - C_n(\frac{1}{\alpha_0})h_n \\ =& \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})u + \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})\Big(-h_{n-1} - \frac{2}{A+B}LSL(u-h_{n-1})\Big) \\ &- C_{n-2}(\frac{1}{\alpha_0})u + C_{n-2}(\frac{1}{\alpha_0})h_{n-2}, \end{aligned}$$

and finally by (3.10),

$$C_n(\frac{1}{\alpha_0})h_n = \frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})\left(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})\right) - C_{n-2}(\frac{1}{\alpha_0})h_{n-2}.$$

Therefore,

$$h_n = \frac{2}{\alpha_0}\frac{C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})) - \frac{C_{n-2}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}h_{n-2}. \tag{3.13}$$

Now, by using (3.10), we have

$$1-\beta_n = 1 - \frac{2}{\alpha_0}\frac{C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})} = -\frac{C_{n-2}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}.$$

In this case, we can rewrite (3.13) as

$$h_n = \beta_n\left(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})\right) + (1-\beta_n)h_{n-2}$$

or equivalently

$$h_n = \beta_n\left(h_{n-1} - h_{n-2} + \frac{2}{A+B}LSL(u-h_{n-1})\right) + h_{n-2}.$$

□

Proposition 3.6. *If* $\beta_n = \frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$, *then the following recurrence holds*

$$\beta_n = (1 - \frac{\alpha_0^2}{4}\beta_{n-1})^{-1}.$$

Proof. By the assumption and the recursive formula, we have

$$\beta_n = \left(\frac{\alpha_0 C_n(\frac{1}{\alpha_0})}{2C_{n-1}(\frac{1}{\alpha_0})}\right)^{-1} = \left(\frac{\alpha_0}{2}\frac{\frac{2}{\alpha_0}C_{n-1}(\frac{1}{\alpha_0}) - C_{n-2}(\frac{1}{\alpha_0})}{C_{n-1}(\frac{1}{\alpha_0})}\right)^{-1} = \left(1 - \frac{\alpha_0^2}{4}\beta_{n-1}\right)^{-1}.$$

□

Now, based on the above argument, we design the following algorithm in order to give an iterative method for solving the equation (1.1). First we note that since L is a positive definite operator, there exists $m > 0$ such that $m\|f\|_H \leq \|Lf\|_H$ for all $f \in H$.

Suppose that $(\psi_\lambda)_{\lambda\in\Lambda}$ is a frame with frame operator S with A and B as the bounds of the frame $(L\psi_\lambda)_{\lambda\in\Lambda}$. The following algorithm solves the operator equation $Lu = f$ with given boundary conditions on the frame space by Chebyshev method. The tolerance value ϵ is chosen such that $\|u - \bar{u}\| \le \epsilon$ in which $\bar{u}$ is the approximated value of u.

Algorithm 1 $[L, \epsilon, f, A, B, m] \to u_\epsilon$

(i) Let $\alpha_0 = \frac{B-A}{B+A}$, $\sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$

(ii) Put $h_0 := 0$, $h_1 := \frac{2}{A+B} LSf$, $\beta_1 = 2$, $n = 1$

(iii) While $\frac{2\sigma^n}{1+\sigma^{2n}} \frac{\|f\|_H}{m} > \epsilon$ do

(1) $n := n + 1$

(2) $\beta_n = (1 - \frac{\alpha_0^2}{4}\beta_{n-1})^{-1}$

(3) $h_n = \beta_n(h_{n-1} - h_{n-2} + \frac{2}{A+B} LS(f - Lh_{n-1})) + h_{n-2}, \quad n \ge 2$

end do

(iv) $u_\epsilon = h_{n-1}$.

Theorem 3.7. *The approximated solution h_n in Algorithm 1 satisfies*

$$\|u - h_n\|_H \le \frac{2\sigma^n}{1+\sigma^{2n}} \frac{\|f\|_H}{m}.$$

Also the output u_ϵ satisfies $\|u - u_\epsilon\|_H \le \epsilon$.

Proof. Combining (3.8) and Lemma 3.3 with $u_0 = h_0 = 0$, we obtain

$$\|u - h_n\|_H \le \frac{1}{C_n(\frac{1}{\alpha_0})} \|u - u_0\|_H = \frac{1}{C_n(\frac{1}{\alpha_0})} \|u\|_H \le \frac{1}{C_n(\frac{1}{\alpha_0})} \frac{\|f\|_H}{m}. \tag{3.14}$$

On the order hand, by (3.11) we have

$$\begin{aligned}
C_n\left(\tfrac{1}{\alpha_0}\right) &= C_n\left(\tfrac{B+A}{B-A}\right) \\
&= \tfrac{1}{2}\left(\left(\tfrac{B+A}{B-A} + \sqrt{\tfrac{(B+A)^2}{(B-A)^2} - 1}\right)^n + \frac{1}{\left(\frac{B+A}{B-A} + \sqrt{\frac{(B+A)^2}{(B-A)^2} - 1}\right)^n}\right) \\
&= \tfrac{1}{2}\left(\left(\tfrac{B+A}{B-A} + \tfrac{\sqrt{4BA}}{B-A}\right)^n + \frac{1}{\left(\frac{B+A}{B-A} + \frac{\sqrt{4BA}}{B-A}\right)^n}\right) \\
&= \tfrac{1}{2}\left(\left(\tfrac{(\sqrt{B}+\sqrt{A})^2}{B-A}\right)^n + \frac{1}{\left(\frac{(\sqrt{B}+\sqrt{A})^2}{B-A}\right)^n}\right) = \tfrac{1}{2}\left(\left(\tfrac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}\right)^n + \frac{1}{\left(\frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}\right)^n}\right) \\
&= \tfrac{1}{2}(\tfrac{1}{\sigma^n} + \sigma^n) = \tfrac{1+\sigma^{2n}}{2\sigma^n}.
\end{aligned}$$

This equality with (3.14) yields that $\|u - u_\epsilon\|_H \le \epsilon$. $\square$

4. **Conjugate gradient method**

In this section, we verify how to use the frames for solving an operator equation with given boundary conditions by applying the conjugate gradient method. Moreover, by using the bounds of the frame, we present a stop criteria.

For Chebyshev method to be effective, a knowledge of an interval $[a, b]$ enclosing the spectrum of L is required. If this interval is too crude, the process loses efficiency. An important advantage of conjugate gradient method is that no a priori information about the location of the spectrum is required. Also in contrast to Chebyshev method, the conjugate gradient method is adaptive. The hidden polynomials Q_n in (3.7) depend nonlinearly on u and arise from a minimization problem.

Suppose that S is the frame operator of the frame $(\psi_\lambda)_{\lambda\in\Lambda}$. Since LSL is positive definite then we can define the LSL norm for the space H by

$$\|f\|_{LSL} = \langle f, LSLf\rangle^{\frac{1}{2}}, \qquad \forall f \in H,$$

corresponding to the inner product

$$\langle f, g\rangle_{LSL} = \langle f, LSLg\rangle, \quad \forall f, g \in H.$$

In this case if u is the solution of the equation (1.1) then by (2.2) we have

$$\|u\|^2_{LSL} = \langle u, LSLu\rangle = \langle Lu, SLu\rangle = \langle f, Sf\rangle \leq B\|f\|^2_H. \tag{4.1}$$

Considering the problem (1.1), define $v_{-1} = 0$, $v_0 = LSf$ and

$$v_{n+1} = LSLv_n - \frac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle} v_n - \frac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1}. \tag{4.2}$$

Assume that u is the solution of the problem (1.1), then the following lemma holds.

Lemma 4.1. *Let $H_n = \operatorname{span}\{(LSL)^j u : \ j = 1, 2, 3, \ldots, n\}$, then*

$$\{v_0, v_1, \ldots, v_{n-1}\} \subseteq H_n.$$

Proof. We verify this claim by induction .Clearly it is true for $n = 1$. Now assume that, it is true for all $k \leq n$. For $n + 1$ we have

$$v_n = LSLv_{n-1} - \frac{\langle LSLv_{n-1}, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1} - \frac{\langle LSLv_{n-1}, LSLv_{n-2}\rangle}{\langle v_{n-2}, LSLv_{n-2}\rangle} v_{n-2} \subseteq H_{n+1}$$

as we desired. □

Lemma 4.2. *The system $\{v_0, v_1, \ldots, v_{n-1}\}$ forms an orthogonal basis for H_n with respect to the LSL inner product*

$$\langle f, g\rangle_{LSL} = \langle f, LSLg\rangle \qquad \forall f, g \in H.$$

Proof. Since dim $H_n \leq n$ and span$\{v_0, v_1, \dots, v_{n-1}\}$ is a subspace of H_n, then it is enough to show that $\{v_0, v_1, \dots, v_{n-1}\}$ is an orthogonal system. Since, $v_{-1} = 0$ and $v_0 = LSf$, then the claim is obvious for $n = 1$. Also, for $n = 2$, we have

$$v_1 = LSLv_0 - \frac{\langle LSLv_0, LSLv_0\rangle}{\langle v_0, LSLv_0\rangle} v_0,$$

and so,

$$\langle v_1, v_0\rangle_{LSL} = \langle v_1, LSLv_0\rangle = \langle LSLv_0, LSLv_0\rangle - \frac{\langle LSLv_0, LSLv_0\rangle}{\langle v_0, LSLv_0\rangle}\langle v_0, LSLv_0\rangle = 0.$$

For $n > 2$, arguing by induction on j, we assume that $\langle v_n, LSLv_j\rangle = 0$ for $j = 1, 2, \dots, n-1$ and $\{v_0, v_1, \dots, v_n\}$ is an LSL orthogonal basis for H_{n+1}. Then we have to show that $\langle v_{n-1}, LSLv_j\rangle = 0$ for $j = 0, 1, 2, \dots, n$. For $j = n$ we have

$$\begin{aligned}
\langle v_{n+1}, LSLv_n\rangle &= \left\langle LSLv_n - \tfrac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle} v_n - \tfrac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle} v_{n-1}, LSLv_n\right\rangle \\
&= \langle LSLv_n, LSLv_n\rangle - \tfrac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle}\langle v_n, LSLv_n\rangle \\
&- \tfrac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle}\langle v_{n-1}, LSLv_n\rangle = 0.
\end{aligned}$$

A similar argument holds for $j = n-1$. Now, for $j < n-1$, we observe that $LSLv_j \in LSL(H_{n-1}) \subset H_n$. On the other hand, the induction hypothesis implies that $\{v_0, \dots, v_{n-1}\}$ is a basis for H_n, hence,

$$LSLv_j = \sum_{i=0}^{n-1} c_i v_i.$$

Thus,

$$\langle v_{n+1}, LSLv_j\rangle = \langle LSLv_n - \alpha v_n - \beta v_{n-1}, LSLv_j\rangle = \langle LSLv_n, LSLv_j\rangle - \alpha\langle v_n, LSLv_j\rangle$$

$$-\beta\langle v_{n-1}, LSLv_j\rangle = \langle LSLv_n, \sum_{i=0}^{n-1} c_i v_i\rangle = \sum_{i=0}^{n-1} c_i\langle LSLv_n, v_i\rangle = \sum_{i=0}^{n-1} c_i\langle v_n, LSLv_i\rangle = 0,$$

for every $j < n-1$. □

Now, using the above argument, we can design the following algorithm based on the conjugate gradient method in order to obtain an approximated solution for the problem (1.1). Let $(\psi_\lambda)_{\lambda\in\Lambda}$ be a frame for a Hilbert space H with frame operator S, and let A, B be the frame bounds of the frame $(L\psi_\lambda)_{\lambda\in\Lambda}$. The following algorithm solves the operator equation $Lu = f$ with given boundary conditions on the frame space by applying the conjugate gradient method. The tolerance value ϵ is chosen such that $\|u - \bar{u}\| \leq \epsilon$ in which $\bar{u}$ is the approximated value of u.

Algorithm 2 $[L, f, \epsilon, A, B] \to u_\epsilon$

(i) Put $h_0 = 0, \quad v_{-1} = 0, \quad r_0 = v_0 = LSf, \quad n = 0$

(ii) While $\frac{2\sigma^n}{1+\sigma^{2n}}\|f\|_H\sqrt{B} > \epsilon$ do

(1) $\lambda_n = \frac{\langle r_n, v_n\rangle}{\langle v_n, LSLv_n\rangle}$

(2) $h_{n+1} = h_n + \lambda_n v_n$

(3) $r_{n+1} = r_n - \lambda_n LSLv_n$

(4) $v_{n+1} = LSLv_n - \frac{\langle LSLv_n, LSLv_n\rangle}{\langle v_n, LSLv_n\rangle}v_n - \frac{\langle LSLv_n, LSLv_{n-1}\rangle}{\langle v_{n-1}, LSLv_{n-1}\rangle}v_{n-1}$

(5) $n = n+1$

end do

(iii)$u_\epsilon = h_{n-1}$.

Theorem 4.3. *The approximated solution h_n in Algorithm 2 is the orthogonal projection of the solution u of the problem (1.1) onto H_n. That is*

$$\|u - h_n\|_{LSL} \leq \|u - g\|_{LSL} \quad \forall g \in H_n.$$

Proof. It is enough to show that $\langle u - h_n, h_n\rangle_{LSL} = 0$. To do this, we note that by step (2) in Algorithm 2, $h_n = \sum_{j=0}^{n-1}\lambda_j v_j \in H_n$. Then, by Lemma 4.2,

$$\langle h_n, v_n\rangle_{LSL} = \langle \sum_{j=0}^{n-1}\lambda_j v_j, v_n\rangle_{LSL} = 0. \tag{4.3}$$

Also by step (3) in Algorithm 2, we have

$$r_n = r_{n-1} - \lambda_{n-1}LSLv_{n-1} = \cdots = r_0 - \sum_{j=0}^{n-1}\lambda_j LSLv_j$$

$$= r_0 - LSL(\sum_{j=0}^{n-1}\lambda_j v_j) = LSLu - LSLh_n = LSL(u - h_n).$$

So, the step (1) in Algorithm 2 implies

$$\lambda_n = \frac{\langle r_n, v_n\rangle_{LSL}}{\langle v_n, LSLv_n\rangle} = \frac{\langle u - h_n, v_n\rangle_{LSL}}{\langle v_n, v_n\rangle_{LSL}}$$

and

$$\begin{aligned}
\langle u - h_n, h_n\rangle_{LSL} &= \langle u - \textstyle\sum_{j=0}^{n-1}\lambda_j v_j, \sum_{j=0}^{n-1}\lambda_j v_j\rangle_{LSL} \\
&= \textstyle\sum_{j=0}^{n-1}\overline{\lambda_j}\left(\langle u, v_j\rangle_{LSL} - \lambda_j\langle v_j, v_j\rangle_{LSL}\right) \\
&= \textstyle\sum_{j=0}^{n-1}\overline{\lambda_j}\left(\langle u, v_j\rangle_{LSL} - \frac{\langle u-h_j, v_j\rangle_{LSL}}{\langle v_j, v_j\rangle_{LSL}}\langle v_j, v_j\rangle_{LSL}\right) \\
&= \textstyle\sum_{j=0}^{n-1}\overline{\lambda_j}\langle h_j, v_j\rangle_{LSL} = 0,
\end{aligned}$$

where the last equality results from (4.3). □

Theorem 4.4. *Given an arbitrary accuracy $\epsilon > 0$ in the Algorithm 2, the output u_ϵ satisfies $\|u - u_\epsilon\|_{LSL} \leq \epsilon$.*

Proof. The definition of h_n implies that $h_n = q_{n-1}(LSL)LSLu$, where $q_{n-1}(x)$ is a polynomial of degree $n-1$. Then

$$u - h_n = (I - q_{n-1}(LSL)LSL)u = \Phi_n(I - LSL)u,$$

where $\Phi_n(x) = 1-(1-x)q_{n-1}(1-x)$ is a polynomial of degree n with $\Phi_n(1) = 1$. Thus

$$\|u - h_n\|_{LSL} = \|\Phi_n(I - LSL)u\|_{LSL} \leq \|P_n(I - LSL)u\|_{LSL}$$

for all polynomials P_n of degree n with $P_n(1) = 1$. Using Lemma 3.3, with $\alpha_0 = \frac{B-A}{B+A}$, $a = -\alpha_0$ and $b = \alpha_0$, we can minimize this error. In fact

$$\begin{aligned}
\|u - h_n\|_{LSL} &\leq \|P_n(I - LSL)u\|_{LSL} = \|(LSL)^{\frac{1}{2}} P_n(I - LSL)(LSL)^{\frac{-1}{2}} (LSL)^{\frac{1}{2}} u\|_H \\
&\leq \|(LSL)^{\frac{1}{2}} P_n(I - LSL)(LSL)^{\frac{-1}{2}}\|_{H\to H} \|(LSL)^{\frac{1}{2}} u\|_H \\
&= \|P_n(I - LSL)\|_{H\to H} \|u\|_{LSL} \\
&\leq \max_{-\alpha_0 \leq x \leq \alpha_0} |P_n(x)| \|u\|_{LSL},
\end{aligned}$$

and by Lemma 3.3, $P_n(x) = \frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})}$ minimizes this error. Specially, the maximum value of this error is

$$\max_{-\alpha_0 \leq x \leq \alpha_0} \left| \frac{C_n(\frac{x}{\alpha_0})}{C_n(\frac{1}{\alpha_0})} \right| = \frac{1}{C_n(\frac{1}{\alpha_0})} = \frac{2\sigma^n}{1 + \sigma^{2n}},$$

where $\sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$. Therefore, by (4.1) we conclude that

$$\|u - h_n\|_{LSL} \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \|u\|_{LSL} = \frac{2\sigma^n}{1 + \sigma^{2n}} \sqrt{B} \|f\|_H$$

that is, by step (ii) in Algorithm 2, $\|u - u_\epsilon\|_{LSL} \leq \epsilon$ as we desired. □

5. Numerical experiments

In this section, we present two examples to confirm the theoretical results given in the previous sections.

Example 5.1. Consider the boundary value problem

$$\begin{cases} -u'' = f & \text{in } \Omega = (0, 1), \\ u(0) = u(1) = 0, & \end{cases}$$

on the space $H = \text{span}\{x^i(1-x)^i : \ i = 1, 1, 2, 2, 3, 3, \ldots, 20, 20\}$. The functions $x^i(1-x)^i$ constitute a frame for H with lower and upper bounds 1 and 2, respectively. The function f is chosen such that $u(x) = 4x^3(1-x)^3 - 3x^4(1-x)^4$ is the exact solution. The value σ is derived $\frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.1716$ that enables the algorithm to converge at limited iterations. Table 1 shows the error $\|u - \bar{u}\|_{L^2([0,1])}$, where $\bar{u}$ denotes the approximated solution given by conjugate gradient method. As is seen, after 58 iterations in 65 seconds, the proposed method would be converged.

Table 1. L2-norm of error between the exact and approximated solutions

n	2	15	30	58	CPU(sec.)
$\|u-\bar{u}\|_{L^2([0,1])}$	1.39	0.87	0.017	0.001	65

Example 5.2. Consider the boundary value problem

$$\begin{cases} -u'' + 2u = f & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

on the space $H = \operatorname{span}\{\sin(i\pi x) : \quad i = 1,1,1,2,2,2,3,4,5,\ldots,30\}$. The functions $\sin(i\pi x)$ constitute a frame for H with lower and upper bounds 1 and 3, respectively. The function f is chosen such that $u(x) = 3\sin(2\pi x) - \sin(8\pi x)$ is the exact solution. The value σ is derived $\frac{\sqrt{3}-1}{\sqrt{3}+1} \approx 0.2679$ that enables the algorithm to converge at limited iterations. Table 2 shows the error $\|u-\bar{u}\|_{L^2([0,1])}$, where $\bar{u}$ denotes the approximated solution given by conjugate gradient method. As is seen, after 96 iterations in 137 seconds, the proposed method would be converged.

Table 2. L2-norm of error between the exact and approximated solutions

n	2	20	40	96	CPU(sec.)
$\|u-\bar{u}\|_{L^2([0,1])}$	1.28	0.967	0.021	0.001	137

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