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SOME TOPOLOGIES ON THE SPACE OF QUASI-MULTIPLIERS

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Abstract. Assume that A is a Banach algebra. We define the $\beta-$topology and the $\gamma-$topology on the space $QM_{el}(A^*)$ of all bounded extended left quasi-multipliers of A^*. We establish further properties of $(QM_{el}(A^*),\gamma)$ when A is a C^*-algebra. In particular, we characterize the $\gamma-$dual of $QM_{el}(A^*)$ and prove that $(QM_{el}(A^*),\gamma)^*$, under the topology of bounded convergence, is isomorphic to A^{***}.

Keywords: Quasi-multiplier, multiplier, Banach algebra, Arens regularity, strict topology.
MSC(2010): Primary: 47B48; Secondary: 46H25.

1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [3] for C^*-algebras. McKennon [13] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

Let $QM(A)$ denote the set of all bounded quasi-multipliers on A. It is showed in [13] that $QM(A)$ is a Banach space for the norm

$$\|m\| = \sup\{\|m(a, b)\|; a, b \in A, \|a\| = \|b\| = 1\}.$$

For some classical Banach algebras, the Banach space of quasi-multipliers may be identified with some other known spaces or algebras. For instance, by [13, Corollary of Theorem 22], one can identify $QM(L^1(G))$, where G is a locally compact Hausdorff group, with the measure algebra $M(G)$.

In [1] we extended the notion of quasi-multipliers to the dual of a Banach algebra A whose second dual has a mixed identity. We considered algebras

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satisfying a weaker condition than Arens regularity. Among others we proved that for an Arens regular Banach algebra A with a b.a.i., $QM_r(A^*)$ (see the definition below) is isometrically isomorphic to A^{**}. We also proved some results concerning Arens regularity of the Banach algebra $QM_r(A^*)$ of all bilinear and bounded right quasi-multipliers of A^*. In this paper, we define extended left (right) quasi-multipliers on the dual of a Banach algebra. We establish some properties of $QM_{el}(A^*)$ of all bounded extended left quasi-multipliers of A^*. In particular, we characterize the $\gamma-$dual of $QM_{el}(A^*)$ and prove that $(QM_{el}(A^*), \gamma)^*$, under the topology of bounded convergence, is isomorphic to A^{***}.

Before we state our main results the basic notation is introduced. We mainly adopt the notation from the monograph [6]. The reader is referred to this book for some results used in this paper, as well.

For a Banach space X, let X^* be its topological dual. The pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. We always consider X naturally embedded into X^{**} through the mapping π, which is given by $\langle \pi(x), \xi \rangle = \langle \xi, x \rangle$ $(x \in X,\ \xi \in X^*)$. Let A be a Banach algebra. It is well known that on the second dual A^{**} there are two algebra multiplications called the first and the second Arens product, respectively. Since in the paper we use mainly the first Arens product, we recall its definition. Let $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$ be arbitrary. Then one defines $\xi \cdot a$ and $G \cdot \xi$ by $\langle \xi \cdot a, b \rangle = \langle \xi, ab \rangle$ and $\langle G \cdot \xi, b \rangle = \langle G, \xi \cdot b \rangle$, where $b \in A$ is arbitrary. Now, the first Arens product of F and G is an element $F \triangleleft G$ in A^{**} which is given by $\langle F \triangleleft G, \xi \rangle = \langle F, G \cdot \xi \rangle$, where $\xi \in A^*$ is arbitrary. The second Arens product, which we denote by $\triangleright$, is defined in a similar way.

The space A^{**} equipped with the first (or second) Arens product is a Banach algebra. When A^{**} is endowed with $\triangleleft$ we denote the algebra by $A^{**}_{\triangleleft}$. Similarly, $A^{**}_{\triangleright}$ is the algebra obtained with A^{**} endowed with the second Arens product $\triangleright$. Since $F \triangleleft a = F \triangleright a$ and $a \triangleleft F = a \triangleright F$ hold for all $a \in A$ and $F \in A^{**}$ the algebra A is a subalgebra of $A^{**}_{\triangleleft}$ and $A^{**}_{\triangleright}$. It is said that A is Arens regular if the equality $F \triangleleft G = F \triangleright G$ holds for all $F,\ G \in A^{**}$, i.e., when $A^{**}_{\triangleleft} = A^{**}_{\triangleright}$. For example, every C^*-algebra is Arens regular, see [5].

An element E in the second dual A^{**} is said to be a mixed identity if it is a right identity for the first and a left identity for the second Arens product. Note that A^{**} has a mixed identity if and only if A has a b.a.i. By [6, Proposition 2.6.21], an element $E \in A^{**}$ is a mixed identity if and only if $E \cdot \xi = \xi = \xi \cdot E$, for every $\xi \in A^*$. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors.

2. Main results

Let A be a complex Banach algebra. Note that A^* is a Banach $A^{**}_{\triangleleft}$-A-bimodule and a Banach A-$A^{**}_{\triangleright}$-bimodule. But in general it is not a Banach $A^{**}_{\triangleleft}$-$A^{**}_{\triangleright}$-bimodule.

Definition 2.1. Let A be a complex Banach algebra. Consider A^* as a Banach $A^{**}_{\triangleleft}$-A-bimodule. A bilinear map

$$m : A^{**} \times A^* \to A^*$$

is a left quasi-multiplier of A^* if

$$m(F \triangleleft G, \xi) = F \cdot m(G, \xi) \quad \text{and} \quad m(F, \xi \cdot a) = m(F, \xi) \cdot a \tag{2.1}$$

hold for all $a \in A$, $\xi \in A^*$ and $F, G \in A^{**}$.

Consider A^* as a Banach A-$A^{**}_{\triangleright}$-bimodule. A bilinear map

$$m : A^* \times A^{**} \to A^*$$

is a right quasi-multiplier of A^* if

$$m(\xi, F \triangleright G) = m(\xi, F) \cdot G \quad \text{and} \quad m(a \cdot \xi, F) = a \cdot m(\xi, F) \tag{2.2}$$

hold for all $a \in A$, $\xi \in A^*$ and $F, G \in A^{**}$.

Let $QM_r(A^*)$ (respectively, $QM_l(A^*)$) be the set of all bounded right (respectively, left) quasi-multipliers of A^*.

Although in our investigation we do not assume Arens regularity of A, we usually have to assume that A satisfies the following weaker condition.

Definition 2.2. A Banach algebra A is weakly Arens regular if

$$(F \cdot \xi) \cdot G = F \cdot (\xi \cdot G) \qquad (F,\ G \in A^{**},\ \xi \in A^*).$$

Of course, every Arens regular Banach algebra is weakly Arens regular. However, the class of weakly Arens regular Banach algebras is larger. It contains, for instance, every Banach algebra A which is an ideal in its second dual. Namely, in this case, we have

$$\begin{aligned} \langle (F \cdot \xi) \cdot G, a \rangle &= \langle \pi(a), (F \cdot \xi) \cdot G \rangle = \langle G \triangleright \pi(a), F \cdot \xi \rangle = \langle (G \triangleright \pi(a)) \triangleleft F, \xi \rangle \\ &= \langle G \triangleright (\pi(a) \triangleleft F), \xi \rangle = \langle \pi(a) \triangleleft F, \xi \cdot G \rangle = \langle F \cdot (\xi \cdot G), a \rangle \qquad (a \in A), \end{aligned}$$

for arbitrary $F,\ G \in A^{**}$ and $\xi \in A^*$. Note that a unital Banach algebra is weakly Arens regular if and only if it is Arens regular.

It is not hard to see that A^* is a Banach $A^{**}_{\triangleleft}$-$A^{**}_{\triangleright}$-bimodule if and only if A is weakly Arens regular.

Definition 2.3. Let A be a weakly Arens regular Banach algebra. Consider A^* as a Banach $A^{**}_{\triangleleft}$-$A^{**}_{\triangleright}$-bimodule. A bilinear map

$$m : A^{**} \times A^* \to A^*$$

is an extended left quasi-multiplier of A^* if

$$m(F \triangleleft G, \xi) = F \cdot m(G, \xi) \quad \text{and} \quad m(F, \xi \cdot G) = m(F, \xi) \cdot G \tag{2.3}$$

hold for all $\xi \in A^*$ and $F, G \in A^{**}$.

Similarly, a bilinear map

$$m : A^* \times A^{**} \to A^*$$

is an extended right quasi-multiplier of A^* if

$$m(\xi, F \triangleright G) = m(\xi, F) \cdot G \quad \text{and} \quad m(G \cdot \xi, F) = G \cdot m(\xi, F) \tag{2.4}$$

hold for all $\xi \in A^*$ and $F, G \in A^{**}$.

Let $QM_{er}(A^*)$ (respectively, $QM_{el}(A^*)$) denote the set of all bounded extended right (respectively, left) quasi-multipliers of A^*.

Proposition 2.4. *If A is a weakly Arens regular Banach algebra, then a map $m : A^{**} \times A^* \to A^*$ is an extended left quasi-multiplier of A^* if and only if it is a left quasi-multiplier of A^*.*

Proof. It is obvious that every extended left quasi-multiplier is a left quasi-multiplier. For the converse observe that for all $G \in A^{**}$ and $\xi \in A^*$ the mapping $G \to \xi \cdot G$ is weak*-weak* continuous. Indeed, assume that a net $\{b_\alpha\}_{\alpha \in I} \subseteq A$ converges to G in the weak* topology. Then for each $x \in A$,

$$\begin{aligned} \lim_\alpha \langle \xi \cdot b_\alpha, x \rangle &= \lim_\alpha \langle \xi, b_\alpha \cdot x \rangle = \langle \xi, \lim_\alpha b_\alpha x \rangle = \langle \xi, G \cdot x \rangle \\ &= \langle G \cdot x, \xi \rangle = \langle x, \xi \cdot G \rangle = \langle \xi \cdot G, x \rangle. \end{aligned}$$

It follows that for each $F \in A^{**}$ we have

$$\begin{aligned} m(F, \xi \cdot G) &= m(F, \lim_\alpha (\xi \cdot b_\alpha)) = \lim_\alpha m(F, \xi \cdot b_\alpha) = \lim_\alpha (m(F, \xi) \cdot b_\alpha) \\ &= m(F, \xi) \cdot \lim_\alpha b_\alpha = m(F, \xi) \cdot G, \end{aligned}$$

which means that m is an extended left quasi-multiplier of A^*. □

A simple computation shows that if A is a weakly Arens regular Banach algebra, then the products

$$H * m(G, \xi) = m(G, H \cdot \xi), \qquad m * H(G, \xi) = m(G \triangleleft H, \xi)$$

$$(m \in QM_l(A^*),\ H \in A^{**},\ \xi \in A^*,\ G \in A^{**})$$

make $QM_{el}(A^*)$ a two-sided $A^{**}_{\triangleleft}$-bimodule. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\xi, F)||; \quad \xi \in A^*,\ F \in A^{**},\ ||\xi|| \leq 1,\ ||F|| \leq 1\}.$$

Beside the norm topology, there are two other useful topologies on $QM_{el}(A^*)$.

Definition 2.5. Let A be a weakly Arens regular Banach algebra. The strict topology β on $QM_{el}(A^*)$ is defined as the locally convex topology which is given by the seminorms

$$m \to \|m * F\| \qquad (F \in A^{**},\ m \in QM_{el}(A^*)).$$

The quasi-strict topology γ on $QM_{el}(A^*)$ is defined as the locally convex topology which is given by the seminorms

$$m \to \|m(F, \xi)\| \qquad (\xi \in A^*,\ F \in A^{**},\ m \in QM_{el}(A^*)).$$

Let τ denote the topology on $QM_{el}(A^*)$ generated by the norm.

Proposition 2.6. *If A is a weakly Arens regular Banach algebra such that $A^{**}_{\triangleleft} = (A^{**}_{\triangleleft})^2$, then $\gamma \subseteq \beta \subseteq \tau$.*

Proof. Let a net $\{m_\alpha\}_{\alpha \in I} \subseteq QM_{el}(A^*)$ converge to $m \in QM_{el}(A^*)$ in the topology β and let $\xi \in A^*$ be arbitrary. Since $A^{**}_{\triangleleft} = (A^{**}_{\triangleleft})^2$, for arbitrary $F \in A^{**}$, there exist $G, H \in A^{**}$ such that $F = G \triangleleft H$. It follows, by the definition of the topology β, that $||m_\alpha * H - m * H|| \to 0$. Thus

$$\begin{aligned} ||m_\alpha(F, \xi) - m(F, \xi)|| &= ||m_\alpha(G \triangleleft H, \xi) - m(G \triangleleft H, \xi)|| \\ &= ||(m_\alpha * H)(G, \xi) - (m * H)(G, \xi)|| \to 0, \end{aligned}$$

which means that $\{m_\alpha\}_{\alpha \in I}$ converges to m in the topology γ. It is obvious that $\beta \subseteq \tau$. □

Corollary 2.7. *If A is a weakly Arens regular Banach algebra such that A^{**} has a mixed identity, then $\gamma \subseteq \beta \subseteq \tau$.*

Proof. Since A^{**} has a mixed identity we have $A^{**}_{\triangleleft} = (A^{**}_{\triangleleft})^2$. □

Recall that a map $T : A^* \to A^*$ is a left multiplier of A^* if

$$T(\xi \cdot F) = T(\xi) \cdot F,$$

for all $\xi \in A^*, F \in A^{**}$. With $M_l(A^*)$ we denote the space of all bounded linear left multipliers of A^*.

Theorem 2.8. *Let A be a weakly Arens regular Banach algebra. Then*

(i) *the space $(QM_{el}(A^*), \gamma)$ is complete;*
(ii) *if A^{**} has a mixed identity of norm one, then $(QM_{el}(A^*), \beta)$ is complete.*

Proof. (i) Let $\{m_\alpha\}_{\alpha \in I}$ be a γ-Cauchy net in $QM_{el}(A^*)$. Then, for arbitrary $\xi \in A^*$ and $F \in A^{**}$, we have a Cauchy net $\{m_\alpha(F, \xi)\}_{\alpha \in I}$ in the norm topology of A^*. Let $m(F, \xi) = \lim_\alpha m_\alpha(F, \xi)$. It is obvious that in this way we have defined a bilinear mapping m on $A^* \times A^{**}$ satisfying condition (2.3). Also by uniform boundedness principle ([11, p. 172] and [7, p. 489]), m is bounded and therefore $m \in QM_{el}(A^*)$.

(ii) Let $\{m_\alpha\}_{\alpha \in I}$ be a $\beta-$Cauchy net in $QM_{el}(A^*)$. For each $F \in A^{**}$, the mapping $T_F^\alpha : A^* \to A^*$ which is given by $T_F^\alpha(\xi) = m_\alpha(F, \xi)$ defines elements in

$M_l(A^*)$. Define a mapping $\rho : M_l(A^*) \to QM_{el}(A^*)$ by $\rho_T(F,\xi) = F \cdot T\xi$. It is easy to show that $\rho_{T_F^\alpha} = m_\alpha * F$. It follows from the definition of the $\beta-$topology that $\{\rho_{T_F^\alpha}\}_{\alpha\in I}$ is a Cauchy net in the norm of $QM_{el}(A^*)$. By [1, Theorem 2.3], ρ is an isometry and therefore $\{T_F^\alpha\}$ is a Cauchy net in the norm of $M_l(A^*)$. By the completeness of $M_l(A^*)$, there exists $T_F \in M_l(A^*)$ such that $||T_F^\alpha - T_F|| \to 0$. Since $\gamma \subseteq \beta$ the net $\{m_\alpha\}_{\alpha\in I}$ is a Cauchy net in γ topology. By the first part of this theorem, $(QM_{el}(A^*), \gamma)$ is complete. Hence there exists $m \in QM_{el}(A^*)$ such that

$$\lim_\alpha m_\alpha(F,\xi) = m(F,\xi) \quad \text{for all} \quad \xi \in A^* \quad \text{and} \quad F \in A^{**}.$$

For each $G \in A^{**}$,

$$\begin{aligned}\rho_{T_F}(G,\xi) &= \lim_\alpha \rho_{T_F^\alpha}(G,\xi) = \lim_\alpha (m_\alpha * F)(G,\xi) = \lim_\alpha m_\alpha(G \triangleleft F, \xi) \\ &= m(G \triangleleft F,\xi) = (m * F)(G,\xi).\end{aligned}$$

It follows that

$$||m_\alpha * F - m * F|| = ||\rho_{T_F^\alpha} - \rho_{T_F}|| = ||T_F^\alpha - T_F|| \to 0,$$

which implies that m is the $\beta-$limit of the net $\{m_\alpha\}_{\alpha\in I}$, i.e., $QM_{el}(A^*)$ is complete in β topology. □

Theorem 2.9. *Let A be a weakly Arens regular Banach algebra.*

(i) $(QM_{el}(A^*), \tau)$ *and* $(QM_{el}(A^*), \gamma)$ *have the same bounded sets.*

(ii) *If A^{**} has a mixed identity, then* $(QM_{el}(A^*), \gamma)$, $(QM_{el}(A^*), \tau)$ *and* $(QM_{el}(A^*), \beta)$ *have the same bounded sets.*

Proof. (i) Since $\gamma \subseteq \tau$, each $\tau-$bounded set is $\gamma-$bounded. On the other hand, let H be a $\gamma-$bounded subset of $QM_{el}(A^*)$. Then for each $\xi \in A^*$ and $F \in A^{**}$, there exists a real number $r = r(F,\xi) > 0$ such that

$$||m(F,\xi)|| \leq r \tag{2.5}$$

for all $m \in H$. For each $\xi \in A^*$ and $m \in H$, define $M_\xi : A^{**} \to A^*$ by

$$M_\xi(F) := m(F,\xi) \ \ (F \in A^{**}).$$

Consider the family $\mathcal{H} = \{M_\xi : m \in H\}$. By (2.5), for each $G \in A^{**}$,

$$||M_\xi(G)|| = ||m(G,\xi)|| \leq r(G,\xi) \ \ (m \in H).$$

Hence, $\mathcal{H}$ is pointwise bounded. By the principle of uniform boundedness, there exists a constant $c = c(F) > 0$ such that

$$||M_f|| \leq c \ \ (m \in H). \tag{2.6}$$

Consider the family $P = \{p_m : m \in H\}$ of semi-norms on A^* defined by

$$p_m(\xi) = ||M_\xi|| = \sup_{||\xi||\leq 1} ||M_\xi(F)|| = \sup_{||\xi||\leq 1} ||m(F,\xi)|| \ \ (\xi \in A^*).$$

In the following we prove that p_m is continuous on A^* for each m. Let $\{\xi_n\} \subseteq A^*$ be a sequence in A^* converging to $\xi_0 \in A^*$, then

$$|p_m(\xi_n) - p_m(\xi_0)| \leq p_m(\xi_n - \xi_0) = \sup_{||F|| \leq 1} ||M_{\xi_n - \xi_0}(F)|| = \sup_{||F|| \leq 1} ||m(F, \xi_n - \xi_0)|| \to 0,$$

which implies that p_m is continuous. It follows from (2.6) that the family P is pointwise bounded. Hence, by [8, p. 142], there exist a closed $B(\xi_0, r) = \{\xi \in A^* : ||\xi - \xi_0|| \leq r\}$ and a constant K_0 such that $p_m(\xi) \leq K_0$ for all $f \in B(\xi_0, r)$. For $\xi \in A^*$ with $||\xi|| \leq 1$, we have

$$p_m(\xi) = \frac{p_m(r\xi + \xi_0 - \xi_0)}{r} \leq \frac{1}{r}(p_m(r\xi + \xi_0) + p_m(\xi_0)) \leq \frac{2K_0}{r}.$$

This implies that

$$||m|| = \sup_{||\xi|| \leq 1, ||F|| \leq 1} ||m(F, \xi)|| = \sup_{||\xi|| \leq 1} p_m(\xi) \leq \frac{2K_0}{r}$$

and so the set H is $\tau-$bounded, as required.
(ii) Since, $\gamma \subseteq \beta \subseteq \tau$, by (i), $(QM_{el}(A^*), \gamma)$, $(QM_{el}(A^*), \tau)$ and $(QM_{el}(A^*), \beta)$ have the same bounded sets. □

For the remainder of this section we assume that A is a C^*-algebra. We characterize the $\gamma-$dual of $QM_{el}(A^*)$.

Theorem 2.10. *Let A be a C^*-algebra. Then*

$$(QM_{el}(A^*), \gamma)^* = \{f \cdot F : \ f \in (QM_{el}(A^*), \tau)^*, F \in A^{**}\},$$

where

$$(f \cdot F)(m) := \langle f, m * F \rangle \qquad (m \in QM_{el}(A^*)).$$

Proof. Let $f \in (QM_{el}(A^*), \tau)^*$. It is obvious that for each $F \in A^{**}$ the mapping $f \cdot F$ is a linear functional. Let us prove that $f \cdot F$ is $\gamma-$continuous. Assume that $m \in QM_{el}(A^*)$ is arbitrary. Since f is $\tau-$continuous, given $\epsilon > 0$, there is $\delta > 0$ such that $|\langle f, m \rangle| < \epsilon$ whenever $||m|| < \delta$. Consider the $\gamma-$neighborhood of 0 in $QM_{el}(A^*)$ given by

$$N(F, \delta) = \{m \in QM_{el}(A^*) \ : \ ||m * F|| < \delta\}.$$

Let $m \in N(F, \delta)$. Now,

$$|(f \cdot F)(m)| = |\langle f, m * F \rangle| < \epsilon.$$

Hence $f \cdot F$ is $\gamma-$continuous.

Conversely, suppose that $g \in (QM_{el}(A^*), \gamma)^*$. Since $\gamma \subseteq \tau$ we have $g \in (QM_{el}(A^*), \tau)^*$. Every C^*-algebra A is (weakly) Arens regular and its second dual A^{**} is a unital von Neumann algebra, hence Arens regular, as well. By [1, Theorem (2.6)], $QM_{el}(A^*)$ is Arens regular and so $(QM_{el}(A^*), \tau)^*$ factors (see [14]). Also by [1, Theorem (2.5)], $QM_{el}(A^*)$ is isomorphic to A^{**}. Therefore there exist $f \in (QM_{el}(A^*), \tau)^*$ and $F \in A^{**}$ such that $g = f \cdot F$. □

For each $H \in A^{**}$, define $\varphi(H) \in QM_{el}(A^*)$ by

$$[\varphi(H)](F, \xi) = (F \triangleleft H) \cdot \xi \qquad \text{for all } \xi \in A^*,\ F \in A^{**}.$$

Lemma 2.11. *If A is an Arens regular Banach algebra with a bounded approximate identity, then $\varphi : A^{**} \to QM_{el}(A^*)$ is an isomorphism.*

Proof. Let $m \in QM_{el}(A^*)$. In order to prove that φ is onto, we show that for all $F, H, G \in A^{**}$ one has

$$m^*(H \triangleleft F, G) = H \triangleleft m^*(F, G)$$

where $m^* : A^{**} \times A^{**} \to A^{**}$ is an extension of m. Let $\xi \in A^*$. Then

$$\begin{aligned}\langle m^*(H \triangleleft F, G), \xi\rangle &= \langle H \triangleleft F, m(G, \xi)\rangle = \langle F, m(G, \xi) \cdot H\rangle = \langle F, m(G, \xi \cdot H)\rangle \\ &= \langle m^*(F, G), \xi \cdot H\rangle = \langle H \triangleleft m^*(F, G), \xi\rangle.\end{aligned}$$

Let E be the mixed identity in A^{**} and suppose that $\xi \in A^*$, $F \in A^{**}$ and $x \in A$ are arbitrary. Then

$$\begin{aligned}\langle \varphi(m^*(E, E))(F, \xi), x\rangle &= \langle (F \triangleleft m^*(E, E)) \cdot \xi, x\rangle = \langle F \triangleleft m^*(E, E), \xi \cdot x\rangle \\ &= \langle m^*(F, E), \xi \cdot x\rangle = \langle F, m(E, \xi \cdot x)\rangle \\ &= \langle F, m(E, \xi) \cdot x\rangle = \langle x \triangleleft F, m(E, \xi)\rangle \\ &= \langle x, F \cdot m(E, \xi)\rangle = \langle x, m(F, \xi)\rangle.\end{aligned}$$

Now, let us prove that φ is one to one. Assume that $\varphi(H) = 0$. Then for each $\xi \in A^*$, one has

$$H \cdot \xi = (E \cdot H) \cdot \xi = 0.$$

Which implies that for each $x \in A$,

$$\langle H, \xi \cdot x\rangle = \langle H \cdot \xi, x\rangle = 0.$$

Since, A is Arens regular, A^* factors. Thus $H = 0$. □

Definition 2.12. Let A be a Banach algebra. The topology of bounded convergence u on $(QM_{el}(A^*), \gamma)^*$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$M(D, G) = \{f \in (QM_{el}(A^*), \gamma)^* : f(D) \subseteq G\},$$

where D is a $\gamma-$bounded subset of $(QM_{el}(A^*), \gamma)$ and G is a neighborhood of 0.

The topology ν on $(QM_{el}(A^*), \gamma)^*$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(D, G) = \{f \in (QM_{el}(A^*), \gamma)^* : f(D) \subseteq G\},$$

where D is a norm-bounded subset of $(QM_{el}(A^*), \gamma)$ and G is a neighborhood of 0.

Theorem 2.13. *Let A be a C^*-algebra. Then $((QM_{el}(A^*),\gamma)^*,u)$ is isomorphic to A^{***}.*

Proof. Since $\gamma\subseteq\tau$ we have $(QM_{el}(A^*),\gamma)^*\subseteq(QM_{el}(A^*),\tau)^*$. By Theorem 2.9, γ and τ have the same bounded sets in $QM_{el}(A^*)$, it follows that the topology u coincides with the norm topology ν on $(QM_{el}(A^*),\gamma)^*$. Therefore $((QM_{el}(A^*),\gamma)^*,u)$ is a normed subspace of $((QM_{el}(A^*),\tau)^*,\nu)$. We will show that $((QM_{el}(A^*),\gamma)^*,u)$ is isomorphic to the subspace $((\varphi(A^{**}),\tau)^*,\nu)$ of $((QM_{el}(A^*),\tau)^*,\nu)$. Consider the map ψ which maps each element $g\in((QM_{el}(A^*),\gamma)^*,u)$ onto its restriction to $\varphi(A^{**})$, that is, $g|\varphi(A^{**})$. Since $\gamma\subseteq\tau$, for each $g\in(QM_{el}(A^*),\gamma)^*$, the map $\psi(g)$ is $\tau-$continuous. It is clear that ψ is linear. Suppose that $\psi(g)=0$. Then $g(\varphi(H))=0$ for all $H\in A^{**}$. By Lemma 2.11, the mapping φ is onto. Hence, $g(m)=0$ for all $m\in QM_{el}(A^*)$ which means that ψ is one to one. Assume that $f\in(\varphi(A^{**}),\tau)^*$. It is easy to see that $\varphi(A^{**})$ is an Arens regular Banach algebra. Hence, by using the same arguments as those in the proof of Theorem 2.10, there exist $h\in(\varphi(A^{**}),\tau)^*$ and $F\in A^{**}$ such that $f=h\cdot F$. By Hahn-Banach theorem, h can be extended to an element $\bar{h}\in(QM_{el}(A^*),\tau)^*$. Then, by Theorem 2.10, the functional $\bar{h}\cdot F$ belongs to $(QM_{el}(A^*),\gamma)^*$. Also, for all $G\in A^{**}$, we have

$$\begin{aligned}\psi(\bar{h}\cdot F)(\varphi(G))&=(\bar{h}\cdot F)(\varphi(G))=\langle\bar{h},\varphi(G)*F\rangle=\langle\bar{h},\varphi(F\ G)\rangle\\&=\langle h,\varphi(F\triangleleft G)\rangle=\langle h,\varphi(G)*F\rangle=(h\cdot F)(\varphi(G))\\&=f(\varphi(G)).\end{aligned}$$

Therefore $\psi(\bar{h}\cdot F)=f$ and so ψ is onto. □

Example 2.14. Let H be a Hilbert space and let $A=K(H)$, the algebra of all compact operators on H. The dual of the space of compact operators is the space of trace-class operators, $C_1(H)$. The second dual of A is $B(H)$. Since $K(H)$ is a C^*-algebra we have $((QM_{el}(C_1(H)),\gamma)^*,u)\cong(B(H))^*$.

Example 2.15. Let $A=c_0(\mathbb{N})$, the space of all complex sequences which converge to 0. The dual of c_0 is l_1 and its second dual is l_∞. Since c_0 is a C^*-algebra, by Theorem 2.13, $((QM_{el}(l_1),\gamma)^*,u)\cong ba(\mathbb{N},2^{\mathbb{N}},\mu)$, the space of all finitely additive finite signed measure which are absolutely continuous with respect to the counting measure μ equipped with the total variation norm. Since the space l_∞ is isometrically isomorphic to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, one can identifies $((QM_{el}(l_1),\gamma)^*,u)$ also with the dual $C(\beta\mathbb{N})^*$.

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