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# INVERSE STURM–LIOUVILLE PROBLEMS USING THREE SPECTRA WITH FINITE NUMBER OF TRANSMISSIONS AND PARAMETER DEPENDENT CONDITIONS

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Abstract. In this manuscript, we study various uniqueness results for inverse spectral problems of Sturm–Liouville operators by using three spectra with a finite number of discontinuities at interior points which we impose the usual transmission conditions. We consider both the cases of classical Robin and eigenparameter dependent boundary conditions.
**Keywords:** Inverse Sturm–Liouville problem, eigenparameter dependent boundary conditions, internal discontinuities, three spectra, Weyl–Titchmarsh m-function.
**MSC(2010):** Primary: 34A55; Secondary: 34B24, 34B08.

## 1. **Introduction**

The aim of this paper is to investigate the inverse problem of Sturm–Liouville equations. In inverse spectral problems, the task is to find a coefficient in the equation using the spectral data. We discuss the uniqueness of spectral problem by developing the Gesztesy–Simon's result for inverse Sturm–Liouville problem using three spectra with a finite number of transmission conditions.

Gesztesy, Simon [9] and Pivovarchik [12, 13] proved if the three spectra are pairwise disjoint, then the potential $q$ can be uniquely determined by the three spectra of the problems defined on three intervals $[0, 1]$, $[0, d]$ and $[d, 1]$ for some $d \in (0, 1)$. Furthermore, Gesztesy and Simon [9] gave a counterexample to show that the pairwise disjoint conditions are necessary. Recently, in the other papers Drignei [3–5] proved a similar result in the case for the Sturm–Liouville problems with Dirichlet and Dirichlet–Robin boundary conditions. In [5], Drignei offered a numerical method for construction the potential function $q(x)$. More recently, Fu, Xu, and Wi [7, 8] generalized the Gesztesy, Simon [9] and Pivovarchik [12, 13] for discontinuous Sturm–Liouville and indefinite

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Sturm–Liouville problems. The purpose of the present paper is to show how to handle an arbitrary finite number of transmission conditions and to use the asymptotic formulas to prove several uniqueness results. In particular, we will introduce a Weyl–Titchmarsh $m$-function which uniquely determines the parameters of the problem. We also show that this Weyl–Titchmarsh function is a meromorphic Herglotz function which is uniquely determined by its poles and residues, as well as by its poles and zeros. This generalizes the results of [8, 9, 12] and [13] to the case of a finite number of transmission and eigenparameter dependent boundary conditions.

Sturm–Liouville problems with transmission conditions at interior points arise in a variety of applications in engineering and we refer to [2] for a nice discussion and further information. Here we only want to mention that they also appear in the description of delta interactions (which play an important role in quantum mechanics [1]) and of radially symmetric quantum trees (cf. the discussion in [14, Section 4] and the references therein). For general background on inverse Sturm–Liouville problems we refer (e.g.) to the monographs [6, 11, 16, 17].

## 2. The Hilbert space formulation and properties of the spectrum

In the first part of our paper, we consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y \tag{2.1}$$

subject to the Robin boundary conditions

$$U(y) := y'(0) + h\,y(0) = 0, \qquad V(y) := y'(\pi) + H\,y(\pi) = 0 \tag{2.2}$$

with transmission (discontinuous) conditions

$$\begin{aligned} U_i(y) &:= y(d_i + 0) - a_i y(d_i - 0) = 0, \\ V_i(y) &:= y'(d_i + 0) - b_i y'(d_i - 0) - c_i y(d_i - 0) = 0, \end{aligned} \tag{2.3}$$

where $q(x)$ is real–valued function in $L^1[0,\pi]$. We also assume that $h$, $H$ and $a_i$, $b_i$, $c_i$ $d_i$, $i = 1, 2, \ldots, m-1$ (with $m \geq 2$) are real numbers, satisfying $a_i b_i > 0$, $d_0 = 0 < d_1 < d_2 < \cdots < d_{m-1} < d_m = \pi$. For simplicity we use the notation $L = L(q(x); h; H; d_i)$, for the problems (2.1)–(2.3). Suppose $d = d_k$ is one of transmission point, for $1 \leq k \leq m-1$, $k$ is an integer fixed number and $c_k = 0$.

Let $L_1 = L(q_1(x); h; H_1; d_i)$ for $i = 1, 2, \cdots, k-1$ and $L_2 = L(q_2(x); h; H_2; d_i)$ for $i = k+1, k+2, \cdots, m-1$ be the following discontinuous Sturm-Liouville problems

$$\ell_1 y := -y'' + q_1\,y = \lambda y \tag{2.4}$$

subject to the Robin boundary conditions

$$y'(0) + h\,y(0) = 0, \qquad y'(d) + H_1\,y(d) = 0 \tag{2.5}$$

with transmission (discontinuous) conditions

$$
\begin{aligned}
&U_i(y) := y(d_i+0) - a_i y(d_i-0) = 0,\\
&V_i(y) := y'(d_i+0) - b_i y'(d_i-0) - c_i y(d_i-0) = 0,
\end{aligned} \tag{2.6}
$$

for $i = 1, 2, \cdots, k-1$ and

$$
\ell_2 y := -y'' + q_2\, y = \lambda y \tag{2.7}
$$

subject to the Robin boundary conditions

$$
y'(d) + H_2\, y(d) = 0, \qquad y'(\pi) + H\, y(\pi) = 0 \tag{2.8}
$$

with transmission (discontinuous) conditions

$$
\begin{aligned}
&U_i(y) := y(d_i+0) - a_i y(d_i-0) = 0,\\
&V_i(y) := y'(d_i+0) - b_i y'(d_i-0) - c_i y(d_i-0) = 0,
\end{aligned} \tag{2.9}
$$

for $i = k+1, k+2, \cdots, m-1$. Where $q_1 = q|_{[0,d)}$ and $q_2 = q|_{(d,\pi]}$. By using the jump conditions (2.3) we obtain $H_2 = \frac{b_k}{a_k} H_1$ for $H_1,\ H_2 \in (0,\, \infty)$.

To obtain a self-adjoint operator we introduce the following weight function

$$
w(x) = \begin{cases} 1, & 0 \le x < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < x < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < x \le \pi, \end{cases} \tag{2.10}
$$

$w_1(x) = w(x)|_{[0,d)}$, and $w_2(x) = w(x)|_{(d,\pi]}$. Now, our Hilbert spaces will be $\mathcal{H} := L_2((0,\pi); w)$, $\mathcal{H}_1 := L_2((0,d); w_1)$, and $\mathcal{H}_2 := L_2((d,\pi); w_2)$, and associated with the weighted inner products

$$
\langle f, g\rangle_{\mathcal{H}} := \int_0^\pi f\overline{g} w, \qquad \langle f, g\rangle_{\mathcal{H}_1} := \int_0^d f\overline{g} w_1, \tag{2.11}
$$

and

$$
\langle f, g\rangle_{\mathcal{H}_2} := \int_d^\pi f\overline{g} w_2. \tag{2.12}
$$

The corresponding norms will be denoted by $\|f\|_{\mathcal{H}} = \langle f, f\rangle_{\mathcal{H}}^{1/2}$, $\|f\|_{\mathcal{H}_1} = \langle f, f\rangle_{\mathcal{H}_1}^{1/2}$, and $\|f\|_{\mathcal{H}_2} = \langle f, f\rangle_{\mathcal{H}_2}^{1/2}$. In this Hilbert spaces we construct the operators

$$
A : \mathcal{H} \to \mathcal{H}, \quad A_1 : \mathcal{H}_1 \to \mathcal{H}_1 \quad \text{and} \quad A_2 : \mathcal{H}_2 \to \mathcal{H}_2 \tag{2.13}
$$

with domain

$$
\operatorname{dom}(A) = \left\{ f \in \mathcal{H} \;\middle|\; \begin{array}{c} f, f' \in AC\big(\cup_0^{m-1} (d_i, d_{i+1})\big), \\ \ell f \in L^2(0,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\}, \tag{2.14}
$$

(2.15) $$\operatorname{dom}(A_1) = \left\{ f \in \mathcal{H}_1 \;\middle|\; \begin{array}{c} f, f' \in AC\big( \cup_0^{k-1} (d_i, d_{i+1})\big), \\ \ell_1 f \in L^2(0,d),\ U_i(f) = V_i(f) = 0 \end{array} \right\},$$

and

(2.16) $$\operatorname{dom}(A_2) = \left\{ f \in \mathcal{H}_2 \;\middle|\; \begin{array}{c} f, f' \in AC\big( \cup_k^{m-1} (d_i, d_{i+1})\big), \\ \ell_2 f \in L^2(d,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\},$$

respectively by

$$Af = \ell f \text{ with } f \in \operatorname{dom}(A)\,, \text{ and } A_j f = \ell_j f \text{ with } f \in \operatorname{dom}(A_j)\,,\ j = 1, 2.$$

Throughout this paper $AC\big( \cup_0^{m-1} (d_i, d_{i+1})\big)$ denotes the set of all functions whose restriction to $(d_i, d_{i+1})$ is absolutely continuous for all $i = 0, \cdots, m-1$. In particular, the limits of these functions exist at each boundary points $d_i$ $i = 1, 2, \cdots, m-1$.

**Lemma 2.1.** *The operators $A$ and $A_j$ are self-adjoint.*

In particular, the eigenvalues of $A$, $A_j$ and hence of $L$, $L_j$ are real and simple. To see that they are simple it suffices to observe that the associated Cauchy problem (2.1) subject to the initial conditions $f(x_0 \pm 0) = f_0$, $f'(x_0 \pm 0) = f_1$ (with $x_0 \in (0, \pi)$) have a unique solution.

*Remark* 2.2. For any function $f \in \operatorname{dom}(A)$ we will denote by $f_i$, $1 \le i \le m$, the restriction of $f$ to the subinterval $(d_{i-1}, d_i)$. Moreover, we will set $f_i(d_{i-1}) = f(d_{i-1} + 0)$ and $f_i(d_i) = f(d_i - 0)$.

## 3. Uniqueness results for Robin boundary conditions

In this section we investigate the inverse problem of the reconstruction of a boundary value problem $L$ from its spectral characteristics. We consider statement of the inverse problem of the reconstruction of the boundary-value problem $L$ from three spectra $\{\lambda_n,\, \mu_n,\, \nu_n\}_{n\ge 0}$. The technique which used to prove these theorems is an adaptation of the method discussed by F. Gesztesy and B. Simon in [9]. We need to the following lemma on asymptotic, poles and residues determining a meromorphic Herglotz function, see [9, Theorem 2.3].

**Lemma 3.1.** *Let $f_1(z)$ and $f_2(z)$ be two meromorphic Herglotz functions with identical sets of poles and residues, respectively. If*

$$f_1(ix) - f_2(ix) \to 0, \quad \text{as } x \to \infty,$$

*then $f_1 = f_2$.*

Consider the interlacing of the sequences between DSLP (2.1)–(2.3) and two DSLP's (SLP's) on subinterval $[0, d)$ and $(d, \pi]$ which are imposed the boundary condition at $d$.

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (2.1) under the initial conditions

$$\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=-h, \tag{3.1}$$

and

$$\psi(\pi,\lambda)=1, \qquad \psi'(\pi,\lambda)=-H, \tag{3.2}$$

as well as the jump conditions (2.3), respectively. It is easy to see that equation (2.1) under the initial conditions (3.1) or (3.2) has a unique solution $\varphi_1(x,\lambda)$ or $\psi_m(x,\lambda)$, which is an entire function of $\lambda \in \mathbb{C}$ for each fixed point $x \in [0,d_1)$ or $x \in (d_{m-1},\pi]$. It is known [17] that $\varphi(x,\lambda)$, $\varphi'(x,\lambda)$, $\psi(x,\lambda)$ and $\psi'(x,\lambda)$ are entire functions of $\lambda$ of order $\frac{1}{2}$ for any fixed $x$. In this section, we obtain the asymptotic form of solutions and characteristic function.

**Theorem 3.2** (see [15]). *Let $\lambda=\rho^2$ and $\tau := \mathrm{Im}\rho$. For equation (2.1) with boundary conditions (2.2) and jump conditions (2.3) as $|\lambda|\to\infty$, the following asymptotic formulas hold:*

(3.3)

$$\varphi(x,\lambda)=\begin{cases}\cos\rho x+O\left(\frac{\exp(|\tau|x)}{\rho}\right), \qquad 0\le x<d_1,\\ \alpha_1\cos\rho x+\alpha_1'\cos\rho(x-2d_1)+O\left(\frac{\exp(|\tau|x)}{\rho}\right), \qquad d_1<x<d_2,\\ \alpha_1\alpha_2\cos\rho x+\alpha_1'\alpha_2\cos\rho(x-2d_1)+\alpha_1\alpha_2'\cos\rho(x-2d_2)\\ \quad +\alpha_1'\alpha_2'\cos\rho(x+2d_1-2d_2)+O\left(\frac{\exp(|\tau|x)}{\rho}\right), \qquad d_2<x<d_3,\\ \quad\vdots\\ \alpha_1\alpha_2\ldots\alpha_{m-1}\cos\rho x+\\ \quad +\alpha_1'\alpha_2\ldots\alpha_{m-1}\cos\rho(x-2d_1)+\cdots\\ \quad +\alpha_1\alpha_2\ldots\alpha_{m-1}'\cos\rho(x-2d_{m-1})+\\ \quad +\alpha_1'\alpha_2'\alpha_3\cdots\alpha_{m-1}\cos\rho(x+2d_1-2d_2)+\cdots\\ \quad +\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\cos\rho(x+2d_i-2d_j)\\ \quad +\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\cos\rho(x-2d_i+2d_j-2d_k)+\cdots\\ \quad +\alpha_1'\alpha_2'\ldots\alpha_{m-1}'\cos\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})\\ \quad +O\left(\frac{\exp(|\tau|x)}{\rho}\right), \qquad d_{m-1}<x\le\pi,\end{cases}$$

*and*

(3.4)

$$\varphi'(x,\lambda)=\begin{cases}\rho[-\sin\rho x]+O(\exp(|\tau|x)), \qquad 0\le x<d_1,\\ \rho[-\alpha_1\sin\rho x-\alpha_1'\sin\rho(x-2d_1)]+O(\exp(|\tau|x)), \qquad d_1<x<d_2,\\ \rho[-\alpha_1\alpha_2\sin\rho x-\alpha_1'\alpha_2\sin\rho(x-2d_1)-\\ \qquad -\alpha_1\alpha_2'\sin\rho(x-2d_2)-\alpha_1'\alpha_2'\sin\rho(x+2d_1-2d_2)]\\ \qquad +O(\exp(|\tau|x)), \qquad d_2<x<d_3,\\ \quad\vdots\\ \rho[-\alpha_1\alpha_2\ldots\alpha_{m-1}\sin\rho x-\alpha_1'\alpha_2\ldots\alpha_{m-1}\sin\rho(x-2d_1)\\ \quad -\cdots-\alpha_1\alpha_2\ldots\alpha_{m-1}'\sin\rho(x-2d_{m-1})\\ \quad -\alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\sin\rho(x+2d_1-2d_2)-\cdots\\ \quad -\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\sin\rho(x+2d_i-2d_j)\\ \quad -\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\sin\rho(x-2d_i+2d_j-2d_k)+\cdots\\ \quad -\alpha_1'\alpha_2'\ldots\alpha_{m-1}'\sin\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})]\\ \quad +O(\exp(|\tau|x)), \qquad d_{m-1}<x\le\pi,\end{cases}$$

*where*

$$\alpha_i=\frac{a_i+b_i}{2} \quad \textit{and} \quad \alpha_i'=\frac{a_i-b_i}{2}, \tag{3.5}$$

*for* $i=1,2,\ldots,m-1$.

It follows from the above theorem that

$$|\varphi^{(\nu)}(x,\lambda)|=O(|\rho|^\nu\exp(|\tau|x)), \qquad 0\le x\le\pi,\ \nu=0,1. \tag{3.6}$$

By changing $x$ to $\pi-x$ one can obtain the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. In particular,

$$|\psi^{(\nu)}(x,\lambda)|=O\left(|\rho|^\nu\exp(|\tau|(\pi-x))\right), \qquad 0\le x\le\pi,\ \nu=0,1. \tag{3.7}$$

From the linear differential equations we obtain that the modified Wronskian

$$W(u,v)=w(x)\big(u(x)v'(x)-u'(x)v(x)\big) \tag{3.8}$$

is constant on $x\in[0,d_1)\cup_1^{m-2}(d_i,d_i+1)\cup(d_{m-1},\pi]$ for two solutions $\ell u=\lambda u$, $\ell v=\lambda v$ satisfying the transmission conditions (2.3). Moreover, from Eqs. (2.2) and Remark 2.2 we set

$$\begin{aligned}\Delta(\lambda):&=W(\varphi(\lambda),\psi(\lambda))\\ &=U(\psi(\lambda))\\ &=-w(\pi)V(\varphi(\lambda))\\ &=w(d)\left(b_k\varphi(d,\lambda)\psi'(d,\lambda)-a_k\varphi'(d,\lambda)\psi(d,\lambda)\right).\end{aligned} \tag{3.9}$$

Since $\Delta(\lambda)$ is composition of the solutions and from [10] it is known that each solution is an entire function of order $\frac{1}{2}$. Consequently $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$ whose roots $\lambda_n$ coincide with the eigenvalues of $L$. The asymptotic form of characteristic function satisfies

$$\begin{aligned}\Delta(\lambda) =&\rho w(\pi)\left[\alpha_1\alpha_2\ldots\alpha_{m-1}\sin\rho\pi + \alpha_1'\alpha_2\ldots\alpha_{m-1}\sin\rho(\pi-2d_1)\right.\\ &+\cdots+\alpha_1\alpha_2\ldots\alpha_{m-1}'\sin\rho(\pi-2d_{m-1})\\ &+\alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\sin\rho(\pi+2d_1-2d_2)+\cdots\\ &+\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_{m-1}\sin\rho(\pi+2d_i-2d_j)\\ &+\alpha_1\ldots\alpha_i'\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\sin\rho(\pi-2d_i+2d_j-2d_k)+\cdots\\ &\left.+\alpha_1'\alpha_2'\ldots\alpha_{m-1}'\sin\rho(\pi+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})\right]\\ &+O(\exp(|\tau|\pi)).\end{aligned} \tag{3.10}$$

Define the Weyl–Titchmarsh m-function

$$m_+(\lambda)=\frac{\psi'(d,\lambda)}{\psi(d,\lambda)},\qquad m_-(\lambda)=-\frac{\varphi'(d,\lambda)}{\varphi(d,\lambda)}. \tag{3.11}$$

As a consequence of [9, Theorem 2.1], we obtain:

**Lemma 3.3.** *The functions $m_\pm(\lambda)$ are the Herglotz functions,* (*i.e., it maps the upper half plane to the upper half plane*)*.*

We consider the DSLPs(SLPs) (2.4)–(2.6) and DSLPs(SLPs) (2.7)–(2.9). Whose increasing sequences of eigenvalues are denoted by $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ respectively. Now we are ready to prove our main uniqueness theorem for the solutions of the problems (2.1)–(2.9). For this purpose we agree that together with $L$ and $L_j$ we consider a boundary value problem $\tilde{L}$ and $\tilde{L}_j$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{d}_i$. If a certain symbol $\eta$ denotes an object related to $L$, then $\tilde{\eta}$ will denote the analogous object related to $\tilde{L}$.

**Theorem 3.4.** *If $\lambda_n=\tilde{\lambda}_n$, $\mu_n=\tilde{\mu}_n$, and $\nu_n=\tilde{\nu}_n$ for $n=0,1,2,\ldots,$ and $w(x)=\tilde{w}(x)$, $h=\tilde{h}$, and $H=\tilde{H}$. Moreover, if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L=\tilde{L}$.*

This result is an extension of [8, 9] to the discontinuous problems with a finite number of transmission (jump) points.

*Proof.* Define a meromorphic function

$$g(\lambda):=\begin{cases}-\frac{\Delta(\lambda)}{w(d)\varphi(d,\lambda)\psi(d,\lambda)}, & H_1=\infty,\\ \\ -\frac{\Delta(\lambda)}{w(d)[\varphi'(d,\lambda)+H_1\varphi(d,\lambda)][\psi'(d,\lambda)+H_2\psi(d,\lambda)]}, & H_1\neq\infty.\end{cases} \tag{3.12}$$

It is clear that the set of poles of $g(\lambda)$ is precisely $\{\mu_n\}_{n=1}^{\infty}\cup\{\nu_n\}_{n=1}^{\infty}$. It should be noted that

$$\Delta(\lambda)=w(d)\left(b_k\varphi(d,\lambda)\psi'(d,\lambda)-a_k\varphi'(d,\lambda)\psi(d,\lambda)\right).$$

So,

$$g(\lambda) = \begin{cases} -b_k \frac{\psi'(d,\lambda)}{\psi(d,\lambda)} + a_k \frac{\varphi'(d,\lambda)}{\varphi(d,\lambda)}, & H_1 = \infty, \\ -b_k \frac{\varphi(d,\lambda)}{\varphi'(d,\lambda)+H_1\,\varphi(d,\lambda)} + a_k \frac{\psi(d,\lambda)}{\psi'(d,\lambda)+H_2\,\psi(d,\lambda)}, & H_1 \neq \infty, \end{cases}$$

$$= M_+(\lambda) + M_-(\lambda), \tag{3.13}$$

where from (3.11)

(3.14)

$$M_+(\lambda) = \begin{cases} -b_k m_+(\lambda), & H_2 = \infty, \\ \frac{a_k}{H_2+m_+(\lambda)}, & H_2 \in \mathbb{R}, \end{cases} \qquad M_-(\lambda) = \begin{cases} -a_k m_-(\lambda), & H_1 = \infty, \\ \frac{b_k}{m_-(\lambda)-H_1}, & H_1 \in \mathbb{R}. \end{cases}$$

It is easy to see that both $M_+(\lambda)$ and $M_-(\lambda)$ are Herglotz functions. Define $\tilde{m}_-(\lambda)$, $\tilde{m}_+(\lambda)$, $\tilde{M}_-(\lambda)$, $\tilde{M}_+(\lambda)$, and $\tilde{g}(\lambda)$ in an analogous manner with $L$ replaced by $\tilde{L}$. Define the function

$$F(\lambda) := \frac{g(\lambda)}{\tilde{g}(\lambda)},$$

$F(\lambda)$ is an entire function from the above argument, since $g$ has the same zeros and poles as $\tilde{g}$. Using Theorems A.1 and A.2, for $H_1 = \infty$ and $H_1 \neq \infty$, respectively, we deduced that

$$F(\lambda) = \frac{g(\lambda)}{\tilde{g}(\lambda)} = 1 + o(1)$$

holds in sector of $\varepsilon \leq \arg\lambda \leq 2\pi - \varepsilon$. By using the Liouville's theorem, we have

$$F(\lambda) = 1$$

which therefore concludes

$$g(\lambda) = \tilde{g}(\lambda).$$

From (3.14) and (3.12), we see that the poles of $M_-(\lambda)$ and $M_+(\lambda)$ are precisely $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$, respectively. So we have

$$\operatorname{res} M_-(\mu_n) = \operatorname{res} g(\mu_n) \text{ and } \operatorname{res} M_+(\nu_n) = \operatorname{res} g(\nu_n), \text{ for } n = 1, 2, 3, \dots .$$

Which means that

$$\operatorname{res} M_-(\mu_n) = \operatorname{res} \tilde{M}_-(\mu_n) \text{ and } \operatorname{res} M_+(\nu_n) = \operatorname{res} \tilde{M}_+(\nu_n), \text{ for } n = 1, 2, 3, \dots .$$

This, together with Lemma 3.1 and Theorem A.2 we get

$$M_-(\lambda) = \tilde{M}_-(\lambda), \qquad M_+(\lambda) = \tilde{M}_+(\lambda).$$

Therefore by Borg's theorem [11] we get

$$L = \tilde{L}.$$

□

## 4. Uniqueness results for eigenparameter dependent boundary conditions

In this last section we will replace the Robin boundary conditions (2.2), (2.5), and (2.8) by the following eigenparameter dependent boundary conditions respectively

$$\begin{aligned}&\mathcal{U}(y) := \lambda\,(y'(0) + h_1 y(0)) - h_2 y'(0) - h_3 y(0) = 0,\\ &\mathcal{V}(y) := \lambda\,(y'(\pi) + H_1 y(\pi)) - H_2 y'(\pi) - H_3 y(\pi) = 0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\lambda\,(y'(0) + h_1 y(0)) - h_2 y'(0) - h_3 y(0) = 0,\\ &y'(d) + \mathfrak{H}_1\, y(d) = 0,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&y'(d) + \mathfrak{H}_2\, y(d) = 0,\\ &\lambda\,(y'(\pi) + H_1 y(\pi)) - H_2 y'(\pi) - H_3 y(\pi) = 0,\end{aligned} \tag{4.3}$$

where we assume that $h_j,\ H_j,\ j = 1, 2, 3$, and $\mathfrak{H}_1,\ \mathfrak{H}_2$ are real numbers, satisfying

$$r_1 := h_3 - h_1 h_2 > 0 \quad \text{and} \quad r_2 := H_1 H_2 - H_3 > 0. \tag{4.4}$$

Using the transmission conditions (2.3) we obtain $\mathfrak{H}_2 = \frac{b_k}{a_k}\mathfrak{H}_1$. In order to obtain a self-adjoint problem we will use the following Hilbert spaces $\mathcal{H} := L_2((0,\pi); w) \oplus \mathbb{C}^2$, $\mathcal{H}_1 := L_2((0,d); w_1) \oplus \mathbb{C}$, and $\mathcal{H}_2 := L_2((d,\pi); w_2) \oplus \mathbb{C}$ with inner product defined by

$$\langle F, G\rangle_{\mathcal{H}} := \int_0^{\pi} f\overline{g}w + \frac{w(0)}{r_1} f_1\overline{g_1} + \frac{w(\pi)}{r_2} f_2\overline{g_2}, \qquad F = \begin{pmatrix} f(x)\\ f_1\\ f_2\end{pmatrix},\ G = \begin{pmatrix} g(x)\\ g_1\\ g_2\end{pmatrix}, \tag{4.5}$$

$$\langle F_1, G_1\rangle_{\mathcal{H}_1} := \int_0^{d} f\overline{g}w_1 + \frac{w_1(0)}{r_1} f_1\overline{g_1}, \qquad F_1 = \begin{pmatrix} f(x)\\ f_1\end{pmatrix},\ G_1 = \begin{pmatrix} g(x)\\ g_1\end{pmatrix}, \tag{4.6}$$

and

$$\langle F_2, G_2\rangle_{\mathcal{H}_2} := \int_d^{\pi} f\overline{g}w_2 + \frac{w_2(\pi)}{r_2} f_2\overline{g_2}, \quad F_2 = \begin{pmatrix} f(x)\\ f_2\end{pmatrix},\ G_2 = \begin{pmatrix} g(x)\\ g_2\end{pmatrix}. \tag{4.7}$$

Again the associated norms will be denoted by $\|F\|_{\mathcal{H}} = \langle F, F\rangle_{\mathcal{H}}^{1/2}$, $\|F_1\|_{\mathcal{H}_1} = \langle F_1, F_1\rangle_{\mathcal{H}_1}^{1/2}$, and $\|F_2\|_{\mathcal{H}_2} = \langle F_2, F_2\rangle_{\mathcal{H}_2}^{1/2}$, respectively. Next we introduce

$$R_1(y) := y'(0) + h_1 y(0), \qquad R_1'(y) := h_2 y'(0) + h_3 y(0),$$

$$R_2(y) := y'(\pi) + H_1 y(\pi), \qquad R_2'(y) := H_2 y'(\pi) + H_3 y(\pi).$$

In this Hilbert space, we construct the operators

$$A : \mathcal{H} \to \mathcal{H}, \qquad A_j : \mathcal{H}_j \to \mathcal{H}_j, \quad j = 1, 2 \tag{4.8}$$

with domains

$$\text{(4.9)}\quad \operatorname{dom}(A) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \middle| \begin{array}{c} f, f' \in AC\big(\cup_0^{m-1}(d_i, d_{i+1})\big),\ \ell f \in L^2(0,\pi) \\ U_i(f) = V_i(f) = 0,\ f_1 = R_1(f),\ f_2 = R_2(f) \end{array} \right\},$$

(4.10)

$$\operatorname{dom}(A_1) = \left\{ F_1 = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \middle| \begin{array}{c} f, f' \in AC\big(\cup_0^{k-1}(d_i, d_{i+1})\big),\ \ell_1 f \in L^2(0,d) \\ U_i(f) = V_i(f) = 0,\ f_1 = R_1(f) \end{array} \right\},$$

and

(4.11)

$$\operatorname{dom}(A_2) = \left\{ F_2 = \begin{pmatrix} f(x) \\ f_2 \end{pmatrix} \middle| \begin{array}{c} f, f' \in AC\big(\cup_k^{m-1}(d_i, d_{i+1})\big),\ \ell_2 f \in L^2(d,\pi) \\ U_i(f) = V_i(f) = 0,\ f_2 = R_2(f) \end{array} \right\},$$

by

$$AF = \begin{pmatrix} \ell f \\ R_1'(f) \\ R_2'(f) \end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in \operatorname{dom}(A),$$

and

$$A_j F = \begin{pmatrix} \ell f \\ R_j'(f) \end{pmatrix} \quad \text{with } F_j = \begin{pmatrix} f(x) \\ R_j(f) \end{pmatrix} \in \operatorname{dom}(A_j).$$

By construction, the eigenvalue problems for $A$ and $A_j$,

$$\text{(4.12)}\qquad AY = \lambda Y, \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in \operatorname{dom}(A),$$

$$\text{(4.13)}\qquad A_j Y_j = \lambda Y_j, \qquad Y_j := \begin{pmatrix} y(x) \\ R_j(y) \end{pmatrix} \in \operatorname{dom}(A_j),$$

are equivalent to the eigenvalue problems (2.1), (2.3), and (4.1) for $L$, and (2.1), (2.3), and (4.2) or (4.3) for $L_j$, respectively. A straightforward calculation shows:

**Lemma 4.1.** *The operators $A$ and $A_j$ for $j = 1, 2$, are symmetric.*

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (2.1) under the initial conditions

$$\text{(4.14)}\qquad \varphi(0,\lambda) = \lambda - h_2,\ \varphi'(0,\lambda) = h_3 - \lambda h_1,$$

and

$$\text{(4.15)}\qquad \psi(\pi,\lambda) = H_2 - \lambda,\ \psi'(\pi,\lambda) = \lambda H_1 - H_3,$$

as well as the jump conditions (2.3), respectively.

**Theorem 4.2** (see [15])**.** *Let $\lambda = \rho^2$ and $\tau := \mathrm{Im}\rho$. For equation (2.1) with boundary conditions (4.1) and jump conditions (2.3) as $|\lambda| \to \infty$, the following asymptotic formulas hold:*

(4.16)

$$\varphi(x,\lambda) = \begin{cases} \rho^2 \cos\rho x + O(\rho\exp(|\tau|x)), \qquad 0 \le x < d_1, \\ \rho^2\,[\alpha_1 \cos\rho x + \alpha'_1 \cos\rho(x-2d_1)] + O(\rho\exp(|\tau|x)), \qquad d_1 < x < d_2, \\ \rho^2[\alpha_1\alpha_2 \cos\rho x + \alpha'_1\alpha_2 \cos\rho(x-2d_1) + \alpha_1\alpha'_2 \cos\rho(x-2d_2) \\ \quad +\alpha'_1\alpha'_2 \cos\rho(x+2d_1-2d_2)] + O(\rho\exp(|\tau|x)), \qquad d_2 < x < d_3, \\ \quad \vdots \\ \rho^2[\alpha_1\alpha_2\ldots\alpha_{m-1}\cos\rho x + \\ \quad +\alpha'_1\alpha_2\ldots\alpha_{m-1}\cos\rho(x-2d_1) + \cdots \\ \quad +\alpha_1\alpha_2\ldots\alpha'_{m-1}\cos\rho(x-2d_{m-1}) + \\ \quad +\alpha'_1\alpha'_2\alpha_3\ldots\alpha_{m-1}\cos\rho(x+2d_1-2d_2) + \cdots \\ \quad +\alpha_1\ldots\alpha'_i\ldots\alpha'_j\ldots\alpha_{m-1}\cos\rho(x+2d_i-2d_j) \\ \quad +\alpha_1\ldots\alpha'_i\ldots\alpha'_j\ldots\alpha'_k\ldots\alpha_{m-1}\cos\rho(x-2d_i+2d_j-2d_k) + \cdots \\ \quad +\alpha'_1\alpha'_2\ldots\alpha'_{m-1}\cos\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})] \\ \quad +O(\rho\exp(|\tau|x)), \qquad d_{m-1} < x \le \pi, \end{cases}$$

*and*

(4.17)

$$\varphi'(x,\lambda) = \begin{cases} \rho^3[-\sin\rho x] + O(\rho^2\exp(|\tau|x)), \qquad 0 \le x < d_1, \\ \rho^3[-\alpha_1\sin\rho x - \alpha'_1\sin\rho(x-2d_1)] + O(\rho^2\exp(|\tau|x)), \qquad d_1 < x < d_2, \\ \rho^3[-\alpha_1\alpha_2\sin\rho x - \alpha'_1\alpha_2\sin\rho(x-2d_1) - \\ \qquad -\alpha_1\alpha'_2\sin\rho(x-2d_2) - \alpha'_1\alpha'_2\sin\rho(x+2d_1-2d_2)] \\ \qquad +O(\rho^2\exp(|\tau|x)), \qquad d_2 < x < d_3, \\ \quad \vdots \\ \rho^3[-\alpha_1\alpha_2\ldots\alpha_{m-1}\sin\rho x - \alpha'_1\alpha_2\ldots\alpha_{m-1}\sin\rho(x-2d_1) \\ \quad - \cdots - \alpha_1\alpha_2\ldots\alpha'_{m-1}\sin\rho(x-2d_{m-1}) \\ \quad -\alpha'_1\alpha'_2\alpha_3\ldots\alpha_{m-1}\sin\rho(x+2d_1-2d_2) - \cdots \\ \quad -\alpha_1\ldots\alpha'_i\ldots\alpha'_j\ldots\alpha_{m-1}\sin\rho(x+2d_i-2d_j) \\ \quad -\alpha_1\ldots\alpha'_i\ldots\alpha'_j\ldots\alpha'_k\ldots\alpha_{m-1}\sin\rho(x-2d_i+2d_j-2d_k) + \cdots \\ \quad -\alpha'_1\alpha'_2\ldots\alpha'_{m-1}\sin\rho(x+2(-1)^{m-1}d_1+2(-1)^{m-2}d_2+\cdots-2d_{m-1})] \\ \quad +O(\rho^2\exp(|\tau|x)), \qquad d_{m-1} < x \le \pi, \end{cases}$$

*where*

$$\alpha_i = \frac{a_i+b_i}{2} \quad \text{and} \quad \alpha'_i = \frac{a_i-b_i}{2}, \tag{4.18}$$

*for $i = 1, 2, \ldots, m-1$.*

It follows from the above theorem that

$$|\varphi^{(\nu)}(x,\lambda)| = O\left(|\rho|^{\nu+2}\exp(|\tau|x)\right) \quad 0 \le x \le \pi,\ \nu = 0,1, \tag{4.19}$$

and so by substituting $x$ to $\pi - x$ we get the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$ and particularly

$$|\psi^{(\nu)}(x,\lambda)| = O\left(|\rho|^{\nu+2}\exp(|\tau|(\pi - x))\right) \quad 0 \le x \le \pi,\ \nu = 0,1. \tag{4.20}$$

Moreover, from (4.1) and 2.2 we have

$$\begin{aligned}\Delta(\lambda) &= W(\varphi(\lambda),\psi(\lambda)) \\ &= -w(\pi)\mathcal{V}(\varphi(\lambda)) \\ &= w(d)\left(b_k\varphi(d,\lambda)\psi'(d,\lambda) - a_k\varphi'(d,\lambda)\psi(d,\lambda)\right).\end{aligned} \tag{4.21}$$

The asymptotic form of characteristic function satisfies

$$\begin{aligned}\Delta(\lambda) =& \rho^5 w(\pi)\left[\alpha_1\alpha_2\dots\alpha_{m-1}\sin\rho\pi + \alpha_1'\alpha_2\dots\alpha_{m-1}\sin\rho(\pi - 2d_1)\right. \\ &+ \dots + \alpha_1\alpha_2\dots\alpha_{m-1}'\sin\rho(\pi - 2d_{m-1}) \\ &+ \alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\rho(\pi + 2d_1 - 2d_2) + \cdots \\ &+ \alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_{m-1}\sin\rho(\pi + 2d_i - 2d_j) \\ &+ \alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\rho(\pi - 2d_i + 2d_j - 2d_k) + \cdots \\ &\left.+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\rho(\pi + 2(-1)^{m-1}d_1 + 2(-1)^{m-2}d_2 + \cdots - 2d_{m-1})\right] \\ &+ O\left(\rho^4\exp(|\tau|\pi)\right).\end{aligned} \tag{4.22}$$

Then $\Delta(\lambda)$ is an entire function whose roots $\lambda_n$ coincide with the eigenvalues of $L$. Again define the Weyl-Titchmarsh m-functions

$$m_+(\lambda) = \frac{\psi'(d,\lambda)}{\psi(d,\lambda)}, \qquad m_-(\lambda) = -\frac{\varphi'(d,\lambda)}{\varphi(d,\lambda)}. \tag{4.23}$$

It is known from [9, Theorem 2.1] that both $m_\pm(\lambda)$ are the Herglotz functions.

Now we consider the PDSLPs (PSLPs) (2.4), (2.6) and (4.2) and PDSLPs (PSLPs) (2.7), (2.9) and (4.3). Whose increasing sequences of eigenvalues are denoted by $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$, respectively. Now we are ready to prove our main uniqueness theorem for the solutions of the problems (2.1), (2.2) and parameter dependent conditions (4.1)–(4.3) for the problems $L$ and $L_j$. For this purpose we agree that together with $L$ and $L_j$ we consider a boundary value problem $\tilde{L}$ and $\tilde{L}_j$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}_j$, $\tilde{H}_j$, $\tilde{H}_1$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{d}_i$.

**Theorem 4.3.** *If $\lambda_n = \tilde{\lambda}_n$, $\mu_n = \tilde{\mu}_n$, and $\nu_n = \tilde{\nu}_n$ for $n = 0,1,2,\dots$, and $w(x) = \tilde{w}(x)$, $h_j = \tilde{h}_j$, and $H_j = \tilde{H}_j$. Moreover, if $\{\mu_n\}_{n=1}^{+\infty}$ and $\{\nu_n\}_{n=1}^{+\infty}$ are pairwise disjoint, then $L = \tilde{L}$.*

## Appendix A. Asymptotic behavior of $m$–functions for Robin and eigenparameter dependent boundary conditions

**Theorem A.1.** *For any $\varepsilon > 0$, if $\varepsilon < \arg\lambda < 2\pi - \varepsilon$, then $m_\pm(\lambda)$ have the following asymptotic behavior*

$$m_\pm(\lambda) = i\sqrt{\lambda}\,(1 + o(1)), \qquad \text{as } \lambda \to \infty. \tag{A.1}$$

*Specially, when $\lambda \to -\infty$, we have*

$$m_\pm(\lambda) = -\sqrt{|\lambda|}\,(1 + o(1)) \to -\infty. \tag{A.2}$$

**Theorem A.2.** *Fixed $H_1 \in \mathbb{R} \cup \{\infty\}$. For any $\varepsilon > 0$, if $\varepsilon < \arg\lambda < 2\pi - \varepsilon$, then $M_-(\lambda)$ and $M_+(\lambda)$ have the following asymptotic behavior*

$$M_-(\lambda) = \begin{cases} i\,a_k\sqrt{\lambda}\,(1 + o(1)), & H_1 = \infty, \\ \frac{i\,b_k}{\sqrt{\lambda}}(1 + o(1)), & H_1 \in \mathbb{R}, \end{cases} \tag{A.3}$$

*and*

$$M_+(\lambda) = \begin{cases} i\,b_k\sqrt{\lambda}\,(1 + o(1)), & H_2 = \infty, \\ \frac{i\,a_k}{\sqrt{\lambda}}(1 + o(1)), & H_2 \in \mathbb{R}. \end{cases} \tag{A.4}$$

*Proof.* By applying the asymptotic form of solutions $\varphi(x, \lambda)$ and $\varphi'(x, \lambda)$ in (3.3) and (3.4) and similar asymptotic form of solutions $\psi(x, \lambda)$ and $\psi'(x, \lambda)$, it is easy to see that the asymptotic forms of $m_-(\lambda)$, $m_+(\lambda)$, $M_-(\lambda)$, and $M_+(\lambda)$ are satisfying in (A.1)–(A.4). ☐

## Appendix B. Asymptotic behavior of solutions and $m$–functions for eigenparameter dependent boundary conditions

**Theorem B.1.** *For any $\varepsilon > 0$, if $\varepsilon < \arg\lambda < 2\pi - \varepsilon$, then $m_\pm(\lambda)$ have the following asymptotic behavior*

$$m_\pm(\lambda) = i\sqrt{\lambda}\,(1 + o(1)), \quad \text{as } \lambda \to \infty. \tag{B.1}$$

*Specially, when $\lambda \to -\infty$, we have*

$$m_\pm(\lambda) = -\sqrt{|\lambda|}\,(1 + o(1)) \to -\infty. \tag{B.2}$$

**Theorem B.2.** *Fixed $H_1 \in \mathbb{R} \cup \{\infty\}$. For any $\varepsilon > 0$, if $\varepsilon < \arg\lambda < 2\pi - \varepsilon$, then $M_-(\lambda)$ and $M_+(\lambda)$ have the following asymptotic behavior*

$$M_-(\lambda) = \begin{cases} i\,a_k\sqrt{\lambda}\,(1 + o(1)), & \mathfrak{H}_1 = \infty, \\ \frac{i\,b_k}{\sqrt{\lambda}}(1 + o(1)), & \mathfrak{H}_1 \in \mathbb{R}, \end{cases} \tag{B.3}$$

*and*

$$M_+(\lambda) = \begin{cases} i\, b_k \sqrt{\lambda}\,(1+o(1))\,, & \mathfrak{H}_2 = \infty, \\ \frac{i\, a_k}{\sqrt{\lambda}}(1+o(1)), & \mathfrak{H}_2 \in \mathbb{R}. \end{cases} \tag{B.4}$$

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