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A STOCHASTIC VERSION ANALYSIS OF AN M/G/1 RETRIAL QUEUE WITH BERNOULLI SCHEDULE

M. BOUALEM*, M. CHERFAOUI, N. DJELLAB AND D. AÏSSANI

(Communicated by Hamid Pezeshk)

Abstract. In this work, we derive insensitive bounds for various performance measures of a single-server retrial queue with generally distributed inter-retrial times and Bernoulli schedule, under the special assumption that only the customer at the head of the orbit queue (i.e., a FCFS discipline governing the flow from the orbit to the server) is allowed to occupy the server. The methodology is strongly based on stochastic comparison techniques. Instead of studying a performance measure in a quantitative fashion, this approach attempts to reveal the relationship between the performance measures and the parameters of the system. We prove the monotonicity of the transition operator of the embedded Markov chain relative to strong stochastic ordering and increasing convex ordering. We obtain comparability conditions for the distribution of the number of customers in the system. Bounds are derived for the stationary distribution and some simple bounds for the mean characteristics of the system. The proofs of these results are based on the validation of some inequalities for some cumulative probabilities associated with every state (m, n) of the system. Finally, the effects of various parameters on the performance of the system have been examined numerically.
Keywords: Retrial queues, performance measures, stochastic orders, monotonicity, simulation.
MSC(2010): Primary: 60K25; Secondary: 60E15, 60K10.

1. Introduction

Retrial queueing systems or systems with repeated attempts are characterized by the requirement that customers finding the service area busy must join the retrial group and retry for service at random intervals. Retrial queues have been widely used to model many practical problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processing unit, etc. Moreover, retrial queues are also used

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as mathematical models of several computer systems: packet switching networks, collision avoidance star local area networks, cellular mobile networks. A review of the main results can be found in [3, 16, 31].

Priority mechanisms are an invaluable scheduling method that allows customers to receive different qualities of service. Service priority is clearly today a main feature of the operation of almost any manufacturing system. The role of quality and service performance are crucial aspects in customer perceptions, and firms must dedicate special attention to them when designing and implementing their operations. For this reason, the priority queue has received considerable attention in the literature [12, 13, 15].

In almost all the literature for retrial queues, the retrial times are assumed to be exponentially distributed and the results about the retrial queues with non-exponential retrial time are very limited. The main difficulty for analyzing the system with non-exponential retrial times is due to the fact that the model must keep track of the elapsed retrial time for each of possibly a very large number of customers [1, 2]. The first investigation on the $M/G/1$ retrial queue with general retrial times is due to Kapyrin [21], who assumed that each customer in the orbit generates a stream of repeated attempts that are independent of the customer in the orbit and state of the server. However, this methodology was found to be incorrect by Falin [14]. Subsequently, Yang et al. [30] have developed an approximation method to obtain the steady state performance for the model of Kapyrin. The order relation for GI/G/1 retrial queue with PH-retrial times and a stability condition for $BMAP/PH/s/s+K$ retrial queue with PH-retrial times are considered in Liang and Kulkarni [20] and He et al. [18], respectively. Later, Gómez-Corral [17] discussed extensively an $M/G/1$ retrial queue with FCFS discipline and general retrial times. Atencia and Moreno [4], analyze an $M/G/1$ retrial queue with Bernoulli schedule where the retrial times are governed by an arbitrary distribution and only the customer at the head of the orbit is allowed for access to the server, a blocked customer can become of high or low priority according to his choice; so if customers had a cost per unit time in each group, we could research the optimal decision in the sense of minimizing the expected total cost per unit time.

Because of complexity of retrial queueing models, analytic results are generally difficult to obtain. In contrast, there are a great number of numerical and approximation methods which are of practical importance. One important approach is monotonicity which can be investigated using the stochastic comparison method based on the general theory of stochastic orders. Stochastic comparison methods have been used to produce bounds and approximations for queue length processes, waiting times and busy period distributions in many queueing systems. For the detailed results and references about the comparison methods and their applications, see Stoyan [27] and especially, for a constructive method, see Bhaskaran [5], Müller and Stoyan [24] and Massey [23].

Monotonicity properties of queueing systems have become an interesting subject recently. One monotonicity property has been considered by Liang and Kulkarni [20], in which they study how the retrial time distribution affects the system congestion. Khalil and Falin [22] investigate some monotonicity properties of $M/G/1$ retrial queues with exponential retrial times relative to stochastic ordering, convex ordering and Laplace ordering. Liang [19] considers a retrial queue which consists of an orbit with infinite capacity, a service station, and a queue with finite capacity. He shows that if the hazard rate function of the retrial times distribution is decreasing, then stochastically longer service time or less servers will result in more customers in the system. Boualem et al. [8] investigate some monotonicity properties of an $M/G/1$ queue with constant retrial policy in which the server operates under a general exhaustive service and multiple vacation policy relative to strong stochastic ordering and convex ordering. Boualem et al. [9] consider a qualitative analysis to investigate various monotonicity properties for an M/G/1 retrial queue with classical retrial policy and Bernoulli feedback. The results obtained allow us to place in a prominent position the insensitive bounds for both the stationary distribution and the conditional distribution of the stationary queue of the considered model. Recently, Boualem et al. [10] investigate various monotonicity properties of a single server retrial queue with FCFS orbit and general retrial times using the mathematical method based on stochastic comparisons of Markov chains in order to derive performance indice bounds. Bounds are derived for the mean characteristics of the busy period, number of customers served during a busy period, number of orbit busy periods and waiting times. Boualem [6] addresses monotonicity properties of the single server retrial queue with no waiting room and server subject to active breakdowns, that is, the service station can fail only during the service period. The obtained results give insensitive bounds for the stationary distribution of the considered embedded Markov chain related to the model in the study. Numerical illustrations are provided to support the results. Boualem et al. [7] introduce a new analytical approach, namely a qualitative analysis, which is another field of own interest to establish insensitive stochastic bounds on some performance measures of a single server queue with classical retrial policy and service interruptions by using the monotonicity and comparability approach relative to the convex ordering.

This paper looks at a particular retrial queue with Poisson arrivals, general service distribution, retrial times under a Bernoulli schedule that are distributed according to a general distribution that has finite moments, and the special assumption that only the customer at the head of the orbit queue is allowed to occupy the server. The waiting room is assumed to be unlimited and the service discipline FCFS, with priority given to waiting customers over retrial customers. The performance characteristics of such a system are available in explicit form, where the main probabilistic descriptors have been obtained [4]. However, the obtained results are cumbersome (they include integrals of Laplace transform, solutions of functional equations, etc.) and are not very exploitable from the application point of view (e.g. performance evaluation), so in the present study, we consider the model from the viewpoint of stochastic orders (strong stochastic order, increasing convex order and Laplace order), in which the characterization is based on the pointwise comparison of the indefinite integral of distribution function. The proposed approach is quite different from that given by Atencia and Moreno [4], in the sense that it provides from the fact that we can come to a compromise between the role of these qualitative bounds and the complexity of resolution of some complicated systems where some parameters are not perfectly known. Besides, the obtained bounds (lower and upper) in this paper are easy to calculate and seem to be good approximations for performance measures of the considered system.

The rest of the paper is organized as follows. In the next Section, we describe the considered queueing system. In Section 3, we introduce some pertinent definitions and notions of stochastic orders. In Section 4, we present some lemmas that will be used in what follows. Section 5 focusses on monotonicity of the transition operator of the embedded Markov chain and gives comparability conditions of two transition operators. Stochastic bounds for the stationary number of customers in the system are discussed in Section 6. In Section 7, we provide insensitive bounds for the mean characteristics of the system. The last Section is devoted to the practical aspect.

2. **The mathematical model**

We consider a single-server retrial queue in which external customers arrive according to a Poisson stream with rate $\lambda > 0$. Upon arrival, customers examine the availability of the server. If an arriving customer finds the server idle, he commences his service immediately. Otherwise, the arriving customer either with probability p enters the retrial group (called orbit) or with complementary probability q $(= 1 - p)$ joins the waiting space (called priority queue), where he waits to be served. We will assume that only the customer at the head of the orbit is allowed for access to the server. If the server is busy upon retrial, the customer joins the orbit again. Such a process is repeated until the customer finds the server idle and gets the requested service at the time of a retrial. Successive inter-retrial times of any customer follow an arbitrary law with common probability distribution function $A(x)$, Laplace-Stieltjes Transform (LST) $\alpha_A(s)$ and nth moments α_n. The service times are independently and identically distributed with probability distribution function $B(x)$, LST $\beta_B(s)$ and nth moments β_n. Moreover, we suppose that inter-arrival times, retrial times and service times are mutually independent.

At an arbitrary time t, the system can be described by means of the Markov process

$$X(t) = (C(t), Q_1(t), Q_2(t), \xi_o(t), \xi_1(t)),$$

where

- $C(t) = \begin{cases} 0, & \text{if the server is free,} \\ 1, & \text{if the server is busy.} \end{cases}$
- $Q_1(t)$ and $Q_2(t)$ are the number of customers in the priority queue and in the orbit respectively.
- $\xi_o(t)$ represents the elapsed retrial time, if $C(t) = 0$, $Q_1(t) = 0$ and $Q_2(t) > 0$.
- $\xi_1(t)$ corresponds to the elapsed time of the customer currently being served, if $C(t) = 1$.

From the model description, it is clear that the evolution of our retrial queue is described in terms of alternating sequence of idle and busy periods for the server. After each service, the server becomes free only when the priority queue is empty; then the next customer to be served is determined by a competition between an exponential law of rate λ and the general retrial time distribution (that is, a possible new arrival and the one, if any, at the head of the orbit compete for service). This is the main difference with classical waiting lines without retrials.

Let t_l be the time of the lth departure, $Q_{1,l} = Q_1(t_l - 0)$ and $Q_{2,l} = Q_2(t_l - 0)$ the number of customers in the priority queue and in the orbit respectively just before the time t_l. For $Q_{1,l}$ and $Q_{2,l}$ we have the following fundamental recursive equations

$$Q_{1,l} = \begin{cases} Q_{1,l-1} - 1 + w_{1,l}, & \text{if } Q_{1,l-1} \geq 1, \\ w_{1,l}, & \text{if } Q_{1,l-1} = 0, \end{cases} \tag{2.1}$$

$$Q_{2,l} = \begin{cases} Q_{2,l-1} + w_{2,l}, & \text{if } Q_{1,l-1} \geq 1, \\ Q_{2,l-1} - b_l + w_{2,l}, & \text{if } Q_{1,l-1} = 0 \text{ and } Q_{2,l-1} \geq 1, \\ w_{2,l}, & \text{if } Q_{1,l-1} = 0 \text{ and } Q_{2,l-1} = 0, \end{cases} \tag{2.2}$$

where $w_{1,l}$ and $w_{2,l}$ are the number of customers arriving at the priority queue and the orbit respectively during the lth service time, and

$$b_l = \begin{cases} 1, & \text{if the } l\text{th served customer proceeds from the orbit,} \\ 0, & \text{otherwise.} \end{cases}$$

We will denote by

$$k_{m,n} = \int_0^\infty \frac{(\lambda q x)^m}{m!} \frac{(\lambda p x)^n}{n!} e^{-\lambda x} dB(x), \tag{2.3}$$

the joint distribution of the number of customers who arrive at the priority queue and the orbit during a service time.

It is easy to prove that

$$k(z_1, z_2) = \sum_{m=0}^\infty \sum_{n=0}^\infty k_{m,n} z_1^m z_2^n = \beta_B(\lambda - \lambda q z_1 - \lambda p z_2). \tag{2.4}$$

The inequality $\rho < \alpha_A(\lambda)(p + q\alpha_A(\lambda))^{-1}$ is a necessary and sufficient condition for ergodicity, where $\rho = \lambda\beta_1$ is the load of the system [4].

The previous comments imply that the sequence of random variables $X_l = (Q_{1,l}, Q_{2,l})$, $l \in \mathbb{N}$ forms a Markov chain with $\mathbb{N}^2$ as state space, which is the embedded chain for our queueing system. It is not difficult to see that $X_l = (Q_{1,l}, Q_{2,l})$, $l \in \mathbb{N}$ is irreducible and aperiodic.

The one-step transition probabilities of the chain $\{X_l, l \in \mathbb{N}\}$ is defined in the following formulae (see [4]):

(2.5)

$$p_{(j,m)(i,n)} = \begin{cases} k_{i,n}, & \text{if } j = 0, m = 0, \\ [1 - \alpha_A(\lambda)]\, k_{i,n-m} + \alpha_A(\lambda) k_{i,n-m+1}, & \text{if } j = 0, 1 \le m \le n, \\ \alpha_A(\lambda) k_{i,0}, & \text{if } j = 0, m = n+1, \\ k_{i-j+1,n-m}, & \text{if } 1 \le j \le i+1, 0 \le m \le n, \\ 0, & \text{otherwise.} \end{cases}$$

3. Stochastic orders and ageing notions

Stochastic ordering is useful for studying internal changes of performance due to parameter variations, to compare distinct systems, to approximate a system by a simpler one, and to obtain upper and lower bounds for the main performance measures of systems.

First, let us recall some stochastic orders and ageing notions which are most pertinent to the main results to be developed in this paper.

3.1. Definitions of some univariate stochastic orders.

Definition 3.1. For two non-negative random variables X and Y with densities f and g and cumulative distribution functions F and G, respectively, let $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ denote the survival functions. X is said to be smaller than Y in:

(1) Usual stochastic order ($\le_{st}$) iff $\overline{F}(x) \le \overline{G}(x)$, $\forall x \ge 0$.
(2) Increasing convex ordering ($\le_{icx}$) iff $\int_x^{+\infty} \overline{F}(u)d(u) \le \int_x^{+\infty} \overline{G}(u)d(u)$.
(3) Laplace ordering ($\le_L$) iff $\int_0^{+\infty} e^{-sx}dF(x) \ge \int_0^{+\infty} e^{-sx}dG(x)$, $\forall s \ge 0$.

If the random variables of interest are of discrete type and $\omega = (\omega_n)_{n\ge0}$, $\nu = (\nu_n)_{n\ge0}$ are the corresponding distributions, then the above definitions can be given in the following form:

(1) $\omega \le_{st} \nu$ iff $\overline{\omega}_m = \sum_{n\ge m} \omega_n \le \overline{\nu}_m = \sum_{n\ge m} \nu_n$, $\forall m$.
(2) $\omega \le_{icx} \nu$ iff $\overline{\overline{\omega}}_m = \sum_{n\ge m} \sum_{k\ge n} \omega_k \le \overline{\overline{\nu}}_m = \sum_{n\ge m} \sum_{k\ge n} \nu_k$, $\forall m$.
(3) $\omega \le_L \nu$ iff $\sum_{n\ge0} \omega_n z^n \ge \sum_{n\ge0} \nu_n z^n$, $\forall z \in [0, 1]$.

3.2. **Some multivariate extensions.** Multi-dimensional stochastic processes are used for the modeling of complex systems such as queueing networks. Since a direct analysis of such systems is very difficult, stochastic comparison has been a standard tool in their analysis and there has been an increasing interest on this technique in the last years [11, 29].

In this section we recall some multivariate extension of the stochastic orders considered in the previous section.

Definition 3.2. Given two random vectors X and Y, we say that X is less than Y in:

(1) Multivariate stochastic order iff $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$,
(2) Multivariate increasing convex order iff $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$,
(3) Multivariate Laplace ordering iff $\mathbb{E}[\exp\{-S^T X\}] \geq \mathbb{E}[\exp\{-S^T Y\}]$,

for all $S \in \mathbb{R}_+^n$ and for all increasing function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, for which the previous expectations exist.

Definition 3.3. Let X be a positive random variable with distribution function F:

(1) F is $HNBUE$ (Harmonically New Better than Used in Expectation) iff $F \leq_{icx} F^*$,
(2) F is of class $\mathcal{L}$ iff $F \geq_L F^*$,

where F^* is the exponential distribution function with the same mean as F.

The ageing classes are linked by the inclusion chain:

$$NBU \subset NBUE \subset HNBUE \subset \mathcal{L}.$$

For a comprehensive discussion on these stochastic orders and their applications, one may refer to [23, 25–28].

4. **Preliminary results**

This section presents several useful lemmas which will be used later in establishing the main results in Section 5.

Now, let $\Sigma^{(1)}$ and $\Sigma^{(2)}$ be two $M/G/1$ retrial queues with general retrial times and Bernoulli schedule defined by $\lambda^{(1)}, p^{(1)}, B^{(1)}, k_{m,n}^{(1)}$ and $\lambda^{(2)}, p^{(2)}, B^{(2)}$, $k_{m,n}^{(2)}$, respectively.

Lemma 4.1. (1) *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)}=p^{(2)}$ *and* $B^{(1)} \leq_{st} B^{(2)}$, *then* $\{k_{m,n}^{(1)}\} \leq_{st} \{k_{m,n}^{(2)}\}$.

(2) *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$ *and* $B^{(1)} \leq_{icx} B^{(2)}$, *then* $\{k_{m,n}^{(1)}\} \leq_{icx} \{k_{m,n}^{(2)}\}$.

Proof. By definition,

$$\overline{k}_{m,n}^{(i)} = \sum_{l \geq m} \sum_{k \geq n} k_{l,k}^{(i)} = \int_0^{+\infty} \sum_{l \geq m} \sum_{k \geq n} \frac{(\lambda^{(i)} q^{(i)} x)^l}{l!} \frac{(\lambda^{(i)} p^{(i)} x)^k}{k!} e^{-\lambda^{(i)} x} dB^{(i)}(x),$$

$$\overline{\overline{k}}_{m,n}^{(i)} = \sum_{j\geq m}\sum_{h\geq n}\overline{k}_{j,h}^{(i)} = \sum_{j\geq m}\sum_{h\geq n}\sum_{l\geq j}\sum_{k\geq h} k_{l,k}^{(i)},\ i=1,2.$$

To prove that $\{k_{m,n}^{(1)}\} \leq_s \{k_{m,n}^{(2)}\}$, we have to establish the usual numerical inequalities

$$\begin{aligned}\overline{k}_{m,n}^{(1)} = \sum_{l\geq m}\sum_{k\geq n} k_{l,k}^{(1)} &\leq \overline{k}_{m,n}^{(2)},\ (\text{for } \leq_s=\leq_{st}),\\ \overline{\overline{k}}_{m,n}^{(1)} = \sum_{j\geq m}\sum_{h\geq n}\overline{k}_{j,h}^{(1)} &\leq \overline{\overline{k}}_{m,n}^{(2)},\ (\text{for } \leq_s=\leq_{icx}).\end{aligned}$$

(1) Consider the function

$$\begin{aligned} f_{m,n}(x,\lambda,p) &= \sum_{l\geq m}\sum_{k\geq n}\frac{(\lambda qx)^l}{l!}\frac{(\lambda px)^k}{k!}e^{-\lambda x}\\ &= \sum_{l\geq m}\frac{(\lambda qx)^l}{l!}e^{-\lambda qx}\sum_{k\geq n}\frac{(\lambda px)^k}{k!}e^{-\lambda px}.\end{aligned}$$

This is an increasing function with respect to λ and x.
With the help of [27, Theorem 1.2.2] and by monotonicity of $f_{m,n}(x,\lambda,p)$ with respect to λ, one can find that

$$\begin{aligned}\int_0^\infty f_{m,n}(x,\lambda^{(1)},p^{(1)})dB^{(1)}(x) &\leq \int_0^\infty f_{m,n}(x,\lambda^{(1)},p^{(1)})dB^{(2)}(x)\\ &\leq \int_0^\infty f_{m,n}(x,\lambda^{(2)},p^{(2)})dB^{(2)}(x).\end{aligned}$$

(2) Consider also $\bar{f}_{m,n}(x,\lambda,p) = \sum_{j\geq m}\sum_{h\geq n} f_{j,h}(x,\lambda,p)$.
This is an increasing function with respect to λ and an increasing and convex function with respect to x.

Similarly, with the help of [27, Theorem 1.3.1] and by monotonicity of $\bar{f}_{m,n}(x,\lambda,p)$ with respect to λ, we obtain the result. □

Lemma 4.2. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$ and $B^{(1)} \leq_L B^{(2)}$, then $\{k_{m,n}^{(1)}\} \leq_L \{k_{m,n}^{(2)}\}$.*

Proof. We have

$$k^{(i)}(z_1,z_2) = \sum_{m\geq 0}\sum_{n\geq 0} k_{m,n}^{(i)} z_1^m z_2^n = \beta_{B^{(i)}}(\lambda^{(i)} - \lambda^{(i)}q^{(i)}z_1 - \lambda^{(i)}p^{(i)}z_2),\ i=1,2,$$

where $k^{(i)}(z_1,z_2)$ be the corresponding probability generating function of the joint distributions of the number of customers who arrive at the priority queue and the orbit during a service time in the ith system, $i=1,2$.

Let $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$ and $B^{(1)} \leq_L B^{(2)}$. To prove that $\{k^{(1)}_{m,n}\} \leq_L \{k^{(2)}_{m,n}\}$, we have to establish that

$$\beta_{B^{(1)}}(\lambda^{(1)}-\lambda^{(1)}q^{(1)}z_1-\lambda^{(1)}p^{(1)}z_2) \geq \beta_{B^{(2)}}(\lambda^{(2)}-\lambda^{(2)}q^{(2)}z_1-\lambda^{(2)}p^{(2)}z_2). \tag{4.1}$$

The inequality $B^{(1)} \leq_L B^{(2)}$ implies that $\beta_{B^{(1)}}(s) \geq \beta_{B^{(2)}}(s)$ for all $s \geq 0$.
In particular, for $s = \lambda^{(1)} - \lambda^{(1)}q^{(1)}z_1 - \lambda^{(1)}p^{(1)}z_2$ we have

$$\beta_{B^{(1)}}(\lambda^{(1)}-\lambda^{(1)}q^{(1)}z_1-\lambda^{(1)}p^{(1)}z_2) \geq \beta_{B^{(2)}}(\lambda^{(1)}-\lambda^{(1)}q^{(1)}z_1-\lambda^{(1)}p^{(1)}z_2). \tag{4.2}$$

The function $\lambda - \lambda q z_1 - \lambda p z_2$ is increasing in λ.
Since any Laplace transform is a decreasing function, $\lambda^{(1)} \leq \lambda^{(2)}$ and $p^{(1)} = p^{(2)}$, implies that

$$\beta_{B^{(2)}}(\lambda^{(1)}-\lambda^{(1)}q^{(1)}z_1-\lambda^{(1)}p^{(1)}z_2) \geq \beta_{B^{(2)}}(\lambda^{(2)}-\lambda^{(2)}q^{(2)}z_1-\lambda^{(2)}p^{(2)}z_2). \tag{4.3}$$

By transitivity, (4.2) and (4.3) give (4.1). □

5. Monotonicity properties of the embedded Markov chain

Now we study monotonicity properties of the embedded Markov chain $\{X_l, l \in \mathbb{N}\}$ relative to the strong stochastic ordering $\leq_{st}$ and the increasing convex ordering $\leq_{icx}$.
Let T be the transition operator of our embedded Markov chain $\{X_l, l \in \mathbb{N}\}$, which associates to every distribution $\phi = (\phi_{(j,m)})$, a distribution $T_\phi = (\delta_{i,n})$ such that $\delta_{i,n} = \sum_j \sum_m \phi_{(j,m)} p_{(j,m)(i,n)}$ (where $p_{(j,m)(i,n)}$ are one-step transition probabilities of the considered chain).

Theorem 5.1. *The transition operator of the embedded Markov chain $\{X_l, l \in \mathbb{N}\}$ is monotone with respect to the order $\leq_{st}$, that is, for any two distributions $\phi^{(1)}$ and $\phi^{(2)}$, the inequality $\phi^{(1)} \leq_{st} \phi^{(2)}$ implies that $T\phi^{(1)} \leq_{st} T\phi^{(2)}$.*

Proof. From [27], the transition operator T is monotone with respect to $\leq_{st}$ if and only if

$$\overline{p}_{(j-1,m-1)(i,n)} \leq \overline{p}_{(j,m)(i,n)} \text{ for all } i, n,\ j > 0 \text{ and } m > 0. \tag{5.1}$$

To prove (5.1), we have

$$\overline{p}_{(j,m)(i,n)} = \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} p_{(j,m)(l,h)} = \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} k_{l-j+1,h-m}.$$

Consequently,

$$\begin{aligned}
\overline{p}_{(j,m)(i,n)} \quad - \quad \overline{p}_{(j-1,m-1)(i,n)} &= \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} k_{l-j+1,h-m} - \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} k_{l-j+2,h-m+1} \\
&= \sum_{l=i}^{\infty}\left[k_{l-j+1,n-m} + \sum_{h=n+1}^{\infty} k_{l-j+1,h-m}\right] - \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} k_{l-j+2,h-m+1} \\
&= k_{i-j+1,n-m} + \sum_{l=i+1}^{\infty}\sum_{h=n+1}^{\infty} k_{l-j+1,h-m} - \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} k_{l-j+2,h-m+1} \\
&= k_{i-j+1,n-m} \ \geq \ 0.
\end{aligned}$$

Finally, T is monotone with respect to the stochastic ordering ($\leq_{st}$). □

Theorem 5.2. *The transition operator of the embedded Markov chain* $\{X_l, l \in \mathbb{N}\}$ *is monotone with respect to* $\leq_{icx}$*, that is, for any two distributions* $\phi^{(1)}$ *and* $\phi^{(2)}$*, the inequality* $\phi^{(1)} \leq_{icx} \phi^{(2)}$ *implies that* $T\phi^{(1)} \leq_{icx} T\phi^{(2)}$*.*

Proof. The transition operator T is monotone with respect to $\leq_{icx}$ if and only if

(5.2) $\quad 2\overline{\overline{p}}_{(j,m)(i,n)} \leq \overline{\overline{p}}_{(j-1,m-1)(i,n)} + \overline{\overline{p}}_{(j+1,m+1)(i,n)}, \forall i, n,$ and $j > 0, m > 0.$

To prove (5.2), we have

$$\overline{\overline{p}}_{(j,m)(i,n)} \quad = \quad \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{p}_{(j,m)(l,h)} = \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j+1,h-m}.$$

Thus,

$$\begin{aligned}
&\overline{\overline{p}}_{(j-1,m-1)(i,n)} + \overline{\overline{p}}_{(j+1,m+1)(i,n)} - 2\overline{\overline{p}}_{(j,m)(i,n)} = \\
&= \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j+2,h-m+1} + \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j,h-m-1} - 2\sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j+1,h-m} \\
&= \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j+2,h-m+1} + \overline{k}_{i-j,n-m-1} - \sum_{l=i}^{\infty}\sum_{h=n}^{\infty} \overline{k}_{l-j+1,h-m} \\
&= \overline{k}_{i-j,n-m-1} - \overline{k}_{i-j+1,n-m} = k_{i-j,n-m-1} \ \geq \ 0.
\end{aligned}$$

Finally, T is monotone with respect to the increasing convex ordering. □

In particular, this theorem implies that if at time $t = 0$ the system was empty then the number of customers in the system, at departure times, form a monotonically increasing sequence with respect to the above orderings.

Remark 5.3. The operator T is not monotone with respect to the Laplace ordering ($\leq_L$). Indeed, for a distributions $\phi^{(1)} = (1, 0, 0, ...)$ and $\phi^{(2)} = (0, 1, 0, ...)$, we have $\phi^{(1)} \leq_L \phi^{(2)}$ but $T\phi^{(1)} \nleq_L T\phi^{(2)}$.

Now we add the transition operators $T^{(1)}$ and $T^{(2)}$ to models $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively. Theorems 5.4–5.5 give comparability conditions of two transition operators.

Theorem 5.4. *If $\lambda^{(1)} \le \lambda^{(2)}$, $p^{(1)} = p^{(2)}$, $A^{(1)} \le_L A^{(2)}$ and $B^{(1)} \le_s B^{(2)}$, where $\le_s$ is either $\le_{st}$ or $\le_{icx}$, then $T^{(1)} \le_s T^{(2)}$, that is, for any distribution ϕ, one has $T^{(1)}\phi \le_s T^{(2)}\phi$.*

Proof. The demonstration is based on [27, Theorem 4.2.3]. We want to establish that

$$\overline{p}^{(1)}_{(j,m)(i,n)} \le \overline{p}^{(2)}_{(j,m)(i,n)}, \forall i,n, 0 \le j \le i+1, 0 \le m \le n, \text{ (for } \le_s = \le_{st}), \tag{5.3}$$

$$\overline{\overline{p}}^{(1)}_{(j,m)(i,n)} \le \overline{\overline{p}}^{(2)}_{(j,m)(i,n)}, \forall i,n, 0 \le j \le i+1, 0 \le m \le n, \text{ (for } \le_s = \le_{icx}). \tag{5.4}$$

To prove inequality (5.3), we have

Case 1: If $j = m = 0$, then we have

$$\overline{p}^{(1)}_{(0,0)(i,n)} = \sum_{l\ge i}\sum_{h\ge n} p^{(1)}_{(0,0)(l,h)} = \overline{k}^{(1)}_{i,n} \le \overline{k}^{(2)}_{i,n} = \overline{p}^{(2)}_{(0,0)(i,n)}. \tag{5.5}$$

Case 2: If $1 \le j \le i+1$ and $0 \le m \le n$, then we obtain

$$\overline{p}^{(1)}_{(j,m)(i,n)} = \sum_{l\ge i}\sum_{h\ge n} p^{(1)}_{(j,m)(l,h)} = \overline{k}^{(1)}_{i-j+1,n-m} \le \overline{k}^{(2)}_{i-j+1,n-m} = \overline{p}^{(2)}_{(j,m)(i,n)}. \tag{5.6}$$

Inequalities (5.5)-(5.6) follow from Lemma 4.1 (for $\le_s = \le_{st}$).

Case 3: If $j = 0$ and $1 \le m \le n$, then we get

$$\begin{aligned}
\overline{p}^{(1)}_{(0,m)(i,n)} &= \sum_{l\ge i}\sum_{h\ge n} p^{(1)}_{(0,m)(l,h)} \\
&= (1-\alpha_{A^{(1)}}(\lambda^{(1)}))\overline{k}^{(1)}_{i,n-m} + \alpha_{A^{(1)}}\overline{k}^{(1)}_{i,n-m+1} \\
&= (1-\alpha_{A^{(1)}}(\lambda^{(1)}))[k^{(1)}_{i,n-m} + \overline{k}^{(1)}_{i,n-m+1}] + \alpha_{A^{(1)}}\overline{k}^{(1)}_{i,n-m+1} \\
&= (1-\alpha_{A^{(1)}}(\lambda^{(1)}))k^{(1)}_{i,n-m} + \overline{k}^{(1)}_{i,n-m+1}. \qquad (5.7)
\end{aligned}$$

Since $\lambda^{(1)} \le \lambda^{(2)}$ and $A^{(1)} \le_L A^{(2)}$, then $\alpha_{A^{(1)}}(\lambda^{(1)}) \ge \alpha_{A^{(2)}}(\lambda^{(2)})$ and

$$\overline{p}^{(1)}_{(0,m)(i,n)} \le (1-\alpha_{A^{(2)}}(\lambda^{(2)}))k^{(1)}_{i,n-m} + \overline{k}^{(1)}_{i,n-m+1}.$$

Moreover, we have

$$(1-\alpha_{A^{(2)}}(\lambda^{(2)}))k^{(1)}_{i,n-m} + \overline{k}^{(1)}_{i,n-m+1} = (1-\alpha_{A^{(2)}}(\lambda^{(2)}))\overline{k}^{(1)}_{i,n-m} + \alpha_{A^{(2)}}(\lambda^{(2)})\overline{k}^{(1)}_{i,n-m+1}.$$

By Lemma 4.1 (for $\le_s = \le_{st}$), we have $\overline{k}^{(1)}_{m,n} \le \overline{k}^{(2)}_{m,n}$, $\forall m \ge 0,\ n \ge 0$. Finally, we get:

$$\overline{p}^{(1)}_{(0,m)(i,n)} \le (1-\alpha_{A^{(2)}}(\lambda^{(2)}))\overline{k}^{(2)}_{i,n-m} + \alpha_{A^{(2)}}(\lambda^{(2)})\overline{k}^{(2)}_{i,n-m+1} = \overline{p}^{(2)}_{(0,m)(i,n)}.$$

Following the technique above and using Lemma 4.1 (for $\leq_s = \leq_{icx}$), we establish inequality (5.4). □

Theorem 5.5. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$, $A^{(1)} \leq_L A^{(2)}$ and $B^{(1)} \leq_{st} B^{(2)}$, then $T^{(1)} \leq_L T^{(2)}$, that is, for any distribution ϕ, we have $T^{(1)}\phi \leq_L T^{(2)}\phi$.*

Proof. To prove that $T^{(1)}\phi \leq_L T^{(2)}\phi$, we have to establish the usual numerical inequality

$$\varphi^{(1)}(z_1, z_2) \geq \varphi^{(2)}(z_1, z_2), \tag{5.8}$$

where $\varphi(z_1, z_2) = \sum_{i\geq 0}\sum_{n\geq 0} \delta_{i,n} z_1^i z_2^n$.

Let $\phi = (\phi_{(j,m)})$ be a distribution and $T_\phi = \delta = \{\delta_{i,n}\}$, where

$$\begin{aligned}
\delta_{i,n} &= \sum_{j\geq 0}\sum_{m\geq 0} \phi_{(j,m)} p_{(j,m)(i,n)} \\
&= \sum_{j\geq 0} \phi_{(j,0)} p_{(j,0)(i,n)} + \sum_{j\geq 0}\sum_{m\geq 1} \phi_{(j,m)} p_{(j,m)(i,n)} \\
&= \phi_{(0,0)} p_{(0,0)(i,n)} + \sum_{j\geq 1} \phi_{(j,0)} p_{(j,0)(i,n)} + \sum_{j\geq 0}\sum_{m\geq 1} \phi_{(j,m)} p_{(j,m)(i,n)} \\
&= \phi_{(0,0)} k_{i,n} + \sum_{j\geq 1} \phi_{(j,0)} k_{i-j+1,n} + \sum_{j\geq 0}\sum_{m\geq 1} \phi_{(j,m)} p_{(j,m)(i,n)} \\
&= \phi_{(0,0)} k_{i,n} + \sum_{j\geq 1} \phi_{(j,0)} k_{i-j+1,n} + \phi_{(0,0)} p_{(0,0)(i,n)} + \sum_{m=1}^{n} \phi_{(0,m)} p_{(0,m)(i,n)} \\
&\quad + \sum_{j\geq 1}\sum_{m=0}^{n} \phi_{(j,m)} p_{(j,m)(i,n)} + \phi_{(0,n+1)} p_{(0,n+1)(i,n)} - \sum_{j\geq 0} \phi_{(j,0)} p_{(j,0)(i,n)}.
\end{aligned}$$

After algebraic manipulation, we obtain

$$\begin{aligned}
\delta_{i,n} &= [1-\alpha_A(\lambda)] \sum_{m=1,n\neq 0}^{n} \phi_{(0,m)} k_{i,n-m} + \alpha_A(\lambda) \sum_{m=1}^{n} \phi_{(0,m)} k_{i,n-m+1} \\
&\quad + \sum_{j=1}^{i+1}\sum_{m=0}^{n} \phi_{(j,m)} k_{i-j+1,n-m} + \phi_{(0,n+1)} \alpha_A(\lambda) k_{i,0} + \phi_{(0,0)} k_{i,n}.
\end{aligned}$$

Let $k(z_1, z_2) = \sum_{i\geq 0}\sum_{n\geq 0} k_{i,n} z_1^i z_2^n$ and $\phi(z_1, z_2) = \sum_{i\geq 0}\sum_{n\geq 0} \phi(i,n) z_1^i z_2^n$ be the generating functions of $(k_{i,n})$ and $(\phi(i,n))$, respectively, and the auxiliary generating functions are defined by

$$\phi(z_2) = \sum_{n\geq 0} \phi(0,n) z_2^n, \;\; k(z_1) = \sum_{i\geq 0} k_{i,0} z_1^i.$$

Taking into account the previous generating functions, the generating function of T_ϕ is given by

$$\begin{aligned}\varphi(z_1,z_2) &= \sum_{i\geq 0}\sum_{n\geq 0}\delta_{i,n}z_1^i z_2^n\\ &= \phi_{(0,0)}k(z_1,z_2)+[1-\alpha_A(\lambda)]\sum_{i\geq 0}\sum_{n\geq 1}\sum_{m=1}^{n}\phi_{(0,m)}k_{i,n-m}z_1^i z_2^n\\ &\quad+\alpha_A(\lambda)\sum_{i\geq 0}\sum_{n\geq 0}\sum_{m=1}^{n}\phi_{(0,m)}k_{i,n-m+1}z_1^i z_2^n\\ &\quad+\sum_{i\geq 0}\sum_{n\geq 0}\sum_{j=1}^{i+1}\sum_{m=0}^{n}\phi_{(j,m)}k_{i-j+1,n-m}z_1^i z_2^n\\ &\quad+\alpha_A(\lambda)\sum_{i\geq 0}\sum_{n\geq 0}\phi_{(0,n+1)}k_{i,0}z_1^i z_2^n\\ &= \frac{\alpha_A(\lambda)}{z_2}k(z_1)\phi(z_2)+k(z_1,z_2)\alpha_A(\lambda)\phi_{(0,0)}\\ &\quad+k(z_1,z_2)\left[\frac{\phi(z_1,z_2)}{z_1}+\frac{\phi(z_2)}{z_2}\{z_2+(1-z_2)\alpha_A(\lambda)\}\right].\end{aligned}$$

If the conditions of Theorem 5.5 are fulfilled, then $k^{(1)}(z_1,z_2)\geq k^{(2)}(z_1,z_2)$ by Lemma 4.2 and $(1-z_2)\alpha_{A^{(1)}}(\lambda^{(1)})\geq(1-z_2)\alpha_{A^{(2)}}(\lambda^{(2)})$, $\forall\, z_2\in[0,1]$. One can see that inequality (5.8) takes place. □

6. Stochastic bound for the stationary distribution

Suppose once more that we have two models $\Sigma^{(1)}$ and $\Sigma^{(2)}$ as defined in the previous section. Let $\{X_l^{(1)}, l\in\mathbb{N}\}$, $\{X_l^{(2)}, l\in\mathbb{N}\}$ be the corresponding embedded Markov chains as well as their stationary distributions $\{\pi_{(i,n)}^{(1)}\}$, $\{\pi_{(i,n)}^{(2)}\}$, respectively.

Theorem 6.1. *The inequalities $\lambda^{(1)}\leq\lambda^{(2)}$, $p^{(1)}=p^{(2)}$, $A^{(1)}\leq_L A^{(2)}$ and $B^{(1)}\leq_s B^{(2)}$, where $\leq_s$ is either $\leq_{st}$ or $\leq_{icx}$, imply that $\{\pi_{(i,n)}^{(1)}\}\leq_s\{\pi_{(i,n)}^{(2)}\}$.*

Proof. By Theorem 5.4, the inequalities $\lambda^{(1)}\leq\lambda^{(2)}$, $p^{(1)}=p^{(2)}$, $A^{(1)}\leq_L A^{(2)}$ and $B^{(1)}(x)\leq_s B^{(2)}(x)$, imply that $T^{(1)}\leq_s T^{(2)}$, i.e. for any distribution ϕ we have the following inequality

$$T^{(1)}\phi\leq_s T^{(2)}\phi. \tag{6.1}$$

According to Theorem 5.1 and Theorem 5.2, the operator $T^{(2)}$ is monotone, i.e. for any two distributions $\phi_1^{(2)},\phi_2^{(2)}$ such that $\phi_1^{(2)}\leq_s\phi_2^{(2)}$, we have

$$T^{(2)}\phi_1^{(2)}\leq_s T^{(2)}\phi_2^{(2)}. \tag{6.2}$$

Moreover, from (6.1), one can obtain

$$T^{(1)}\phi^{(1)} \leq_s T^{(2)}\phi^{(1)}. \tag{6.3}$$

There exists a probability $\phi_1^{(2)}$ such that the inequality

$$T^{(2)}\phi^{(1)} \leq_s T^{(2)}\phi_1^{(2)}, \tag{6.4}$$

takes place.

From (6.2)-(6.4), for any two distributions $\phi^{(1)}, \phi^{(2)}$ one can obtain the following result

$$T^{(1)}\phi^{(1)} \leq_s T^{(2)}\phi^{(2)}.$$

Therefore,

$$T^{(1)}\phi^{(1)}_{(i,n)} = P(Q^{(1)}_{1,l} = i,\ Q^{(1)}_{2,l} = n) \quad \leq_s \quad P(Q^{(2)}_{1,l} = i,\ Q^{(2)}_{2,l} = n) = T^{(2)}\phi^{(2)}_{(i,n)},$$

when $l \longrightarrow \infty$, we have $\{\pi^{(1)}_{(i,n)}\} \leq_s \{\pi^{(2)}_{(i,n)}\}$. □

Based on Theorem 6.1 we can establish insensitive stochastic bounds for the stationary distribution of the number of customers in the system.

Theorem 6.2. *If in the $M/G/1$ retrial queue with general retrial times and Bernoulli schedule, the service time distribution $B(x)$ is HNBUE and the retrial time distribution is of class $\mathcal{L}$, then $\{\pi_{(i,n)}\} \leq_{icx} \{\pi^*_{(i,n)}\}$, where $\{\pi^*_{(i,n)}\}$ is the stationary distribution of the number of customers in the $M/M/1$ retrial queue with exponential retrial times and Bernoulli schedule with the same parameters as those of the $M/G/1$ retrial queue with general retrial times and Bernoulli schedule.*

Proof. Denote by $\Sigma^{(1)}$ our system defined in Section 2 (ie. $M/G/1$ retrial queue with general retrial times and Bernoulli schedule) with parameters $A^{(1)} \equiv A,\ B^{(1)} \equiv B,\ \lambda^{(1)} = \lambda,\ p^{(1)} = p,\ \alpha_1^{(1)} = \alpha_1$ and $\beta_1^{(1)} = \beta_1$.

On the other hand, let $\Sigma^{(2)}$ an auxiliary $M/M/1$ retrial queue with exponential retrial times and Bernoulli schedule having the same arrival rate $\lambda^{(2)} = \lambda$, retrial rate $\alpha_1^{(2)} = \alpha_1$, probability $p^{(2)} = p$ and mean service $\beta_1^{(2)} = \beta_1$ as in $\Sigma^{(1)}$ system, but with $B^{(2)} \equiv B^*$ and $A^{(2)} \equiv A^*$ where,

$$B^*(x) = \begin{cases} 1 - e^{-\frac{x}{\beta_1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

$$A^*(x) = \begin{cases} 1 - e^{-\frac{x}{\alpha_1}}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

If $B(x)$ is $HNBUE$, then $B(x) \leq_{icx} B^*(x)$ and, if A is of class $\mathcal{L}$, then $A \leq_L A^*$.

Moreover, the following conditions of Theorem 6.1 are satisfied:
$\lambda^{(1)} = \lambda^{(2)}$, $p^{(1)} = p^{(2)}$, $B^{(1)}(x) \leq_{icx} B^{(2)}(x)$ and $A^{(1)}(x) \leq_L A^{(2)}(x)$.

Thus, the stationary distribution in the $M/G/1$ retrial queue with general retrial times and Bernoulli schedule is less than the corresponding distribution in the $M/M/1$ retrial queue with exponential retrial times and Bernoulli schedule, if $B(x)$ is $HNBUE$ and $A(x)$ is of class $\mathcal{L}$. □

7. Bounds for the mean characteristics of the system

In this section, we show how the theoretical results obtained in the previous section can be used to derive some bounds for the mean characteristics of our considered model. To do so, consider two models $\Sigma^{(1)}$ and $\Sigma^{(2)}$ as defined previously. Let $E(N_2)$, $E(N)$, W_2 and W be the mean number of customers in the retrial group, the mean number of customers in the system, the mean waiting time in the retrial group and the mean time a customer spends in the system (including the service time) respectively. The explicit expressions for these performance measures were obtained in the literature by Atencia and Moreno [4]. That is, under the condition $\rho < \alpha_A(\lambda)(p+q\alpha_A(\lambda))^{-1}$, the authors have given the following results:

$$\begin{aligned}
E(N_2) &= \frac{\lambda p}{2(1-\rho q)}\frac{2\beta_1(1-\rho q)[1-\alpha_A(\lambda)]+\lambda\beta_2[p+q\alpha_A(\lambda)]}{\alpha_A(\lambda)-[p+q\alpha_A(\lambda)]\rho}.\\
E(N) &= \rho+\frac{\lambda^2\beta_2}{2(1-\rho)}+\frac{p[1-\alpha_A(\lambda)]}{\alpha_A(\lambda)-[p+q\alpha_A(\lambda)]\rho}\left[\rho+\frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right].\\
W_2 &= \frac{2\beta_1(1-\rho q)[1-\alpha_A(\lambda)]+\lambda\beta_2[p+q\alpha_A(\lambda)]}{2(1-\rho q)[\alpha_A(\lambda)-[p+q\alpha_A(\lambda)]\rho]}.\\
W &= \beta_1+\frac{\lambda\beta_2}{2(1-\rho)}+\frac{p[1-\alpha_A(\lambda)]}{\alpha_A(\lambda)-[p+q\alpha_A(\lambda)]\rho}\left[\beta_1+\frac{\lambda p\beta_2}{2(1-\rho)(1-\rho q)}\right].
\end{aligned}$$

Theorem 7.1. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$, $B^{(1)} \leq_s B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$E(N^{(1)}) \leq E(N^{(2)}),\ \textit{and}\ E(N_2^{(1)}) \leq E(N_2^{(2)}),$$

where $\leq_s$ is one of the symbols $\leq_{st}$, $\leq_{icx}$, $\leq_L$.

Proof. The quantities $E(N)$ and $E(N_2)$ are increasing with respect to λ, β_1 and β_2, and decreasing with respect to $\alpha_A(.)$. Under conditions of Theorem 7.1, we obtain the desired inequalities. Recall that $X \leq_s Y$ implies $E(X^n) \leq E(Y^n)$ for all n. □

Theorem 7.2. *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} = p^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$W_2^{(1)} \leq W_2^{(2)},\ \textit{and}\ W^{(1)} \leq W^{(2)}.$$

Proof. The quantities W_2 and W are increasing with respect to λ, β_1 and β_2, decreasing with respect to $\alpha_A(.)$. Under the conditions of Theorem 7.2 we obtain the desired inequalities. □

Theorem 7.3. *For any $M/G/1$ retrial queue with general retrial times and Bernoulli schedule*

$$E(N) \leq \rho + \frac{\lambda^2\beta_2}{2(1-\rho)} + \frac{p[1-e^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}\left[\rho + \frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right]. \tag{7.1}$$

If A and B are of class $\mathcal{L}$, then the mean number of customers in the system is bounded as follows

$$E(N)_{Lower} \leq E(N) \leq E(N)_{Upper}, \tag{7.2}$$

where the lower and upper bounds are given respectively by

$$E(N)_{Lower} = \rho + \frac{\lambda^2\beta_2}{2(1-\rho)} + \frac{\lambda p\alpha_1}{1-[q+(\lambda\alpha_1+1)p]\rho}\left[\rho + \frac{\lambda^2 p\beta_2}{2(1-\rho)(1-\rho q)}\right],$$

$$E(N)_{Upper} = \rho + \frac{2\lambda^2\beta_1^2}{2(1-\rho)} + \frac{p[1-e^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}\left[\rho + \frac{2\lambda^2 p\beta_1^2}{2(1-\rho)(1-\rho q)}\right].$$

Proof. For the class of distribution functions with mean m, θ_m is its $\leq_L$-maximum, i.e., $F \leq \theta_m$, where θ_m is the Dirac distribution at m. If $F \in \mathcal{L}$ then $e^{-1/m}$ is its $\leq_L$-minimum, i.e., $e^{-1/m} \leq_L F$ [27]. In our case, consider auxiliary $M/D/1$ and $M/M/1$ retrial queues with the same arrival rates λ, mean service times β_1 and mean retrial times α_1. A is a Dirac distribution at α_1 for the $M/D/1$ system, and is an exponential distribution for the $M/M/1$ system. Using the Theorem 7.1 we obtain the stated results. Recall that if B is of class $\mathcal{L}$ then $\beta_2 \leq 2\beta_1^2$. □

Remark 7.4. Inequality (7.1) gives upper bound on the mean number of customers in the retrial group when the retrial time and service time distributions are unknown, but we have partial information about the first two moments. For the second inequality (7.2) we use the partial information about the ageing class of the retrial time and service time distributions.

Theorem 7.5. *For any $M/G/1$ retrial queue with general retrial times and Bernoulli schedule*

$$E(N_2) \leq \frac{\lambda p}{2(1-\rho q)} \frac{2\beta_1(1-\rho q)[1-e^{-\lambda\alpha_1}] + \lambda\beta_2[p+qe^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}. \tag{7.3}$$

If A and B are of class $\mathcal{L}$, then the mean number of customers in the retrial group is bounded as follows

$$E(N_2)_{Lower} \leq E(N_2) \leq E(N_2)_{Upper}, \tag{7.4}$$

where the lower and upper bounds are given respectively by

$$E(N_2)_{Lower} = \frac{\lambda p}{2(1-\rho q)} \frac{2\beta_1(1-\rho q)\lambda\alpha_1 + \lambda\beta_2[(\lambda\alpha_1+1)p+q]}{1-[q+(\lambda\alpha_1+1)p]\rho},$$

$$E(N_2)_{Upper} = \frac{\lambda p}{2(1-\rho q)} \frac{2\beta_1(1-\rho q)[1-e^{-\lambda\alpha_1}] + 2\lambda\beta_1^2[p+qe^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}.$$

Theorem 7.6. *For any $M/G/1$ retrial queue with general retrial times and Bernoulli schedule*

$$W_2 \leq \frac{2\beta_1(1-\rho q)[1-e^{-\lambda\alpha_1}] + \lambda\beta_2[p+qe^{-\lambda\alpha_1}]}{2(1-\rho q)[e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho]}. \tag{7.5}$$

If A and B are of class $\mathcal{L}$, then the mean waiting time in the retrial group is bounded as follows

$$W_{2,Lower} \leq W_2 \leq W_{2,Upper}, \tag{7.6}$$

where the lower and upper bounds are given respectively by

$$\begin{aligned} W_{2,Lower} &= \frac{2\beta_1(1-\rho q)\lambda\alpha_1 + \lambda\beta_2[q+(\lambda\alpha_1+1)p]}{2(1-\rho q)[1-(q+(\lambda\alpha_1+1)p)\rho]}, \\ W_{2,Upper} &= \frac{2\beta_1(1-\rho q)[1-e^{-\lambda\alpha_1}] + 2\lambda\beta_1^2[p+qe^{-\lambda\alpha_1}]}{2(1-\rho q)[e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho]}. \end{aligned}$$

Theorem 7.7. *For any $M/G/1$ retrial queue with general retrial times and Bernoulli schedule*

$$W \leq \beta_1 + \frac{\lambda\beta_2}{2(1-\rho)} + \frac{p[1-e^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}\left[\beta_1 + \frac{\lambda p\beta_2}{2(1-\rho)(1-\rho q)}\right]. \tag{7.7}$$

If A and B are of class $\mathcal{L}$, then the mean time a customer spends in the system is bounded as follows

$$W_{Lower} \leq W \leq W_{Upper}, \tag{7.8}$$

where the lower and upper bounds are given respectively by

$$\begin{aligned} W_{Lower} &= \beta_1 + \frac{\lambda\beta_2}{2(1-\rho)} + \frac{\lambda p\alpha_1}{1-(q+(\lambda\alpha_1+1)p)\rho}\left[\beta_1 + \frac{\lambda p\beta_2}{2(1-\rho)(1-\rho q)}\right], \\ W_{Upper} &= \beta_1 + \frac{2\lambda\beta_1^2}{2(1-\rho)} + \frac{p[1-e^{-\lambda\alpha_1}]}{e^{-\lambda\alpha_1} - [p+qe^{-\lambda\alpha_1}]\rho}\left[\beta_1 + \frac{2\lambda p\beta_1^2}{2(1-\rho)(1-\rho q)}\right]. \end{aligned}$$

Proof. The proof of Theorems 7.5, 7.6 and 7.7 is similar to that of Theorem 7.3 and thus omitted. □

8. Numerical example

In this section, we give a numerical illustration concerning the mean number of customers in the $M/G/1$ retrial queue with general retrial times and Bernoulli schedule given in Theorem 7.3. To this end, for the service time distribution $B(X)$, we choose exponential (exp), two-stage Erlang (E_2) and Weibull (Wbl) laws; whereas for the retrial time distribution $A(X)$, we consider exponential (exp), two-stage Erlang (E_2) and two-stage hyper-exponential (H_2) distributions. The distributions in question are the most representative. The exact values of the mean number of customers in the system, $E(N)_{(B(X),A(X))}$ (where $B(X)$ and $A(X)$ are of class $\mathcal{L}$), the upper bound $E(N)_{Upper}$ and the lower bound $E(N)_{Lower}$ are represented:

(1) In Figure 1, against β_1 for fixed values of p ($p = 0.5$), λ ($\lambda = 1$) and α_1 ($\alpha_1 = 2$) and the service time distribution $B(X)$ is a Weibull law.
(2) In Figure 2 the quantities $E(N)_{(Exp,A(X))}$, $E(N)_{Upper}$ and $E(N)_{Lower}$ are plotted against the retrial rate $1/\alpha_1$ for fixed values of p ($p = 0.5$), λ ($\lambda = 1$) and β_1 ($\beta_1 = 0.5$).
(3) In Figure 3 the quantities $E(N)_{(Exp,A(X))}$, $E(N)_{Upper}$ and $E(N)_{Lower}$ are plotted against the arrivals rate λ for fixed values of p ($p = 0.5$), β_1 ($\beta_1 = 1$) and α_1 ($\alpha_1 = 1$).
(4) In Figure 4 the quantities $E(N)_{(Exp,A(X))}$, $E(N)_{Upper}$ and $E(N)_{Lower}$ are plotted against the probability p for fixed values of λ ($\lambda = 0.5$), β_1 ($\beta_1 = 1$) and α_1 ($\alpha_1 = 1$).

Note that a similaire type of figures as Figures 1–3 can be obtained for other settings (when we change the service time distribution $B(x)$ or the value of the probability p). Another similar type of figures as Figure 4 can be obtained when changing the service time distribution $B(x)$.

For the above considered situations, we note that:

- The lower bound $E(N)_{Lower}$ is nothing else than the mean number of the customers $E(N)$ in the $M/G/1$ retrial queue with exponential retrial times and Bernoulli schedule.
- The inequality $E(N)_{Lower} \leq E(N)_{(B(x),A(x))} \leq E(N)_{Upper}$ holds.
- If the service rate $1/\beta_1$ or the retrial rate $1/\alpha_1$ are large enough then the mean number of customers in the system is closer to the $E(N)_{(B(x),Exp)}$, in other words, closer to the $E(N)_{Lower}$.
- If the distribution of the retrial time is close to the exponential distribution in the Laplace transform, then the exact value $E(N)_{(B(x),A(x))}$ is closer to the lower bound $E(N)_{Lower}$ (see the case of $E(N)_{((B(x),E_2))}$).
- If p is close to 0 (resp. close to 1), then our system tend to behave as a nominal $M/G/1$ queue (resp. an $M/G/1$ retrial queue with general retrial times). Alternatively, when $p = 1$, our model becomes an $M/G/1$ retrial queue with general retrial times, which was studied in [17].

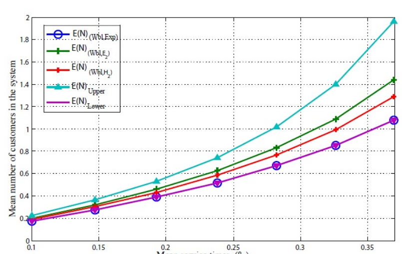

FIGURE 1. Bounds for $E(N)$ in $M/Wbl/1$ queue with general retrial times versus β_1

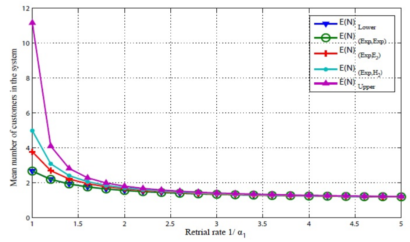

FIGURE 2. Bounds for $E(N)$ in $M/M/1$ queue with general retrial times versus $1/\alpha_1$

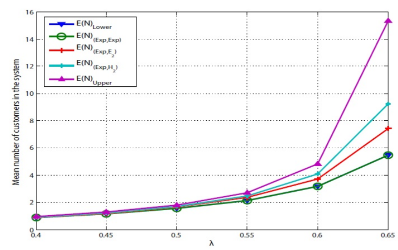

FIGURE 3. Bounds for $E(N)$ in $M/M/1$ queue with general retrial times versus λ

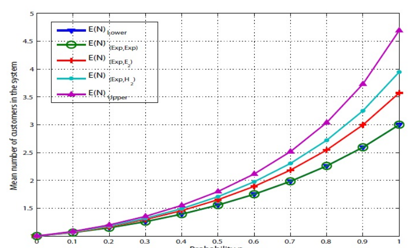

FIGURE 4. Bounds for $E(N)$ in $M/G/1$ queue with general retrial times versus p

References

[1] A.S. Alfa and W. Li, PCS networks with correlated arrival process and retrial phenomenon, *IEEE Trans. Wireless Commun.* **1** (2000) 630–637.

[2] J.R. Artalejo and A. Gómez-Corral, Modelling communication systems with phase type service and retrial times, *IEEE Commun. Lett.* **11** (2007) 955–957.

[3] J.R. Artalejo and A. Gómez-Corral, Retrial Queueing System: A Computational Approach, Springer-Verlag, Berlin, Heidelberg, 2008.

[4] I. Atencia and P. Moreno, A single-server retrial queue with general retrial times and Bernoulli schedule, *Appl. Math. Comput.* **162** (2005) 855–880.

[5] B.G. Bhaskaran, Almost sure comparison of birth and death processeswith application to M/M/s queueing systems, *Queueing Syst.* **1** (1986) 103–127.

[6] M. Boualem, Insensitive bounds for the stationary distribution of a single server retrial queue with server subject to active breakdowns, *Adv. Oper. Res.* (2014), Article ID 985453, 12 pages.

[7] M. Boualem, M. Cherfaoui and D. Aïssani, Monotonicity properties for a single server queue with classical retrial policy and service interruptions, *Proc. Jangjeon Math. Soc.* **19** (2016), no. 2, 225–236.

[8] M. Boualem, N. Djellab and D. Aïssani, Stochastic inequalities for $M/G/1$ retrial queues with vacations and constant retrial policy, *Math. Comput. Modelling* **50** (2009), no. 1-2, 207–212.

[9] M. Boualem, N. Djellab and D. Aïssani, Stochastic approximations and monotonicity of a single server feedback retrial queue, *Math. Probl. Eng.* (2012), Article ID 536982, 13 pages

[10] M. Boualem, N. Djellab and D. Aïssani, Stochastic bounds for a single server queue with general retrial times, *Bull. Iranian Math. Soc.* **40** (2014), no. 1, 183–198.

[11] H. Castel-Taleb and N. Pekergin, Stochastic monotonicity in queueing networks, in: Computer Performance Engineering, pp. 116–130, Lecture Notes in Comput. Sci. 5652, Springer, Berlin, Heidelberg, 2009.

[12] B.D. Choi and Y. Chang, Single server retrial queues with priority calls, *Math. Comput. Modelling* **30** (1999), no. 3-4, 7–32.

[13] B.D. Choi and K.K. Park, The M/G/1 retrial queue with Bernoulli schedule, *Queueing Syst.* **7** (1990), no. 2, 219–227.

[14] G.I. Falin, Single line repeated orders queueing systems, *Optimization* **17** (1986), no. 5, 649–667.

[15] G.I. Falin, J.R. Artalejo and M. Martin, On the single server retrial queue with priority customers, *Queueing Syst.* **14** (1993), no. 3-4, 439–455.

[16] G.I. Falin, and J.G.C. Templeton, Retrial Queues, Chapman and Hall, London, 1997.

[17] A. Gómez-Corral, Stochastic analysis of a single server retrial queue with general retrial times, *Naval Res. Logist.* **46** (1999), no. 5, 561–581.

[18] Q.M. He, H. Li and Y.Q. Zhao, Ergodicity of the $BMAP/PH/s/s+K$ retrial queue with PH-retrial times, *Queueing Syst.* **35** (2000), no. 1-4, 323–347.

[19] H.M. Liang, Service station factors in monotonicity of retrial queues, *Math. Comput. Modelling* **30** (1999), no. 3-4, 189–196.

[20] H.M. Liang and V.G. Kulkarni, Monotonicity properties of single server retrial queues, *Stoch. Models* **9** (1993), no. 3, 373–400.

[21] V.A. Kapyrin, A study of stationary distributions of a queueing system with recurring demands, *Cybernatics* **13** (1977) 548–590.

[22] Z. Khalil and G. Falin, Stochastic inequalities for M/G/1 retrial queues, *Oper. Res. Lett.* **16** (1994) 285–290.

[23] W.A. Massey, Stochastic orderings for Markov processes on partially ordered spaces, *Math. Oper. Res.* **12** (1987), no. 2, 350–367.
[24] A. Müller and D. Stoyan, Comparison Methods for Stochastic Models and Risks, John Wiley & Sons, LTD, 2002.
[25] M. Shaked and J.G. Shanthikumar, Stochastic Orders and Their Applications, Academic Press, San Diego, 1994.
[26] M. Shaked and J.G. Shanthikumar, Stochastic Orders, Springer-Verlag, New York, 2007.
[27] D. Stoyan, Comparison Methods for Queues and Other Stochastic Models, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Chichester, 1983.
[28] R. Szekli, Stochastic Ordering and Dependence in Applied Probability, Lecture Notes in Statist. 97, Springer-Verlag, New York, 1995.
[29] I.M. Verloop, U. Ayesta and S. Borst, Monotonicity properties for multi-class queueing systems, *Discrete Event Dyn. Syst.* **20** (2010), no. 4, 473–509.
[30] T. Yang, M.J.M. Posner, J.G.C. Templeton and H. Li, An approximation method for the M/G/1 retrial queue with general retrial times, *European J. Oper. Res.* **76** (1994) 552–562.
[31] T. Yang and J.G.C. Templeton, A survey on retrial queues, *Queueing Syst.* **2** (1997) 201–233.

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