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AN ALGORITHM FOR APPROXIMATING NONDOMINATED POINTS OF CONVEX MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this paper, we present an algorithm for generating approximate nondominated points of a multiobjective optimization problem (MOP), where the constraints and the objective functions are convex. We provide outer and inner approximations of nondominated points and prove that inner approximations provide a set of approximate weakly nondominated points. The proposed algorithm can be applied for differentiable or nondifferentiable convex MOPs. To illustrate efficiency of the proposed algorithm for convex MOPs, we provide numerical examples.
Keywords: Multiobjective optimization, convexity, nondominated point, efficient solution, approximation algorithm, differentiable problem.
MSC(2010): Primary: 90C29; Secondary: 90C30, 49M37.

1. Introduction

Multiobjective optimization is a research field that is concerned with optimization problems involving several conflicting objective functions to be minimized simultaneously. Various applications of multiobjective optimization have been reported in many areas of science, concerning especially engineering, economics, logistics and medicine. For studying multiobjective optimization we refer to [1,7,15,24,29,34] and the references therein.

Generally, for a multiobjective optimization problem, there does not exist a unique solution that simultaneously optimizes each objective function, but a set of solutions can be identified, by using the concept of Pareto optimality. A Pareto optimal (efficient) solution is defined as a feasible solution for which none of the objective functions can be improved in value, without deterioration in at least one of the other objectives. The image of a Pareto optimal solution in the objective space is called a nondominated point.

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Researchers have developed a variety of methods to obtain Pareto optimal solutions of a given MOP [2, 6, 17, 18, 22, 26, 30]. But, since the number of objectives of an MOP is often much smaller than the number of variables and typically many efficient solutions in decision space are mapped to a single nondominated point in objective space, Benson [4, 5] argued that generating the nondominated set should require less computation than generating the efficient set. Therefore, finding the nondominated set in objective space instead of Pareto optimal set in decision space is more important for the decision maker. While it is theoretically possible to identify the complete set of Pareto optimal solutions, finding an exact description of this set often turns out to be practically impossible or at least computationally too expensive. Therefore, many researchers focus on approximate efficient solutions [3, 9–14, 16, 19–21, 23, 27, 28, 33, 35–38].

Ehrgott et al. [8] proposed an outer approximation algorithm for finding approximate nondominated points of a differentiable convex MOP. They utilized Benson's outer approximation algorithm [4, 5] and a linearization technique for their algorithm. In each iteration of their proposed algorithm a polyhedron is constructed by adding a new hyperplane to the previous polyhedron.
Now, in this paper, motivated by the algorithm given by Thuy et al. [31] for finding nondominated points of a convex MOP, we propose a modified algorithm for finding approximate nondominated points of a convex MOP. In each iteration of the algorithm, we obtain a reverse polyblock from the previous one by cutting out a box. We show that the proposed method yields a set of $\varepsilon-$nondominated points for the given convex MOP. The suggested algorithm works for nondifferentiable MOPs, too. Validity of the algorithm is verified with numerical examples.

The rest of this paper is organized as follows: In Section 2 preliminaries and basic definitions are given. In Section 3, the algorithm of Thuy et al. [31] is presented. Section 4 is devoted to the proposed algorithm for finding approximate nondominated points. In this section, we also prove theorems related to the given algorithm. In Section 5, numerical results are given to clarify the efficiency of the proposed algorithm. Finally, in Section 6 conclusions are given.

2. **Preliminaries and basic definitions**

Let $y^1, y^2 \in \mathbb{R}^p$. With regard to vector inequalities, the following convention will be applied: $y^1 \leqq y^2$ if and only if $y^1_i \leq y^2_i$, for all $i \in \{1, \dots, p\}$ and $y^1 \leq y^2$ if and only if $y^1 \leqq y^2$ and $y^1 \neq y^2$. Moreover, $y^1 < y^2$ if and only if $y^1_i < y^2_i$, for all $i \in \{1, \dots, p\}$. Due to the above-defined componentwise orders, we define $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$.

A multiobjective optimization problem (MOP) may be written as

$$\begin{aligned} \min f(x) &= (f_1(x), \ldots, f_p(x)) \\ s.t.\ x \in \mathcal{X} &= \{x \in \mathbb{R}^n :\ g(x) = (g_1(x), \ldots, g_m(x))^T \leqq 0\}, \end{aligned} \tag{2.1}$$

where $\mathcal{X}$ is the feasible set in the decision space $\mathbb{R}^n$. The image of $\mathcal{X}$ under f is denoted by $\mathcal{Y} = \{f(x) : x \in \mathcal{X}\}$ and is called the feasible set in the objective space $\mathbb{R}^p$. We assume that $\mathcal{X}$ is nonempty and compact.

Definition 2.1 ([24])**.** The multiobjective optimization problem (2.1) is convex if all its objective functions and its feasible region $\mathcal{X}$ are convex.

Definition 2.2 ([7])**.** A feasible solution $\hat{x} \in \mathcal{X}$ is called an efficient (a Pareto optimal) solution for MOP (2.1), if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. The set of all efficient solutions $\hat{x} \in \mathcal{X}$ will be denoted by $\mathcal{X}_E$ and called the efficient set in the decision space. If $\hat{x}$ is efficient, $\hat{y} = f(\hat{x})$ is called nondominated point and the set of all nondominated points in the objective space is denoted by $\mathcal{Y}_N$.

Definition 2.3 ([7])**.** A feasible solution $\hat{x} \in \mathcal{X}$ is called a weakly efficient solution for MOP (2.1), if there is no other $x \in \mathcal{X}$ such that $f(x) < f(\hat{x})$. The set of all weakly efficient solutions $\hat{x} \in \mathcal{X}$ will be denoted by $\mathcal{X}_{WE}$ and called the weakly efficient set in the decision space. If $\hat{x}$ is weakly efficient, $\hat{y} = f(\hat{x})$ is called weakly nondominated point and the set of all weakly nondominated points in the objective space is denoted by $\mathcal{Y}_{WN}$.

Definition 2.4 ([23])**.** Consider $\varepsilon \in \mathbb{R}^p_{\geqq}$. A feasible solution $\hat{x} \in \mathcal{X}$ is called ε-efficient for MOP (2.1), if there is no other $x \in \mathcal{X}$ where $f(x) \leq f(\hat{x}) - \varepsilon$. Also, $\hat{x}$ is ε-weakly efficient if there is no other $x \in \mathcal{X}$ with $f(x) < f(\hat{x}) - \varepsilon$.

Definition 2.5 ([7])**.** The point $y^m = (y_1^m, \ldots, y_p^m)$ in which $y_i^m = \min_{y \in \mathcal{Y}} y_i$, $i = 1, \ldots, p$, is called the ideal point of MOP (2.1).

It is obvious that if $y^m \in \mathcal{Y}$, then $\mathcal{Y}_N = \{y^m\}$ and the problem is solved. Therefore, we assume that $y^m \notin \mathcal{Y}$.

By definition, it is clear that the optimal objective value of the following problem is the kth component of the ideal point [31]:

$$\begin{aligned} y_k^m &= \min y_k \\ s.t.\ f_i(x) - y_i &\leq 0,\ i = 1, \ldots, p, \\ g_j(x) &\leq 0,\ j = 1, \ldots, m. \end{aligned} \tag{2.2}$$

By solving problem (2.2), for $k = 1, \ldots, p$, optimal objective values $y_1^m, \ldots, y_p^m$ are obtained. Assume that $(x^*, y^*) \in \mathbb{R}^{n+p}$ is an optimal solution of problem (2.2) with $y^* = (y_1^k, \ldots, y_p^k)$. As in [31], we define $y^M = (y_1^M, \ldots, y_p^M) \in \mathbb{R}^p$ as $y_i^M = \alpha$ for all $i = 1, \ldots, p$ where α is a real number satisfying:

$$\alpha > \max\{y_i^k | i = 1, \ldots, p,\ k = 1, \ldots, p\}. \tag{2.3}$$

3. The Thuy et al. algorithm for generating nondominated points

Thuy et al. [31] have proposed an algorithm to generate a number of nondominated points of a convex MOP. In this section, we summarize their algorithm.

Consider the multiobjective optimization problem (2.1), where $\mathcal{X} \subseteq \mathbb{R}^n$ is nonempty and compact. In this case, $\mathcal{Y}^0 = \mathcal{Y} + \mathbb{R}^p_{\geqq}$ is a full dimensional convex and nonempty set in $\mathbb{R}^p$. Furthermore, it is obvious that $\mathcal{Y}^0_N = \mathcal{Y}_N$ [15]. For the convex MOP (2.1), let

$$\mathcal{Y}' = \mathcal{Y}^0 \cap (y^M - \mathbb{R}^p_{\geqq}).$$

To find nondominated points of MOP (2.1), Thuy et al. [31] investigate $\mathcal{Y}'$ instead of $\mathcal{Y}$.

Theorem 3.1 ([31])**.** $\mathcal{Y}_N = \mathcal{Y}'_N$.

The set $B^0 = (y^m + \mathbb{R}^p_{\geqq}) \cap (y^M - \mathbb{R}^p_{\geqq})$ is called a block and briefly is denoted by $B^0 = [y^m, y^M]$. It is obvious that B^0 is a cover for $\mathcal{Y}'$, that is $\mathcal{Y}' \subseteq B^o$.

Theorem 3.2 ([31])**.** *Assume that $\bar{y} \in B^0 \setminus \mathcal{Y}'$. Let y^w be the unique point on the boundary of $\mathcal{Y}'$ that belongs to the line segment connecting $\bar{y}$ and y^M. Then, $y^w \in \mathcal{Y}'_N$.*

Consider the following optimization problem:

$$\begin{aligned} &\min \lambda \\ &\text{s.t.} \;\; f(x) - \lambda(y^M - \bar{y}) - \bar{y} \leq 0, \\ &\qquad g_i(x) \leq 0 \;\; i = 1, \ldots, m, \\ &\qquad 0 \leq \lambda \leq 1. \end{aligned} \tag{3.1}$$

If λ^* is the optimal objective value of problem (3.1), then $y^w = \lambda^* y^M + (1-\lambda^*)\bar{y}$ is the boundary point of $\mathcal{Y}'$ that belongs to the line segment connecting $\bar{y}$ and y^M.

For each nondominated point y^w (given in Theorem 3.2), we can determine p new points which belong to $B^0 \setminus \mathcal{Y}'$. To compute these new vertices, the cutting reverse polyblock technique [31, 32] is used. A set of the form $B = \cup_{y \in V}[y, y^M] \subset \mathbb{R}^p$, where $[y, y^M] := \{\hat{y} | y \leq \hat{y} \leq y^M\}$ and $V \subseteq B^0$, is called a reverse polyblock in B^0 with vertex set V.

Theorem 3.3 ([31])**.** *Let $B = \cup_{y \in V}[y, y^M]$ be a reverse polyblock with vertex set V and $\mathcal{Y}' \subseteq B$. Let $v = (v_1, \ldots, v_p) \in V \setminus \mathcal{Y}'$ and y^w be the unique point on the boundary of $\mathcal{Y}'$ that belongs to the line segment connecting v and y^M. Then, $\bar{B} = B \setminus [v, y^w]$ is a reverse polyblock with vertex set $\bar{V} = (V \setminus \{v\}) \cup \{v^1, v^2, \ldots, v^p\}$, where*

$$v^i = y^w - (y^w_i - v_i)e^i, \;\; i = 1, \ldots, p, \tag{3.2}$$

where e^i is the ith unit vector of $\mathbb{R}^p$.

In Algorithm 1, the Thuy et al. [31] algorithm for finding nondominated points of a convex MOP is given.

Step 1.: (Initialization) Construct $B^0 = [y^m, y^M]$. Set $\mathcal{Y}_N = \emptyset$, $\mathcal{X}_E = \emptyset, S = \{y^m\}$,
Nef = the number of needed elements, $B = B^0, k = 0$.

Step 2.: (Iteration).

(a) Set $\bar{S} = \emptyset$.

(b) for each $\bar{y} \in S$ do:

begin

$k := k + 1$.

Find an optimal solution $(x^*, \lambda^*) \in \mathbb{R}^{n+1}$ of problem (3.1) and set:

- $w^k = \bar{y} + \lambda^*(y^M - \bar{y})$,
- $\mathcal{Y}_N = \mathcal{Y}_N \cup \{w^k\}$, $\mathcal{X}_E = \mathcal{X}_E \cup \{x^*\}$,
- $\bar{B} = B \setminus [v, y^w]$ with $v = \bar{y}$ and $y^w = w^k$.
- Obtain p vertices $v^1, \ldots, v^p$ corresponding to y^w from relation (3.2).
- Set $\bar{S} = \bar{S} \cup \{v^1, \ldots, v^p\}$ and $B = \bar{B}$,

end.

(c) If $k \geq Nef$ then terminate the algorithm.
Else set $S = \bar{S}$ and return the Step 2.

Algorithm 1: Thuy et al. [31] algorithm for generating nondominated points

4. An approximation algorithm for Convex MOPs

For convex multiobjective optimization problems, finding the set of nondominated points has many difficulties. For example, the set of nondominated points may be empty or for a large scale nonlinear MOP, the numerical methods can not find nondominated points, easily. Therefore, in recent decades, many researchers have considered approximations of nondominated points. Ehrgott et al. [8], using a linearization technique, extended the outer approximation algorithm of Benson [4, 5] to find ε-nondominated points of a convex MOP. In their algorithm, separating hyperplanes are constructed in each iteration. In the suggested algorithm the objectives and constraints must be differentiable, and the algorithm is not applicable for nondifferentiable MOPs.

Now, in this section we provide an algorithm for finding ε-nondominated points of a convex MOP. The proposed algorithm works for every convex MOP even nondifferentiable MOPs. In the given algorithm, it is not necessary to construct separating hyperplanes and in each iteration, covers are constructed easily and the vertices of the covers can be updated. We construct inner and outer approximations of $\mathcal{Y}'$ and show that using inner approximation, we can find $\varepsilon-$nondominated points. To this end, we consider two sets I and O, with empty values in initialization. Let $\epsilon \in \mathbb{R}$, with $\epsilon > 0$ be a tolerance given by the decision maker (DM). If the Euclidean distance of the boundary point y^w and $\bar{y} \in S$ is lower than or equal to ϵ, then y^w is added to I and $\bar{y}$ is added to O. Otherwise, cutting reverse polyblock technique [32] is utilized to produce new vertices for adding to S.

The set of extreme points of the inner approximation of $\mathcal{Y}'$ is $V^i = I \cup \{A_1, \ldots, A_p, y^M\}$ and the set of extreme points of the outer approximation $\mathcal{Y}'$ is $V^o = O \cup \{A_1, \ldots, A_p, y^M\}$, where $A_i = (A_{i1}, \ldots, A_{ip})$ and

$$A_{ij} = \begin{cases} y_j^M & i \neq j, \quad j = 1, \ldots, m \\ y_j^m & i = j \end{cases}$$

for $i = 1, \ldots, p$.

The convex hull of the set V^i, which is an inner approximation of $\mathcal{Y}'$, is denoted by $\mathcal{Y}'^i$ and the convex hull of the set V^o, which is an outer approximation of $\mathcal{Y}'$, is denoted by $\mathcal{Y}'^o$. In Algorithm 2, the approximation algorithm for convex MOPs is given.

Step 1.: (Initialization) Let $\epsilon \geq 0$ be given. Construct block $B^0 = [y^m, y^M]$ containing $\mathcal{Y}'$.
Set $B = B^0$, $I = O = \emptyset$, $S = \{y^m\}$ and $k = 0$, where y^m and y^M are described in Section 2.

Step 2.: (Iteration).

(a) Set $\bar{S} = \emptyset$.

(b) For each $\bar{y} \in S$ do:

begin

b_1) $k := k + 1$.

b_2) Find an optimal solution (x^*, λ^*) from problem (3.1).

b_3) Set $w^k = \bar{y} + \lambda^*(y^M - \bar{y})$.

b_4) If $d(\bar{y}, w^k) \leq \epsilon$, then set $I = I \cup \{w^k\}$ and $O = O \cup \{\bar{y}\}$. Select another $\bar{y} \in S$ and go to b_1.

b_5) If $d(\bar{y}, w^k) > \epsilon$, then $\bar{B} = B \setminus [v, y^w]$, where $v = \bar{y}$ and $y^w = w^k$.

b_6) Determine p vertices $v^1, \ldots, v^p$ corresponding to y^w via formulation (3.2).

b_7) Set $\bar{S} = \bar{S} \cup \{v^1, \ldots, v^p\}$ and $B := \bar{B}$.

end

(c) If $\bar{S} \neq \emptyset$, set $S = \bar{S}$ and then go to Step 2. Else, stop.

Algorithm 2: Approximation algorithm for finding ε-nondominated points

By the given approximation algorithm, we have the following observations:

Theorem 4.1. (i) *The number of points in V^o is equal to the number of points in V^i.*

(ii) *All points in V^i are on the boundary of $\mathcal{Y}'$ and the points of V^0 are on the boundary of $\mathcal{Y}'$ or outside $\mathcal{Y}'$. Moreover, $y \in V^o$ is not on the boundary of $\mathcal{Y}'$ if and only if $y \notin V^i$.*

Proof. (i) By Algorithm 2, the initial values for I and O are empty sets. Moreover, when we add a point to I (for example w^k), its corresponding point (for example $\bar{y}$) is added to O. Therefore the number of points in I and O is equal. Also, since the elements $\{A_1, \ldots, A_p, y^M\}$ are shared in the definitions of V^o and V^i, therefore the number of vertices of the inner and outer approximations of $\mathcal{Y}'$ is equal.

(ii) By definition of I, it is obvious that all of its members are on the boundary of $\mathcal{Y}'$. Also, by definition of $\mathcal{Y}'$, the points $\{A_1, \ldots, A_p, y^M\}$ are on the boundary of $\mathcal{Y}'$. Furthermore, it is clear that all points in $V^o \setminus \{A_1, \ldots, A_p, y^M\}$ which are selected from S, are different from the boundary points w^k. Therefore, these points are not on the boundary of $\mathcal{Y}'$. □

Theorem 4.2. (i) *If y_o is a point in V^o, then, there exists a point $y_i \in V^i$ corresponding to y_o such that $d(y_o, y_i) \leq \epsilon$, and vice versa.*
(ii) *If $\mathcal{Y}'^i_N$ is the nondominated set of the inner approximation $\mathcal{Y}'^i$ and $\mathcal{Y}'^o_N$ is the nondominated set of the outer approximation $\mathcal{Y}'^o$, then $\mathcal{Y}'^i_N + \mathbb{R}^p_{\geqq} \subseteq \mathcal{Y}'_N + \mathbb{R}^p_{\geqq} \subseteq \mathcal{Y}'^o_N + \mathbb{R}^p_{\geqq}$.*

Proof. (i) $y_o \in V^o$, therefore $y_o \in O$ or $y_o \in \{A_1, \ldots, A_p, y^M\}$. If $y_o \in \{A_1, \ldots, A_p, y^M\}$ then we have $y_o = y_i$ and the proof is completed. Otherwise, if $y_o \in O$ we consider y_i as its corresponding boundary point which is in I. By the steps of the algorithm, it is obvious that since $y_o \in O$ therefore $d(y_o, y_i) \leq \epsilon$, and vice versa.
(ii) Since $\mathcal{Y}'^o$ is the outer approximation of $\mathcal{Y}'$, therefore $\mathcal{Y}' \subseteq \mathcal{Y}'^o$. On the other hand, $\mathcal{Y}'_N \subseteq \mathcal{Y}'$, therefore $\mathcal{Y}'_N \subseteq \mathcal{Y}'^o$. Now, since $\mathcal{Y}'^o$ is $\mathbb{R}^p-$closed and $\mathbb{R}^p-$bounded, therefore $\mathcal{Y}'^o \subseteq \mathcal{Y}'^o_N + \mathbb{R}^p_{\geqq}$. Hence, we have $\mathcal{Y}'_N + \mathbb{R}^p_{\geqq} \subseteq \mathcal{Y}'^o_N + \mathbb{R}^p_{\geqq}$. The other part of the inclusion can be proved similarly. □

The following theorems show that the set of nondominated points of the inner approximation is a set of weakly $\varepsilon-$nondominated points of $\mathcal{Y}'$.

Theorem 4.3. *If y_o is a weakly nondominated point of the outer approximation set $\mathcal{Y}'^o$, then there exists a weakly nondominated point y_i of the inner approximation set $\mathcal{Y}'^i$ such that $d(y_o, y_i) \leq \epsilon$.*

Proof. The proof is similar to that of [28, Proposition 2]. □

Theorem 4.4. *Let $\varepsilon = \epsilon e$ with $e = (1, 1, \ldots, 1) \in \mathbb{R}^p$ be given. Then, $\mathcal{Y}'^i_N$ is a set of weakly ε-nondominated points of $\mathcal{Y}'$.*

Proof. The proof is similar to that of [28, Theorem 5]. □

Theorem 4.4 shows that the proposed algorithm gives us a guaranteed approximation quality for the weakly nondominated set of $\mathcal{Y}'$. Since $\mathcal{Y}_N \subseteq \mathcal{Y}'_{WN}$, we can find an approximation of the nondominated points of $\mathcal{Y}$. However, it is obvious that for approximating the nondominated set of $\mathcal{Y}$ we have to delete some points of the weakly ε-nondominated vertices of $\mathcal{Y}$, since some of them belong to $\mathcal{Y}' \setminus \mathcal{Y}$ and some of them are only approximate weakly nondominated points of $\mathcal{Y}$. To this end, we consider the following definitions:

$$V(\mathcal{Y}^i_N) = \{y \in V^i : y < y^M\},$$

and

$$V(\mathcal{Y}_N^o) = \{y \in V^o : y < y^M\}.$$

These sets present the vertices of the nondominated set of the inner approximation of $\mathcal{Y}$ and the vertices of the nondominated set of the outer approximation of $\mathcal{Y}$, respectively. Using the points in $V(\mathcal{Y}_N^i)$ and $V(\mathcal{Y}_N^o)$ we can construct faces and connect them with each other. The nondominated set $\mathcal{Y}_N$ can be approximated from inside and outside by these two sets.

Theorem 4.5. *Let $v \in B \setminus \mathcal{Y}'$ and w be the boundary point of $\mathcal{Y}'$ on the line segment connecting v and y^M such that $d(v,w) \leq \epsilon$. If $v^i,\ i = 1,\ldots,p$ are new vertices corresponding to w (obtained from formulation (3.2)) and $w^i, i = 1,\ldots,p$ are the corresponding boundary points on the line segment connecting $v^i, i = 1,\ldots,p$, and y^M, then $d(v^i,w^i) \leq \epsilon, \forall i = 1,\ldots,p$.*

Proof. Let v^1 be one of the p vertices obtained from formulation (3.2) and w^1 be the boundary point obtained from connecting v^1 and y^M (see Figure 1 for $p = 2$). Since w and w^1 are nondominated points (by Theorem 3.2), it is obvious that $w^1 \notin (w + \mathbb{R}^p_{\geqq})$. Let k be a point of $w + \mathbb{R}^p_{\geqq}$ on the line connecting y^M and v^1. Then $d(v^1,k) > d(v^1,w^1)$. Now, we show that $d(v,y^M) > d(v^1,y^M)$. Assume that $v = (v_1,\ldots,v_p)$ and $y^M = (\alpha,\ldots,\alpha)$ and $w = (w_1,\ldots,w_p)$ and $v^1 = (v_1,w_2,\ldots,w_p)$. We have $w = \lambda v + (1-\lambda)y^M,\ \lambda \in (0,1)$. Therefore

$$\begin{aligned}
&w_i = \lambda v_i + (1-\lambda)\alpha, \quad \forall\, i = 1,\ldots,p \\
&\Rightarrow (\alpha - w_i) = \lambda(\alpha - v_i), \quad \forall\, i = 1,\ldots,p \\
&\Rightarrow (\alpha - w_i)^2 < (\alpha - v_i)^2, \quad \forall\, i = 1,\ldots,p.
\end{aligned}$$

Hence,

$$\begin{aligned}
d(v,y^M) &= \sqrt{(\alpha - v_1)^2 + (\alpha - v_2)^2 + \ldots + (\alpha - v_p)^2} \\
&> \sqrt{(\alpha - v_1)^2 + (\alpha - w_2)^2 + \ldots + (\alpha - w_p)^2} \\
&= d(v^1,y^M).
\end{aligned}$$

Let L be the line connecting v and v^1 and $\bar{L}$ be the line connecting w and $k \in w + \mathbb{R}^p_{\geqq}$ such that $L \parallel \bar{L}$ (see Figure 1 for $p = 2$). Now, by Thales theorem, in the triangle vy^Mv^1, we have

$$\frac{d(v,w)}{d(v,y^M)} = \frac{d(v^1,k)}{d(v^1,y^M)} \longrightarrow d(v,w) > d(v^1,k).$$

On the other hand $d(v^1,k) > d(v^1,w^1)$ and $d(v,w) \leq \epsilon$, therefore $d(v^1,w^1) \leq \epsilon$. Hence, the proof is completed.

□

FIGURE 1. Theorem 4.5 for $p = 2$

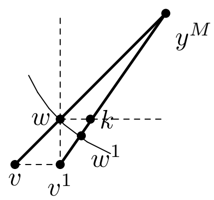
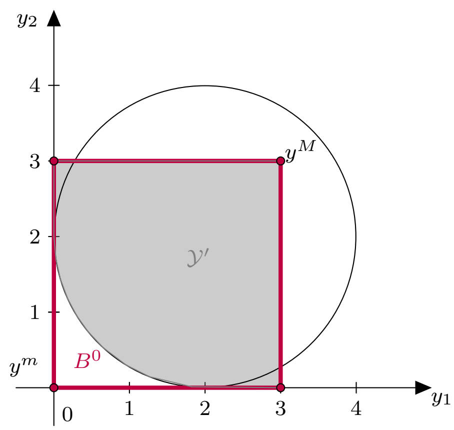


FIGURE 2. $B^0, \mathcal{Y}', y^m$ and y^M in Example 5.1.



5. Numerical results

In this section, we solve convex MOPs using Algorithm 2.

Example 5.1. Consider the following nonlinear convex MOP.

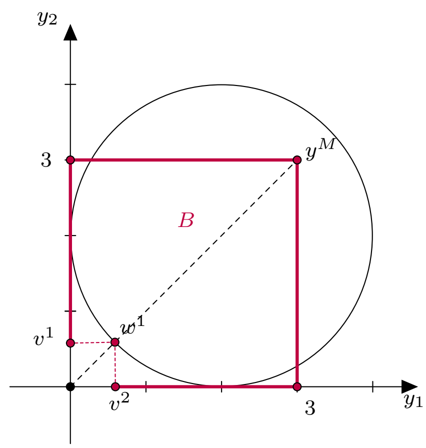
$$\begin{aligned} \min \quad & (f_1(x), f_2(x)) = (x_1, x_2) \\ \text{s.t.} \quad & (x_1 - 2)^2 + (x_2 - 2)^2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Let $\epsilon = 0.1$ be the approximation error given by DM. Using relations (2.2) and (2.3), we have $y^m = (0, 0)$ and $y^M = (3, 3)$. Set $\mathcal{Y}' = (\mathcal{Y} + \mathbb{R}^2_{\geqq}) \cap (y^M - \mathbb{R}^2_{\geqq})$. Initial cover $B^0 = [(0, 0), (3, 3)]$ of $\mathcal{Y}'$, y^m, y^M and $\mathcal{Y}'$ are given in Figure 2. Let $S = \{(0, 0)\}$. Connect $\bar{y} = (0, 0)$ to y^M and obtain the boundary point w^1 of $\mathcal{Y}'$. We have $w^1 = \bar{y} + \lambda^*(y^M - \bar{y}) = (0.5858, 0.5858)$. Since $d(\bar{y}, w^1) > \epsilon$, we construct a new cover of $\mathcal{Y}'$. We have:

$$\bar{B} = B \setminus [v, y^w] = [(0,0),(3,3)] \setminus [(0,0),(0.5858,0.5858)].$$

$\bar{B}$ is the reverse polyblock. Now, by relation (3.2), two vertices corresponding to $y^w = w^1$ are defined as $v^1 = (0, 0.5858)$ and $v^2 = (0.5858, 0)$ and we have $\bar{S} = \{v^1, v^2\}$. Therefore, by letting $B := \bar{B}$ and $S := \bar{S}$ we go to Step 2. Figure 3 shows this process. Figure 4 shows the second iteration. After 4 iterations, in $k = 9$, the first components of I and O appear. The algorithm terminates

FIGURE 3. First iteration in Example 5.1



after 8 iterations and I and O are as follows:

$$I = \Big\{(0.932, 0.315), (0.7666, 0.4256), (0.6392, 0.5343), (1.0726, 0.228), (1.1812, 0.175), (1.264, 0.14), (1.32, 0.116), (1.4221, 0.0853), (0.315, 0.923), (0.425, 0.7666), (0.5343, 0.6392), (0.228, 1.0726), (0.175, 1.1812), (0.14, 1.264), (0.116, 1.32), (0.0853, 1.4221)\Big\},$$

and

$$O = \Big\{(0.8683, 0.244), (0.7019, 0.301), (0.5858, 0.4785), (1.0419, 0.184), (1.162, 0.1455), (1.2512, 0.119), (1.32, 0.0999), (1.376, 0), (0.244, 0.8683), (0.301, 0.7019), (0.4785, 0.5858), (0.184, 1.0419), (0.1455, 1.162), (0.119, 1.2512), (0.0999, 1.32), (0, 1.376)\Big\}.$$

Therefore, we have:

$$V^i = I \cup \{A_1, A_2, y^M\} = I \cup \{(0,3),(3,0),(3,3)\},$$

Figure 4. Second iteration in Example 5.1

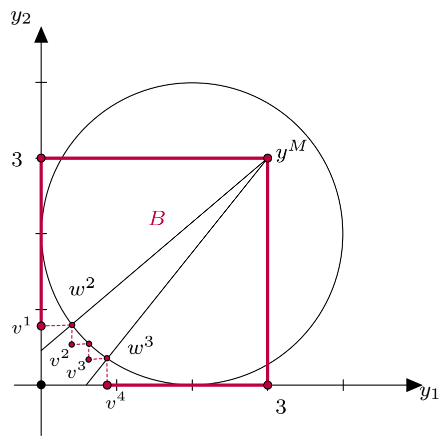
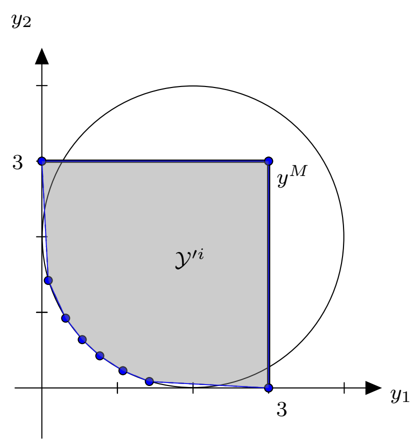


Figure 5. The inner approximation of $\mathcal{Y}'$ in Example 5.1



and

$$V^o = O \cup \{A_1, A_2, y^M\} = O \cup \{(0,3), (3,0), (3,3)\}.$$

Convex hull of V^i (i.e. $\mathcal{Y}'^i$) and convex hull of V^o (i.e. $\mathcal{Y}'^o$) are the inner and outer approximations of $\mathcal{Y}'$, respectively. Figure 5 shows $\mathcal{Y}'^i$ i.e., the inner approximation of $\mathcal{Y}'$. $V(\mathcal{Y}_N^i)$ and $V(\mathcal{Y}_N^o)$ are the sets I and O, respectively, and the inner and outer approximations of $\mathcal{Y}_N$ are constructed by connecting vertices of I and O, sequentially.

Example 5.2. Consider the following convex MOP with differentiable constraints but nondifferentiable objective functions:

$$\begin{array}{ll} \min & (f_1(x), f_2(x)) = (|x_1| + |x_2|, |x_1 - 2| + |x_2|) \\ \text{s.t.} & x_1^2 + x_2^2 - 100 \leq 0. \end{array}$$

Ehrgott et al. [8] considered this MOP and showed that [8, Algorithm 4.2], can not find an ε-nondominated point of this nondifferentiable problem. Now, assume that $\epsilon = 0.2$. We want to find ε-nondominated points using Algorithm 2. For this nondifferentiable MOP we have:

$$y^m = (0,0), \quad y^M = (3,3), \quad B = [(0,0),(3,3)], \ S = \{(0,0)\}.$$

For $k = 1$, we obtain $w^1 = (1,1)$ and $d(\bar{y}, w^1) > \epsilon$. If we repeat the algorithm, after 4 iterations we find $w^8 = (1.479, 0.521)$ in which $d(\bar{y}, w^8) \leq \epsilon$. Therefore, in the 4^{th} iteration we have $I = \{w^8\}$ and $O = \{\bar{y}\}$. After 6 iterations and finding 12 points for I and O, the algorithm terminates. The sets I and O are obtained as follows:

$$\begin{aligned} I = \Big\{ & (1.479, 0.521), (1.087, 0.913), (1.276, 0.724), (0.521, 1.479), (0.724, 1.276), \\ & (0.913, 1.087), (1.651, 0.35), (0.35, 1.651), (1.7575, 0.2425), (1.865, 0.135), \\ & (0.135, 1.865), (0.242, 1.758) \Big\}, \end{aligned}$$

and

$$\begin{aligned} O = \Big\{ & (1.4, 0.3913), (1, 0.8181), (1.1819, 0.6), (0.3913, 1.4), (0.6, 1.1819), \\ & (0.8181, 1), (1.6087, 0.267), (0.267, 1.6087), (1.733, 0.188), (1.812, 0), \\ & (0, 1.812), (0.188, 1.733) \Big\}. \end{aligned}$$

In the following two examples, in order to show some preliminary computational experiments, we implement the proposed algorithm in *Matlab* (R2013a) using *FMINCON* as NLP solver. The test problems were run on a core i5 processor CPU with 2.5 GHz and 4 GB RAM.

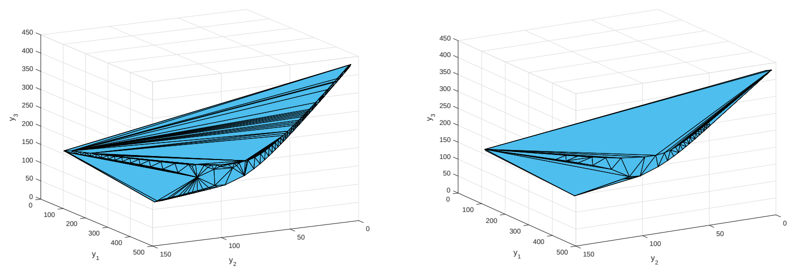


FIGURE 6. The convex hulls of I (left) and O (right) for $\epsilon = 5$ in Example 5.3.

TABLE 1. Obtained results for Example 5.3 with different values of ϵ.

Error	Time (second)	r	$\|I\|$	r_1
$\epsilon = 5$	24.4	71	217	3
$\epsilon = 7$	14.03	43	125	3
$\epsilon = 10$	8.13	24	69	3

Example 5.3. Consider the following convex multiobjective optimization problem:

$$\begin{aligned}
\min f_1(x) &= (x_1-1)^4 + 2(x_2-2)^4 + 3(x_3-3)^4 + 4(x_4-4)^4 \\
f_2(x) &= e^{x_1+x_2+x_3+x_4} + x_1^2 + x_2^2 + x_3^2 + x_4^2 \\
f_3(x) &= 4e^{-x_1} + 6e^{-x_2} + 6e^{-x_3} + 4e^{-x_4}
\end{aligned}$$

$$\text{s.t.} \begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 10 \\ -2 \leq x_1 \leq 2 \\ -2 \leq x_2 \leq 2 \\ -2 \leq x_3 \leq 2 \\ -2 \leq x_4 \leq 2. \end{cases}$$

We solve this problem with approximation errors $\epsilon = 5$, $\epsilon = 7$ and $\epsilon = 10$, respectively. In Figures 6 and 7 the convex hulls of I (inner approximations) and the convex hulls of O (outer approximations) for $\epsilon = 5$ and $\epsilon = 10$ are given, respectively. In Table 1, the computational time of the algorithm, the number of iterations (r), the number of obtained boundary points (size of I) and the iteration in which the first components of I and O appear (r_1), for $\epsilon = 5, 7, 10$ are given. Table 1 shows the effect of the choice of ϵ. The smaller the error parameter, the more iterations and the more boundary points are generated and the longer the computational time.

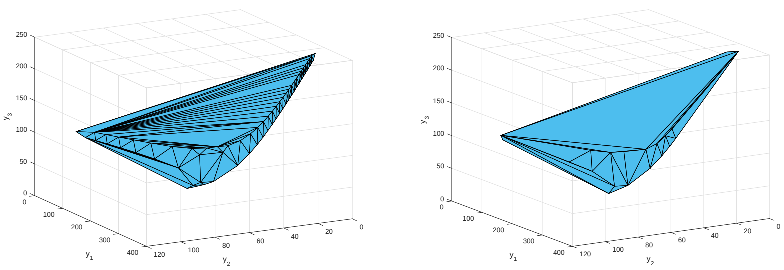


FIGURE 7. The convex hulls of I (left) and O (right) for $\epsilon = 10$ in Example 5.3.

TABLE 2. Obtained results for Example 5.4 with different values of ϵ.

Error	Time (second)	r	$\|I\|$	r_1
$\epsilon = 5$	107.38	67	3091	3
$\epsilon = 10$	76.53	48	1561	3
$\epsilon = 15$	42.04	40	1077	3

Example 5.4 ([8,25])**.** Consider the following convex MOP:

$$\begin{aligned}
\min f_1(x) &= x_1^2 + x_2^2 + x_3^2 + 10x_2 - 120x_3 \\
f_2(x) &= x_1^2 + x_2^2 + x_3^2 + 80x_1 - 448x_2 + 80x_3 \\
f_3(x) &= x_1^2 + x_2^2 + x_3^2 - 448x_1 + 80x_2 + 80x_3
\end{aligned}$$

$$\text{s.t.} \begin{cases} x_1^2 + x_2^2 + x_3^2 \leq 100 \\ 0 \leq x_1 \leq 10 \\ 0 \leq x_2 \leq 10 \\ 0 \leq x_3 \leq 10. \end{cases}$$

We solve this problem with approximation errors $\epsilon = 5$, $\epsilon = 10$ and $\epsilon = 15$, respectively. In Figures 8 and 9 the convex hulls of I (inner approximations) and the convex hulls of O (outer approximations) for $\epsilon = 5$ and $\epsilon = 10$ are given, respectively. In Table 2, the computational time of the algorithm, the number of iterations (r), the number of obtained boundary points (size of I) and the iteration in which the first components of I and O appear (r_1), for $\epsilon = 5, 10, 15$ are given. Table 2 shows the effect of the choice of ϵ. The smaller the error parameter, the more iterations and the more boundary points that are generated and the longer the computational time.

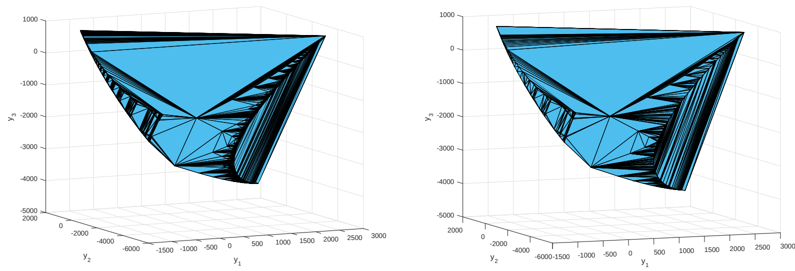


FIGURE 8. The convex hulls of I (left) and O (right) for $\epsilon = 5$ in Example 5.4.

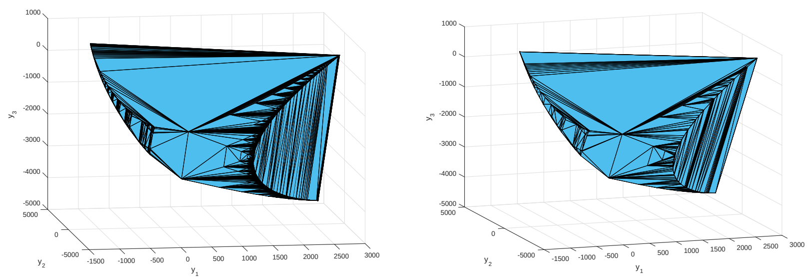


FIGURE 9. The convex hulls of I (left) and O (right) for $\epsilon = 10$ in Example 5.4.

6. Conclusions

In this paper, we have presented an approximation algorithm to find ε-nondominated points of a convex MOP. The proposed algorithm, compared with the existing algorithms, does not require differentiability of the objective functions and constraints. Also, in the suggested algorithm, the nonlinear objectives and constraints are not approximated using linearization techniques and therefore nonlinear and convex objectives and constraints directly play a rule in finding the ε-nondominated points. Performing the steps of the algorithm is very easy and calculating the vertices of the new covers is not difficult.

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