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CARATHEODORY DIMENSION FOR OBSERVERS

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Abstract. In this essay we introduce and study the notion of dimension for observers via Caratheodory structures and relative probability measures. We show that the dimension as a three variables function is an increasing function on observers, and decreasing function on the cuts of an observer. We find observers with arbitrary non-negative dimensions. We show that Caratheodory dimension for observers is an invariant object under conjugate relations. Caratheodory dimension as a mapping, for multi-dimensional observers is considered. News spread is modeled via multi-dimensional observers.
Keywords: Caratheodory structure, observer, news spread, Hausdorff dimension.
MSC(2010): Primary: 54F45; Secondary: 37C45.

1. **Introduction**

Caratheodory was the first scientist who introduced the notion of dimension as an invariant object in dynamical systems [3, 9]. In this paper we extend the notion of Caratheodory dimension for one dimensional observers. One dimensional observer as an extension of random variables [1] has been introduced as a mathematical object in dynamical systems in 2004 [7]. It's extension as a multi-dimensional observer has been considered from topological viewpoint in 2009 [8]. A one dimensional observer of a non-empty set X is a mapping $\mu : X \to [0, 1]$, [7]. One must pay attention to this point that an observer is a fuzzy set, but each fuzzy set is not an observer. In fact the closed interval $[0, 1]$ has essential role in the mathematical modeling of a physical observer, and it can not change with an ordered lattice, because of some physical facts. For more details see [8]. In the physical world, because of the rules of nature we have systems which are sets with a special kind of relations. Hence in this paper we assume that (X, T) is a discrete semi-dynamical system i.e., X is a

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non-empty set and $T : X \to X$ is a mapping. If E is a subset of X, then the relative probability measure of E with respect to an observer μ is the observer $m_\mu^T(E) : X \to [0,1]$ defined by

$$m_\mu^T(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))\mu(T^i(x)).$$

If we assume that (X, B, m) is a probability space, $T : (X, B, m) \to (X, B, m)$ is an ergodic map and E is a member of the σ-algebra B, and if the one dimensional observer $\mu : X \to [0,1]$ is the characteristic function of X, then Birkhoff ergodic theorem [10] implies that
$m_\mu^T(E)(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x))\chi_X(T^i(x)) = m(E)$ a.e., for all $x \in X$. Thus the relative probability measure is an extension of the concept of probability measures.

In the next section the notion of dimension from an observer viewpoint is considered. We show that the dimension of an observer depends on it's relative probability measure and the metrics defined by two set functions on it's observational caratheodory structure. By examples we introduce observers with arbitrary positive dimensions. In section three conjugate relations are considered and we show that dimension is an invariant object under conjugate relations. In section four we extend the notion of Caratheodory dimension to multi-dimensional observers. We show that in this case Caratheodory dimension is a mapping. As an application we consider news spread by using of multi-dimensional observers.

2. Observational dimension

In this section we assume that X is a nonempty set and $\mu : X \to [0,1]$ is an observer. An observer $\lambda : X \to [0,1]$ is called a subset of μ if $\lambda(x) \le \mu(x)$. Suppose that F is a collection of subsets of μ, and f, g are two set functions from F to $[0,\infty]$. Then (F, f, g) is called an observational Caratheodory dimension structure or an OC-structure on μ if f and g satisfy the following three axioms:

A.1 $\chi_\emptyset \in F$ and $f(\chi_\emptyset) = g(\chi_\emptyset) = 0$. Moreover $f(\alpha) > 0$ and $g(\alpha) > 0$ if $\alpha \in F$ and $\alpha \ne \chi_\emptyset$.

A2. For given $\delta > 0$ there is $\epsilon > 0$ such that if $g(\alpha) \le \epsilon$ then $f(\alpha) \le \delta$.

A3. For given $\epsilon > 0$ there is a finite or countable collection $G \subseteq F$ such that $\mu \subseteq \cup_{\alpha\in G}\alpha$ and $g(G) = \sup\{g(\alpha) \ : \ \alpha \in G\} \le \epsilon$.

Let η be a subset of μ and let $d \in R$, $\epsilon > 0$, $c \in [0,1]$ and $x \in X$ be given. Then we define $M_c(\eta, d, \epsilon)(x) = \inf_G \{\Sigma_{\alpha\in G} h(\alpha)(x) f(\alpha)^d\}$, where $h(\alpha)(x) = m_\mu^T(\alpha^{-1}(c,1])(x)$, G is a finite or countable subcollection of F that covers μ and $g(G) \le \epsilon$. We denote $\lim_{\epsilon\to 0} M_c(\eta, d, \epsilon)(x)$ by $M_c(\eta, d)(x)$. This limit can be infinite.

The definition of $M_c(\eta,d)(x)$ implies that $M_c(\chi_\emptyset,d)(x)=0$ for $d>0$ and $M_c(\eta_1,d)(x)\leq M_c(\eta_2,d)(x)$ if $\eta_1\subseteq\eta_2\subseteq\mu$.

Theorem 2.1. *If $x\in X$ and $c\in[0,1]$, then $M_c(\cup_{i=1}^{\infty}\eta_i,d)(x)\leq\Sigma_{i=1}^{\infty}M_c(\eta_i,d)(x)$, when $\eta_i\subseteq\mu$.*

Proof. If for some i, $M_c(\eta_i,d)(x)=\infty$, then the proof is complete. Hence suppose that for each i, $M_c(\eta_i,d)(x)<\infty$. Let $\delta>0$ and $\epsilon>0$ be given. For given i there is $\epsilon_i\leq\epsilon$ such that

$$|M_c(\eta_i,d)(x)-M_c(\eta_i,d,\epsilon_i)(x)|\leq\frac{\delta}{2^{i+1}},$$

and there is a subcollection $G_i=\{\alpha_{ij}\}$ of F which covers η_i, $g(G_i)\leq\epsilon_i$, and

$$|M_c(\eta_i,d,\epsilon_i)(x)-\Sigma_{j=1}^{\infty}h(\alpha_{ij})(x)f(\alpha_{ij})^d|\leq\frac{\delta}{2^{i+1}}.$$

Hence

$$|M_c(\eta_i,d)(x)-\Sigma_{j=1}^{\infty}h(\alpha_{ij})(x)f(\alpha_{ij})^d|\leq\frac{\delta}{2^{i}}.$$

We have $G=\{\eta_{ij}\}$ is a cover of $\eta=\cup_{i=1}^{\infty}\eta_i$ and $g(G)\leq\epsilon$. Thus

$$M_c(\eta,d,\epsilon)(x)\leq\Sigma_{i,j}h(\eta_{ij})(x)f(\eta_{ij})^d\leq\delta+\Sigma_{i=1}^{\infty}M_c(\eta_i,d)(x).$$

Since ϵ and δ are arbitrary, we get $M_c(\cup_{i=1}^{\infty}\eta_i,d)(x)\leq\Sigma_{i=1}^{\infty}M_c(\eta_i,d)(x)$. □

Theorem 2.2. *Let $\eta\subseteq\mu$, $x\in X$ and $c\in[0,1]$. Then there exists $d_c(x)\in[-\infty,\infty]$ such that*

$$M_c(\eta,d)(x)=\begin{cases}\infty & if\ d<d_c(x),\\ 0 & if\ d>d_c(x).\end{cases}$$

Proof. If $M_c(\eta,d)(x)=0$ for all d, then $d_c(x)=-\infty$. If $M_c(\eta,d)(x)=\infty$ for all d, then $d_c(x)=\infty$.

We now prove that if $0\leq M_c(\eta,d)(x)<\infty$ then $M_c(\eta,r)(x)=0$ for all $r>d$. Let $\epsilon>0$ be given then (A2) implies that: for $\delta=\frac{1}{n}$ there is $0<\epsilon_n\leq\epsilon$ such that if $g(\alpha)\leq\epsilon_n$ then $f(\alpha)\leq\frac{1}{n}$. Thus there exists $0<\epsilon'_n\leq\epsilon_n$ and $G_{\epsilon'_n}$ such that

$$|M_c(\eta,d)(x)-\Sigma_{\alpha\in G_{\epsilon'_n}}h(\alpha)(x)f(\alpha)^d|\leq\frac{1}{n},\ and\ g(G_{\epsilon'_n})\leq\epsilon'_n.$$

Hence

$$\Sigma_{\alpha\in G_{\epsilon'_n}}h(\alpha)(x)f(\alpha)^d\leq M_c(\eta,d)(x)+\frac{1}{n}.$$

Thus

$$M_c(\eta,r,\epsilon)(x)\leq\Sigma_{\alpha\in G_{\epsilon'_n}}h(\alpha)(x)f(\alpha)^r\leq(\frac{1}{n})^{r-d}(M_c(\eta,d)(x)+\frac{1}{n})$$

for all $n\in N$. Therefore, $M_c(\eta,r,\epsilon)(x)=0$ for all ϵ and hence $M_c(\eta,r)(x)=0$.

Now we prove that if $M_c(\eta,d)(x)=\infty$, then $M_c(\eta,r)(x)=\infty$ for all $r<d$. Since $M_c(\eta,d)(x)=\infty$, there exists a positive sequence ϵ_n such that $\epsilon_n\to0$ and

$M_c(\eta, d, \epsilon_n)(x) \geq n$. Put $\delta = \frac{1}{n}$, then there exists $\epsilon'_n \leq \epsilon_n$ such that $g(\alpha) \leq \epsilon'_n$ implies that $f(\alpha) \leq \frac{1}{n}$. Thus if $g(G) \leq \epsilon'_n$, then $\Sigma_{\alpha \in G} h(\alpha)(x) f(\alpha)^d \geq n$ and $f(G) \leq \frac{1}{n}$. Hence

$$\sum_{\alpha \in G} h(\alpha)(x) f(\alpha)^r = \sum_{\alpha \in G} h(\alpha)(x) f(\alpha)^d f(\alpha)^{r-d} \geq \frac{1}{n}^{r-d} n = n^{1+d-r},$$

and it follows that $M_c(\eta, r, \epsilon'_n)(x) \geq n^{1+d-r}$. Since $n \to \infty$ implies $\epsilon'_n \to 0$, the previous inequality implies that $M_c(\eta, r)(x) = \infty$.

Now we show that $d_c(x) = \inf\{d : M_c(\eta, d)(x) = 0\}$. Noting that $d_c(x)$ is a real number, because we prove that there exist r, s such that $M_c(\eta, r)(x) = 0$ and $M_c(\eta, s)(x) = \infty$. If $d < d_c(x)$ and $M_c(\eta, d)(x) < \infty$, then we prove that $M_c(\eta, r)(x) = 0$ for all $d < r < d_c(x)$, and this is a contradiction because $d_c(x)$ is the infimum. If $M_c(\eta, d_c)(x) = \infty$, then $M_c(\eta, d)(x) = \infty$ for all $d < d_c$. If $s > d_c$ then $M_c(\eta, d)(x) = \infty$ for all $d_c(x) < d < s$. Thus $d_c(x) = \inf\{d : M_c(\eta, d)(x) = 0\} \geq s$ which is a contradiction. Hence $M_c(\eta, d)(x) = \infty$ for all $d < d_c(x)$. If $0 < M_c(\eta, d)(x) < \infty$ for some $d > d_c(x)$, then $M_c(\eta, r)(x) = \infty$ for all $d_c(x) < r < d$. Thus $d_c(x) \geq d$ which is a contradiction. Thus $M_c(\eta, d)(x) = 0$ for all $d > d_c(x)$. □

One can also prove that $d_c(x) = \sup\{d : M_c(\eta, d)(x) = \infty\}$. $d_c(x)$ is called the Caratheodory dimension of η at x and we also denote it by $\dim_c \eta(x)$.

Theorem 2.3. *If $\eta \subseteq \mu$, $x \in X$, $c_1, c_2 \in [0,1]$ and $c_1 \geq c_2$, then $\dim_{c_2} \eta(x) \geq \dim_{c_1} \eta(x)$.*

Proof. For $i \in \{1, 2\}$ we define $\eta_i : X \to [0,1]$ by

$$\eta_i(t) = \begin{cases} \eta(t) & if\ t > c_i, \\ 0 & if\ t \leq c_i. \end{cases}$$

We have $\eta_1 \subseteq \eta_2 \subseteq \mu$. Thus $M_c(\eta_1, d)(x) \leq M_c(\eta_2, d)(x)$ for all $c \in [0,1]$. Thus $\dim_{c_2} \eta(x) = \dim_{c_2} \eta_2(x) = \inf\{d : M_{c_2}(\eta_2, d)(x) = 0\} = \inf\{d : M_{c_1}(\eta_2, d)(x) = 0\} \geq \inf\{d : M_{c_1}(\eta_1, d)(x) = 0\} = \dim_{c_1} \eta_1(x) = \dim_{c_1} \eta(x)$. □

We assume that X and Y are two non-empty sets, and $T : X \to X$, $U : Y \to Y$ are two bijections. We say that T and U are conjugate if there is a bijection $W : Y \to X$ such that $W \circ U = T \circ W$.

Theorem 2.4. *If E is a subset of X, then $m^U_{\mu o W}(W^{-1}E)(W^{-1}(x)) = m^T_\mu(E)(x)$, for all $x \in X$.*

Proof. Let $x \in X$ be given. Then

$$m^U_{\mu\circ W}(W^{-1}E)(W^{-1}(x))$$
$$= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{W^{-1}(E)}(U^i(W^{-1}(x)))\mu\circ W(U^i(W^{-1}(x)))$$
$$= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{W^{-1}(E)}(W^{-1}\circ T^i\circ W(W^{-1}(x)))\mu\circ W(W^{-1}\circ T^i\circ W(W^{-1}(x)))$$
$$= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_E(T^i(x))\mu(T^i(x)) = m^T_\mu(E)(x).$$

□

If (F, f, g) is an OC-structure on μ, then $(F^W = \{\alpha\circ W \mid \alpha \in F\}, f^W, g^W)$ is an OC-structure on $\mu\circ W$, where $f^W : F^W \to [-\infty,\infty]$, and $g^W : F^W \to [-\infty,\infty]$ are defined by $f^W(\alpha\circ W) = f(\alpha)$ and $g^W(\alpha\circ W) = g(\alpha)$. With these assumptions and notations we have the next theorem.

Theorem 2.5. $\dim_c\eta(x) = \dim_c\eta\circ W(W^{-1}(x))$, *for all* $\eta \subseteq \mu$ *and* $x \in X$.

Proof. Let $d \in R$, $\epsilon > 0$, $c \in [0,1]$ and $x \in X$ be given. Then $M_c(\eta\circ W, d, \epsilon)(W^{-1}(x)) = \inf_{G^W}\{\Sigma_{\alpha\circ W\in G^W} h^W(\alpha\circ W)(W^{-1}(x))(f^W(\alpha\circ W))^d\}$, where $h^W(\alpha\circ W)(W^{-1}(x)) = m^U_{\mu\circ W}((\alpha\circ W)^{-1}(c,1])(W^{-1}(x))$, and $g^W(G^W) \le \epsilon$. Hence Theorem 2.4 implies that

$$M_c(\eta\circ W, d, \epsilon)(W^{-1}(x)) = \inf_G\{\Sigma_{\alpha\in G} h(\alpha)(x)f(\alpha)^d\} = M_c(\eta, d, \epsilon)(x),$$

where $h(\alpha)(x) = m^T_\mu(\alpha^{-1}(c,1])(x)$, and $g(G) \le \epsilon$. Thus $\dim_c\eta(x) = \dim_c\eta\circ W(W^{-1}(x))$, for all $\eta \subseteq \mu$ and $x \in X$. □

3. Examples

In the genesis population model [2], the number of population at time t is $P(t) = e^{kt}$ where k is the birth rate minus the death rate, and it is a positive number. This situation will be true only for a period of time such as $[0,1]$. Hence let $X = [0,1]$ and let $T : X \to X$ be defined by $T(t) = \frac{1}{e^k}e^{kt}$. This normal situation disturb by some natural facts such as illness, earthquake, and so on. We can insert these facts via observers. Let $\mu : X \to X$ be defined by $\mu(t) = e^{-t}$. The mapping T has two fixed points in $(0,1]$ one of them is 1, and we denote the other one by p, this fixed point is an attracting point (see Figure 1). Thus if E is an interval as a subset of $[0,1]$, then

$$m^T_\mu(E)(x) = \begin{cases} e^{-p} & \text{if } p \in E, \\ 0 & \text{otherwise.} \end{cases}$$

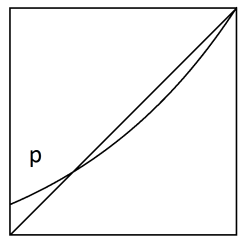

FIGURE 1. p is an attracting fixed point of T.

Example 3.1. For given $n \in N$, and $i \in \{1, 2, \ldots, n\}$ let $\lambda_{n,i} : X \to X$ be defined by

$$\lambda_{n,i}(t) = \begin{cases} \mu(t) - \frac{\mu(t)}{n} & \text{if } t \in [\frac{i-1}{n}, \frac{i}{n}], \\ 0 & \text{otherwise.} \end{cases}$$

We take $F = \{\chi_\emptyset, \mu, \lambda_{n,i} : n \in \mathbb{N} \text{ and } i \in \{1, 2, \ldots, n\}\}$ and we define $f = g : F \to [0, \infty]$ by

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha = \mu, \\ 0 & \text{if } \alpha = \chi_\emptyset, \\ \frac{1}{n} & \text{if } \alpha = \lambda_{n,i}. \end{cases}$$

Then we have

$$M_c(\mu, d, \frac{1}{n})(x) \cong \begin{cases} \frac{e^{-p}}{n^d} & \text{if } c \le p, \\ 0 & \text{if } c > p. \end{cases}$$

Thus

$$\dim_c \mu(x) = \begin{cases} 0 & \text{if } c \le p, \\ -\infty & \text{if } c > p. \end{cases}$$

Example 3.2. For given $n \in \mathbb{N}$, let $\lambda_n : X \to X$ be defined by

$$\lambda_n(t) = \begin{cases} \mu(t) - \frac{\mu(t)}{n} & \text{if } t \in [\frac{1}{n}, 1], \\ 0 & \text{otherwise.} \end{cases}$$

If $F = \{\chi_\emptyset, \mu, \lambda_n : n \in \mathbb{N}\}$ and if $f = g : F \to [0, \infty]$ is the mapping

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha = \mu, \\ 0 & \text{if } \alpha = \chi_\emptyset, \\ \frac{1}{n^s} & \text{if } \alpha = \lambda_n, \end{cases}$$

where s is a constant positive number, then for large enough n

$$M_c(\mu, d, \frac{1}{n})(x) \cong \begin{cases} \frac{ne^{-p}}{n^{sd}} & \text{if } c \le p, \\ 0 & \text{if } c > p. \end{cases}$$

Thus

$$\dim_c \mu(x) = \begin{cases} \frac{1}{s} & \text{if } c \leq p. \\ -\infty & \text{if } c > p. \end{cases}$$

4. **Caratheodory dimension for multi-dimensional observers**

We assume that X is a non-empty set, I is an index set and

$$\prod_{i \in I} [0,1] = \{g : I \to [0,1] : g \text{ is a mapping}\}.$$

A mapping $\mu : X \to \prod_{i \in I} [0,1]$ is called a multi-dimensional observer [8].
Information systems are examples of multi-dimensional observers. We know that a triple (X, I, F) is called an information system if X, and I are non-empty finite sets, and $F = \{f_i \mid f_i \text{ is a map on } X, \text{ and } i \in I\}$ [5,11]. The finiteness of X implies to the finiteness of the image of each f_i, so it corresponds to a finite subset of the interval $[0,1]$ via a one to one mapping g_i. Thus an information system (X, I, F) can be denoted by a finite dimensional observer

$$\mu : X \to \prod_{i \in I} [0,1]$$

defined by

$$\mu(x) = (g_1(f_1(x)), g_2(f_2(x)), \ldots, g_{|I|}(f_{|I|}(x)),$$

where X and I are finite sets, and $x \in X$.
Fuzzy information systems are other examples of multi-dimensional observers. In fact in a fuzzy information system each f_i allows to take it's values in the interval $[0,1]$ [6]. Hence it is a multi-dimensional observer.
Stochastic (or random) processes on finite spaces are other examples of multi-dimensional observers. A stochastic process (Amigo, Kennel 2007) on a finite space X is a sequence (S_n), where S_n is a random variable on X with values in $A = \{a_1, \ldots, a_{|A|}\}$, $n \in I$, and I is $\mathbb{N}$ or $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ or $\mathbb{Z}$. If we correspond the image of each S_n with a finite subset of $[0,1]$ via a one to one mapping g_n, then (S_n) can be considered as a multi-dimensional observer $\mu : X \to \prod_{i \in I} [0,1]$ defined by $\mu = (g_n \circ S_n)$, where I is $\mathbb{N}$ or $\mathbb{N}_0 = N \bigcup \{0\}$ or $\mathbb{Z}$. A multi-dimensional observer μ of X can be denoted by $\mu = \prod_{i \in I} \mu_i$, where each $\mu_i : X \to [0,1]$ is a one dimensional observer defined by $\mu_i(x) = \mu(x)(i)$.

Now we assume that E is a subset of X, and $T : X \to X$ is a mapping. The relative probability measure of E with respect to a multi-dimensional observer

μ is the multi-dimensional observer $m_\mu^T(E) : X \to \prod_{i\in I}[0,1]$ defined by

$$m_\mu^T(E)(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_E(T^i(x))\mu(T^i(x)).$$

Let $\mu = \prod_{i\in I}\mu_i$ and $\eta = \prod_{i\in I}\eta_i$ be two multi-dimensional observers of X. Then we say that $\eta \subseteq \mu$ if $\eta_i(x) \le \mu_i(x)$ for all $x \in X$ and $i \in I$. $\eta\cap\mu$ and $\eta\cup\mu$ are two multi-dimensional observers of X defined by $(\eta\cap\mu)_i(x) = \min\{\eta_i(x), \mu_i(x)\}$ and $(\eta\cup\mu)_i(x) = \max\{\eta_i(x), \mu_i(x)\}$.

Now let F be a collection of subsets of μ, and let f, g be two set functions from F to $[0,\infty]$. We say that (F, f, g) is an observational Caratheodory dimension structure or OC-structure on multi-dimensional observer μ if f and g satisfy the following three axioms:

(AM1) $\prod_{i\in I}\chi_\emptyset \in F$ and $f(\prod_{i\in I}\chi_\emptyset) = g(\prod_{i\in I}\chi_\emptyset) = 0$. Moreover, $f(\alpha) > 0$ and $g(\alpha) > 0$ if $\alpha \in F$ and $\alpha \ne \prod_{i\in I}\chi_\emptyset$.

(AM2) For given $\delta > 0$ there is $\epsilon > 0$ such that if $g(\alpha) \le \epsilon$ then $f(\alpha) \le \delta$.

(AM3) For given $\epsilon > 0$ there is a finite or countable collection $G \subseteq F$ such that $\mu \subseteq \cup_{\alpha\in G}\alpha$ and $g(G) = \sup\{g(\alpha) : \alpha \in G\} \le \epsilon$.

Now such as section 2 we assume that η is a subset of μ, $d \in R$, $\epsilon > 0$, $c \in [0,1]$ and $x \in X$. Then we define $M_c(\eta, d, \epsilon)(x) = \inf_G\{\Sigma_{\alpha\in G} h(\alpha)(x) f(\alpha)^d\}$, where $h(\alpha)(x) = m_\mu^T(\alpha^{-1}(\prod_{i\in I}(c,1]))(x)$, G is a finite or countable subcollection of F that covers μ and $g(G) \le \epsilon$. We also denote $\lim_{\epsilon\to 0} M_c(\eta, d, \epsilon)(x)$ by $M_c(\eta, d)(x)$.

Theorem 4.1. *Let $\eta \subseteq \mu$, $x \in X$, and $c \in [0,1]$ be given. Then there exists $d_c(x) : I \to [-\infty,\infty]$ such that*

$$M_c(\eta, d)(x)(i) = \begin{cases} \infty & \text{if } d < d_c(x)(i), \\ 0 & \text{if } d > d_c(x)(i). \end{cases}$$

Proof. Since $M_c(\eta, d)(x)$ is a multi-dimensional observer, then we can denote it by $\prod_{i\in I} M_c(\eta_i, d)(x)$. If we apply Theorem 2.2 for $M_c(\eta_i, d)(x)$, then we find $d_c(x)(i) \in [-\infty,\infty]$ such that

$$M_c(\eta_i, d)(x) = \begin{cases} \infty & \text{if } d < d_c(x)(i), \\ 0 & \text{if } d > d_c(x)(i). \end{cases}$$

Thus the mapping $d_c(x) : I \to [-\infty,\infty]$, $i \mapsto d_c(x)(i)$ implies the validity of the theorem. □

We denote the mapping $d_c(x)$ by $\dim_c\eta(x)$ and we call it the Caratheodory dimension of η at x.

5. Modelling of news spread

News can spread in a society by different means such as internet, television, magazine, mobile, radio and so on. We assume that X is the set of communication means, and $\lambda : X \to [0,1]$ is a mapping such that the probability that some news transforms in a small time interval Δt by a means $x \in X$ is equal to $\lambda(x)\Delta t + O((\Delta t)^2)$. Let $N(t)$ be the number of population of a society at time t (we assume that there is a positive constant M such that $N(t) \leq M$ for all $t \in R$). We denote the expected number of population at time t who receive a special news via a means $x \in X$ by $E(N(t))(x)$. Hence

$$E(N(t))(x) = \Sigma_{n=0}^{M} n p_n^x(t),$$

where $p_n^x(t)$ is the probability that exactly n individuals receive that news at time t via the means x. If $I \subseteq R$, then we define $\mu : X \to \prod_{t\in I}[0,1]$ by $\mu(x) = \prod_{t\in I}\frac{1}{M}E(N(t))(x)$. The observer μ determines the news spread by different communication means. To determine μ we first determine the evolution of $p_n^x(t)$. The probability law implies

$$p_n^x(t+\Delta t) = \begin{cases} (n-1)p_{n-1}^x(t)\lambda(x)\Delta t + p_n^x(t) + O((\Delta t)^2) & \text{if } 1 < n \leq M, \\ p_1^x(t) + \lambda(x)\Delta t & \text{if } n = 1, \\ p_0^x(t) + (1-\lambda(x))\Delta t & \text{if } n = 0. \end{cases}$$

Letting $\Delta t \to 0$, we deduce the following system of differential equations

$$\frac{dp_n^x(t)}{dt} = \begin{cases} (n-1)\lambda(x)p_{n-1}^x(t) & \text{if } 1 < n \leq M, \\ \lambda(x) & \text{if } n = 1, \\ (1-\lambda(x)) & \text{if } n = 0. \end{cases}$$

Hence

$$p_n^x(t) = \begin{cases} (n-1)\lambda(x)^n \frac{t^n}{n!} + (n-1)\lambda(x)^{n-1}p_1^x(0)\frac{t^{n-1}}{(n-1)!} + \cdots + & \\ (n-1)\lambda(x)p_{n-1}^x(0)t + p_n^x(0) & \text{if } 1 < n \leq M, \\ \lambda(x)t + p_1^x(0) & \text{if } n = 1, \\ (1-\lambda(x))t + p_0^x(0) & \text{if } n = 0. \end{cases}$$

Thus we can determine $E(N(t))(x) = \sum_{n=0}^{M} n p_n^x(t)$ and $\mu(x) = \prod_{t\in I}\frac{1}{M}E(N(t))(x)$.

Example 5.1. Suppose $X = \{x_1 = \text{Television}, x_2 = \text{Radio}, x_3 = \text{Magazine}, x_4 = \text{Internet}\}$, $I \subseteq R$, $M = 20$, $p_n^{x_i}(0) = \frac{n}{101}$, and $\lambda(x_i) = \frac{1}{i+100}$ for $i \in$

$\{1, 2, 3, 4\}$. If $E \subseteq X$ and $T : X \to X$ is the mapping

$$T(x_i) = \begin{cases} x_{i+3} & \text{if } i = 1, \\ x_{i-3} & \text{if } i = 4, \\ x_i & \text{if } i \in \{2, 3\}, \end{cases}$$

then

$$m_\mu^T(E)(x_1) = \begin{cases} \mu(x_1) & \text{if } x_1, x_4 \in E, \\ \frac{1}{2}\mu(x_1) & \text{if } (x_1 \in E, \text{ and } x_4 \notin E) \text{ or } (x_4 \in E, \text{ and } x_1 \notin E), \\ \prod_{t\in I} 0 & \text{otherwise,} \end{cases}$$

$$m_\mu^T(E)(x_4) = \begin{cases} \mu(x_4) & \text{if } x_1, x_4 \in E, \\ \frac{1}{2}\mu(x_4) & \text{if } (x_1 \in E, \text{ and } x_4 \notin E) \text{ or } (x_4 \in E, \text{ and } x_1 \notin E), \\ \prod_{t\in I} 0 & \text{otherwise,} \end{cases}$$

and for $i \in \{2, 3\}$

$$m_\mu^T(E)(x_i) = \begin{cases} \mu(x_i) & \text{if } x_i \in E, \\ \prod_{t\in I} 0 & \text{otherwise.} \end{cases}$$

Let $\eta_n = (1-\frac{1}{n})\mu$, and $F = \{\prod_{i\in I}\chi_\emptyset, \mu, \eta_n \ : \ n \in N\}$. If $\alpha \in F$, then we define $||\alpha|| = \max\{||\alpha(x_i)|| \mid i = 1, 2, 3, 4\}$. We also define $f = g : F \to [-\infty, \infty]$ by

$$f(\alpha) = \begin{cases} \frac{1}{n^{||\mu||+1}} & \text{if } \alpha = \eta_n, \\ 1 & \text{if } \alpha = \mu, \\ 0 & \text{if } \alpha = \prod_{i\in I}\chi_\emptyset. \end{cases}$$

If $d \in R$, $n \in N$, and $c \in [0, 1]$, then

$$M_c(\mu, d, \frac{1}{n})(x_1) \cong$$

$$\begin{cases} \frac{n}{n^{(||\mu||+1)d}}\mu(x_1) & \text{if } \mu(x_1) > \mu(c), \text{ and } \mu(x_4) > \mu(c), \\ \frac{n}{2n^{(||\mu||+1)d}}\mu(x_1) & \text{if } (\mu(x_1) > \mu(c), \text{ and } \mu(x_4) \le \mu(c)) \text{ or} \\ & \quad (\mu(x_4) > \mu(c), \text{ and } \mu(x_1) \le \mu(c)), \\ \prod_{t\in I} 0 & \text{otherwise,} \end{cases}$$

$$M_c(\mu, d, \frac{1}{n})(x_4) \cong$$

$$\begin{cases} \frac{n}{n^{(||\mu||+1)d}}\mu(x_4) & \text{if } \mu(x_1) > \mu(c), \text{ and } \mu(x_4) > \mu(c), \\ \frac{n}{2n^{(||\mu||+1)d}}\mu(x_4) & \text{if } (\mu(x_1) > \mu(c), \text{ and } \mu(x_4) \le \mu(c)) \text{ or} \\ & \quad (\mu(x_4) > \mu(c), \text{ and } \mu(x_1) \le \mu(c)), \\ \prod_{t\in I} 0 & \text{otherwise,} \end{cases}$$

and for $i \in \{2, 3\}$

$$M_c(\mu, d, \frac{1}{n})(x_i) \cong \begin{cases} \frac{n}{n^{(||\mu||+1)d}}\mu(x_i) & \text{if } \mu(x_i) > \mu(c), \\ \prod_{t\in I} 0 & \text{otherwise.} \end{cases}$$

Thus

$$\dim_c\mu(x_1)(i) = \begin{cases} \frac{1}{||\mu||+1} & \text{if } \mu(x_1)(i) \ne 0 \text{ and } \mu(x_1) > \mu(c) \text{ or } \mu(x_4) > \mu(c), \\ -\infty & \text{if } \mu(x_1)(i) = 0 \text{ or } (\mu(x_1) \le \mu(c), \text{ and } \mu(x_4) \le \mu(c)), \end{cases}$$

$$\dim_c\mu(x_4)(i) = \begin{cases} \frac{1}{||\mu||+1} & \text{if } \mu(x_4)(i) \neq 0 \text{ and } \mu(x_1) > \mu(c) \text{ or } \mu(x_4) > \mu(c), \\ -\infty & \text{if } \mu(x_4)(i) = 0 \text{ or } (\mu(x_1) \leq \mu(c) \text{ and } \mu(x_4) \leq \mu(c)), \end{cases}$$

and for $j \in \{2,3\}$

$$\dim_c\mu(x_j)(i) = \begin{cases} \frac{1}{||\mu||+1} & \text{if } \mu(x_j)(i) \neq 0 \text{ and } \mu(x_j) > \mu(c), \\ -\infty & \text{otherwise.} \end{cases}$$

6. Conclusion

We assume that μ is a multi-dimensional observer of X, and F is a μ-topology [8] i.e. F is a collection of subsets of μ with the following conditions:

(i) $\mu, \prod_{i\in I} \chi_\emptyset \in F$;

(ii) $\lambda \cap \eta \in F$ whenever $\lambda, \eta \in F$;

(iii) $\bigcup_{a\in J} \lambda_a \in F$ whenever $\lambda_a \in F$.

If $T : X \to X$ is a mapping and $c \in [0,1]$, then we define $f = g : F \to [0,\infty]$ by $f(\alpha) = ||m_\alpha^T(E)||$, where $E = \mu^{-1}\prod_{i\in I}(c,1]$. If (F, f, g) is an OC-structure on multi-dimensional observer μ, $x \in X$, and $\eta \subseteq \mu$, then $\dim_c\mu(x)$ is called the Hausdorff dimension of η at x. This dimension is an extension of the notion of Hausdorff dimension for a bounded subset X of the Euclidean space R^n. To see this we assume that B is the σ-algebra generated by open subsets of R^n restricted to X and m is the Lebesgue measure on X. We assume that $T : (X, B, \frac{1}{m(X)}m) \to (X, B, \frac{1}{m(X)}m)$ is an ergodic map, and that the one dimensional observer $\mu : X \to [0,1]$ is the characteristic function of X. If $F = \{\chi_U\ :\ \text{U is open in } X\}$, then simple calculations imply that $\dim_c\mu(x)$ is the Hausdorff dimension of X, for all $x \in X$ and $c \in [0,1)$. Because in this case $m_{\chi_U}^T(E) = m(U)$ a.e., and $k_1\text{diameter}(U) \leq m(U) \leq k_2\text{diameter}(U)$, where k_1 and k_2 are two positive constants.

Consideration of Hausdorff dimension [4] for observers can be a topic for further research.

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