On a p-Laplacian system and a generalization of the Landesman-Lazer type condition

B.Q. Hung and H.Q. Toan
ON A P-LAPLACIAN SYSTEM AND A GENERALIZATION OF THE LANDESMAN-LAZER TYPE CONDITION

B.Q. HUNG* AND H.Q. TOAN

(Communicated by Asadollah Aghajani)

Abstract. This article shows the existence of weak solutions of a resonance problem for nonuniformly p-Laplacian system in a bounded domain in $\mathbb{R}^N$. Our arguments are based on the minimum principle and rely on a generalization of the Landesman-Lazer type condition.

Keywords: Semilinear elliptic equation, non-uniform, Landesman-Lazer condition, minimum principle.


1. Introduction and preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial \Omega$. In the present paper we consider the existence of weak solutions of the following Dirichlet problem at resonance for nonuniformly p-Laplacian system:

\begin{align*}
-\text{div}(h_1(x)|\nabla u|^{p-2}\nabla u) &= \lambda_1|u|^\alpha - 1|v|^\beta - 1 v + f(x, u, v) - k_1(x), &\text{in } \Omega \\
-\text{div}(h_2(x)|\nabla v|^{p-2}\nabla v) &= \lambda_1|u|^\alpha - 1|v|^\beta - 1 u + g(x, u, v) - k_2(x), &\text{in } \Omega \\
u &= 0, &v &= 0 &\text{on } \partial \Omega,
\end{align*}

where

\begin{align*}
p &\geq 2, \quad \alpha \geq 1, \quad \beta \geq 1, \quad \alpha + \beta = p.
\end{align*}

and $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions which will be specified later,

\begin{align*}
h_i(x) &\in L^1_{\text{loc}}(\Omega), \quad h_i(x) \geq 1, \quad \text{for a.e } x \in \Omega, \quad i = 1, 2, \\
k_i(x) &\in L^{p'}(\Omega), \quad p' = \frac{p}{p-1}, \quad k_i(x) > 0, \quad \text{for a.e } x \in \Omega, \quad i = 1, 2.
\end{align*}

$\lambda_1$ denotes the first eigenvalue of the problem:
A generalization of the Landesman-Lazer condition

\begin{equation}
\begin{aligned}
-\Delta_p u &= \lambda |u|^{\alpha-1}|v|^\beta v, \\
-\Delta_p v &= \lambda |u|^\alpha |v|^{\beta-1} u,
\end{aligned}
\end{equation}

and \((u, v) \in W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega), \ p > 2, \alpha > 1, \beta > 1, \alpha + \beta = p.\)

It is well-known that the principle eigenvalue \(\lambda_1 = \lambda_1(p)\) of (1.4) is obtained using the Ljusternick-Schnirelmann theory by minimizing the functional

\[ J(u, v) = \frac{\alpha}{p} \int_\Omega |\nabla u|^p dx + \frac{\beta}{p} \int_\Omega |\nabla v|^p dx, \]

on \(C^1\) - manifold:

\[ S = \left\{ (u, v) \in X = W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega) : \wedge(u, v) = 1 \right\}, \]

where

\[ \wedge(u, v) = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1} u.v dx, \]

that is \(\lambda_1 = \lambda_1(p)\) can be variational characterized as

\begin{equation}
\lambda_1 = \lambda_1(p) = \inf_{\wedge(u, v) > 0} \inf_{(u, v) \in X : u.v > 0} \frac{\frac{\alpha}{p} \int_\Omega |\nabla u|^p dx + \frac{\beta}{p} \int_\Omega |\nabla v|^p dx}{\int_\Omega |u|^{\alpha-1}|v|^{\beta-1} u.v dx}.
\end{equation}

Moreover the eigenpair \((\varphi_1, \varphi_2)\) associated with \(\lambda_1\) is componentwise positive and unique (up to multiplication by nonzero scalar) (see [1, Theorem 2.2] and [15, Remark 5.4]).

We firstly make some comments on the problem (1.1). Observe that the existence of weak solutions of \((p, q)\)-Laplacian systems at resonance in bounded domains with Dirichlet boundary condition, was first considered by Zographopoulos in [20]. Later in [10] Kandilakis and Magiropoulos have studied a quasilinear elliptic system with resonance part and nonlinear boundary condition in an unbounded domain by assuming the nonlinearities \(f\) and \(g\) depending only on variable \(u\) or \(v\). In [14], Ou and Tang have considered the same system as in [10] with Dirichlet condition in a bounded domain. In these papers, the existence of weak solutions is obtained by critical point theory under a Landesman-Lazer type condition. At the same time for nonuniformly nonlinear elliptic equations involving \(p\)-Laplacian \((p \geq 2)\) at resonance we refer the reader to [12, 13, 18].

In this paper by introducing a generalization of Landesman-Lazer type condition we shall prove an existence result for a \(p\)-Laplacian system on resonance in bounded domain with the nonlinearities \(f\) and \(g\) to be functions depending on both variables \(u\) and \(v\).

Note that in [9] we considered system (1.1) in the case \(h_1(x) = h_2(x) = 1\) and shows the existence of weak solutions of (1.1) in \(W^{1,p}_0 \times W^{1,p}_0\). Our arguments are based on the saddle point theorem and rely on a generalization of the Landesman-Lazer type condition.
Recall that due to $h_i(x) \in L^1_{loc}(\Omega)$, $i = 1, 2$, the problem (1.1) now is nonuniformly in sense that the Euler-Lagrange functional associated to the problem may be infinity at some $w_0 = (u_0, v_0) \in X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Hence we must consider the problem (1.1) in some suitable subspace of $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

As usually $W_0^{1,p}(\Omega)$ denotes the Sobolev space which can be defined as the completion of $C_0^\infty(\Omega)$ under the norm:

$$
\|u\| = \left( \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}.
$$

Now we define the following subspaces $E_i$, $i = 1, 2$, of $W_0^{1,p}(\Omega)$ by:

$$
E_i = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega h_i(x)|\nabla u|^p dx < +\infty \right\},
$$

where $h_i(x)$, $i = 1, 2$, satisfy condition (1.2). $E_i$ can be endowed with the norm

$$
\|u\|_{E_i} = \left( \int_\Omega h_i(x)|\nabla u|^p dx \right)^{\frac{1}{p}}.
$$

Applying the arguments as those used in the proof of [8, Proposition 1.1] we can prove the following proposition.

**Proposition 1.1.** For each $i = 1, 2$, $E_i$ is a Banach space and the embeddings $E_i$ into $W_0^{1,p}(\Omega)$ are continuous.

**Proof.** It is clear that $E_i$ is a normed space. Let $\{u_m\}$ be a Cauchy sequence in $E_i$. Then

$$
\lim_{m,k \to +\infty} \|u_m - u_k\|_{E_i}^p = \lim_{m,k \to +\infty} \int_\Omega h_i(x)|\nabla u_m - \nabla u_k|^p dx = 0,
$$

and $\{\|u_m\|_{E_i}\}$ is bounded. By (1.3) : $\|u_m - u_k\|_{W_0^{1,p}(\Omega)} \leq \|u_m - u_k\|_{E_i}$ for $m, k = 1, 2, \ldots$. Hence the sequence $\{u_m\}$ is also a Cauchy sequence in $W_0^{1,p}(\Omega)$ and it converges to some $u$ in $W_0^{1,p}(\Omega)$, i.e:

$$
\lim_{m \to +\infty} \int_\Omega |\nabla u_m - \nabla u|^p dx = 0.
$$

It follows that $\nabla u_m \to \nabla u$ in $L^p(\Omega)$ and there exists a subsequence $\{\nabla u_{m_k}\}$ converging to $\nabla u$ a.e $x \in \Omega$. Applying Fatou’s lemma we get

$$
\int_\Omega h_i(x)|\nabla u|^p dx \leq \liminf_{k \to +\infty} \int_\Omega h_i(x)|\nabla u_{m_k}|^p dx < +\infty
$$
A generalization of the Landesman-Lazer condition

Hence \( u \in E_i \). Applying again Fatou’s lemma we get

\[
0 \leq \lim_{k \to +\infty} \int_{\Omega} h_i(x)|\nabla u_{m_k} - \nabla u|^p dx
\leq \lim_{k \to +\infty} \left\{ \lim_{i \to +\infty} \int_{\Omega} h_i(x)|\nabla u_{m_k} - \nabla u_{m_i}|^p dx \right\} = 0.
\]

Hence \( \{u_{m_k}\} \) converges to \( u \) in \( E_i \). From this, it implies the sequence \( \{u_m\} \) converges to \( u \) in \( E_i, i = 1, 2 \). Thus \( E_i \) is a Banach space and the continuous embedding \( E_i \) into \( W^{1,p}_0 \) holds true. Proposition 1.1 is proved. \( \square \)

**Remark 1.2.** Since the embedding \( W^{1,p}_0(\Omega) \) to \( L^p(\Omega) \) is compact, hence \( E_i \to L^p(\Omega) \) compactly.

Set \( E = E_1 \times E_2 \) and for \( w = (u, v) \in E \):

\[
\|w\|_E = \left( \|u\|_{E_1}^p + \|v\|_{E_2}^p \right)^{\frac{1}{p}}.
\]

Moreover for simplicity of notation denotes by \( X = W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega) \). Then we have \( \|w\|_X \leq \|w\|_E, \forall w = (u, v) \in E \).

**Definition 1.3.** Function \( w = (u, v) \in E \) is called a weak solution of the problem (1.1) if and only if

\[
\alpha \int_{\Omega} h_1(x)\nabla u \nabla \tilde{u} dx + \beta \int_{\Omega} h_2(x)\nabla v \nabla \tilde{v} dx
\]

\[
- \lambda_1 \int_{\Omega} \left( \alpha |u|^{a-1} |v|^{\beta-1} \tilde{v} + \beta |u|^{a-1} |v|^{\beta-1} \tilde{u} \right) dx
\]

\[
- \int_{\Omega} \left( \alpha f(x, u, v) \tilde{u} + \beta g(x, u, v) \tilde{v} \right) dx
\]

\[
+ \int_{\Omega} \left( \alpha k_1(x) \tilde{u} + \beta k_2(x) \tilde{v} \right) dx = 0, \quad \forall \tilde{w} = (\tilde{u}, \tilde{v}) \in E.
\]

Let us introduce the following some conditions on nonlinearities of system (1.1):

(H₁)

(i) \( f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) are Carathéodory functions: \( f(x, 0, 0) = 0, \; g(x, 0, 0) = 0 \).

(ii) There exists function \( \tau(x) \in L^{p'}(\Omega), \; p' = \frac{p}{p-1} \) such that:

\[
|f(x, s, t)| \leq \tau(x), \; |g(x, s, t)| \leq \tau(x), \; \text{for a.e } x \in \Omega, (s, t) \in \mathbb{R}^2.
\]

(iii) For \( (s, t) \in \mathbb{R}^2 \):

\[
(1.6) \quad \alpha \frac{\partial f(x, s, t)}{\partial t} = \beta \frac{\partial g(x, s, t)}{\partial s} \quad \text{for a.e } x \in \Omega.
\]

\[
\frac{\partial f(x, s, t)}{\partial t} = \frac{\partial g(x, s, t)}{\partial s} \quad \text{for a.e } x \in \Omega.
\]

\[
\frac{\partial f(x, s, t)}{\partial t} = \frac{\partial g(x, s, t)}{\partial s} \quad \text{for a.e } x \in \Omega.
\]
Denotes, for \((u, v) \in \mathbb{R}^2\)

\begin{equation}
H(x, u, v) = \frac{\alpha}{2} \int_0^u (f(x, s, v) + f(x, s, 0)) \, ds + \frac{\beta}{2} \int_0^v (g(x, u, t) + g(x, 0, t)) \, dt, \quad \text{for a.e } x \in \Omega.
\end{equation}

**Remark 1.4.** By hypothesis (1.6), from (1.7) with some simple computations we deduce that:

\begin{equation}
\frac{\partial H(x, s, t)}{\partial s} = \alpha f(x, s, t), \quad \frac{\partial H(x, s, t)}{\partial t} = \beta g(x, s, t), \quad \text{a.e } x \in \Omega, \forall (s, t) \in \mathbb{R}^2.
\end{equation}

Now we define, for \(i, j = 1, 2\):

\begin{equation}
F_i(x) = \limsup_{\tau \to +\infty} \frac{\alpha}{\tau} \int_0^\tau \left( f \left( x, (-1)^{1+i}y\varphi_1, (-1)^{1+i}y\varphi_2 \right) + f \left( x, (-1)^{1+i}y\varphi_1, 0 \right) \right) \, dy,
\end{equation}

\begin{equation}
G_j(x) = \limsup_{\tau \to +\infty} \frac{\beta}{\tau} \int_0^\tau \left( g \left( x, (-1)^{1+j}y\varphi_1, (-1)^{1+j}y\varphi_2 \right) + g \left( x, 0, (-1)^{1+j}y\varphi_2 \right) \right) \, dy.
\end{equation}

Assume that

\begin{equation}
(H_2)
\end{equation}

\begin{equation}
\int_\Omega (F_1(x)\varphi_1(x) + G_1(x)\varphi_2(x)) \, dx < 2 \int_\Omega (\alpha k_1(x)\varphi_1(x) + \beta k_2(x)\varphi_2(x)) \, dx < \int_\Omega (F_2(x)\varphi_1(x) + G_2(x)\varphi_2(x)) \, dx.
\end{equation}

**Remark 1.5.** For example, we can take functions \(f(x, u, v), g(x, u, v)\) as following:

\begin{align*}
f(x, u, v) &= \tau_1(x) \sin \left( \frac{u}{\beta} + \frac{v}{\alpha} \right) + \eta_1(x) \frac{u}{\sqrt{1 + u^2}}, \\
g(x, u, v) &= \tau_1(x) \sin \left( \frac{u}{\beta} + \frac{v}{\alpha} \right) + \eta_2(x) \frac{v}{\sqrt{1 + v^2}},
\end{align*}

where \(\tau_1(x), \eta_1(x), \eta_2(x)\) are functions in \(L^p(\Omega)\) and \(\eta_1(x) < 0, \eta_2(x) < 0\) for \(x \in \Omega\).

By some simple computations we get:

\begin{align*}
F_1(x) &= 2\alpha \eta_1(x), \\
F_2(x) &= -2\alpha \eta_1(x), \\
G_1(x) &= 2\beta \eta_2(x), \\
G_2(x) &= -2\beta \eta_2(x).
\end{align*}

Therefore, hypothesis (1.10) is satisfied whenever

\begin{align*}
-\eta_1(x) > k_1(x) \quad \text{and} \quad -\eta_2(x) > k_2(x).
\end{align*}

Our main result is given by the following theorem:
Theorem 1.1. Assume that the conditions (H1) and (H2) are fulfilled. Then the problem (1.1) has at least a nontrivial weak solution in E.

Proof of Theorem 1.1 is based on variational techniques and the Minimum Principle.

2. Proof of the main result

We define the Euler-Lagrange functional associated to the problem (1.1) by

\[ I(w) = \frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla u|^p dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla v|^p dx - \lambda_1 \int_{\Omega} |v|^{\alpha-1}|v|^{\beta-1}uv dx \\
- \int_{\Omega} H(x,u,v) dx + \int_{\Omega} (\alpha k_1(x)u + \beta k_2(x)v) dx \]

(2.1)

= J(w) + T(w), \quad \forall w = (u,v) \in E,

where

\[ J(w) = \frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla u|^p dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla v|^p dx, \]

(2.2)

\[ T(w) = -\lambda_1 \int_{\Omega} |v|^{\alpha-1}|v|^{\beta-1}uv dx - \int_{\Omega} H(x,u,v) dx + \int_{\Omega} (\alpha k_1(x)u + \beta k_2(x)v) dx. \]

(2.3)

Firstly we note that due to \( h_i(x) \in L^1_{\text{loc}}(\Omega), \) \( i = 1, 2, \) in general the functional \( J(w) \) may not belong to \( C^1(E) \). Therefore we need some modifications in order to apply the critical point theory to our problem.

Definition 2.1. (see [6, Definition 2.1]) Let \( I \) be a map from a Banach space \( X \) to \( R \). We say that \( I \) is weakly continuously differentiable on \( X \) if the following conditions are satisfied:

(i) \( I \) is continuous on \( X \)

(ii) For any \( u \in X \) there exists a linear map \( I'(u) \) from \( X \) into \( R \) such that:

\[ \lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = (I'(u), v), \quad \forall v \in X. \]

(iii) For any \( v \in X \) the map \( u \to (I'(u), v) \) is continuous on \( X \).

Denotes by \( C^1_w(X) \) the set of weakly continuously differentiable functionals on \( X \). It is clear that \( C^1(X) \subset C^1_w(X) \), where we denote by \( C^1(X) \) the set of all continuously Fréchet differentiable functionals on \( X \).

Let \( I \in C^1_w(X) \) we put:

\[ ||I'(u)|| = \sup \{ |(I'(u),h)| : h \in X : ||h|| = 1 \}, \quad \forall u \in X \]
We say that $I \in C^1_w(X)$ satisfies the Palais-Smale condition on $X$ if any sequence $\{u_m\} \subset X$ for which $\{I(u_m)\}$ is bounded and $\lim_{m \to +\infty} \|I'(u_m)\|_{X^*} = 0$ has a convergent subsequence in $X$.

**Theorem 2.2** (The minimum Principle, see in [12,13, Theorem 2.3]). Let $X$ be a Banach space and $I \in C^1_w(X)$. Assume that:

(i) $I$ is bounded from below, $c = \inf_X I(u)$

(ii) $I$ satisfies the Palais-Smale condition on $X$.

Then there exists $u_0 \in X$ such that $I(u_0) = c$.

The following proposition concerns the smoothness of the functional $I = J + T$ given by (2.1).

**Proposition 2.3.** Assuming hypothesis (H$_1$) and (H$_2$) are fulfilled. We assert that:

(i) The functional $T(w), w \in E$ given by (2.3) is continuous on $E$. Moreover, $T$ is weakly continuously differentiable on $E$ and

$$
(T'(w), \bar{w}) = -\lambda_1 \int_{\Omega} \left( \alpha |u|^{{\alpha}-1}|v|^{{\beta}-1}v\bar{u} + \beta |u|^{{\alpha}-1}|v|^{{\beta}-1}u\bar{v} \right) dx
$$

$$
- \int_{\Omega} (\alpha f(x,w)\bar{u} + \beta g(x,w)\bar{v}) dx
$$

$$
+ \int_{\Omega} (\alpha k_1(x)\bar{u} + \beta k_2(x)\bar{v}) dx, \quad \forall w = (u,v); \quad \bar{w} = (\bar{u},\bar{v}) \in E.
$$

(ii) The functional $J(w), w \in E$ given by (2.2) is weakly continuously differentiable on $E$ and we have: $\forall w = (u,v), \bar{w} = (\bar{u},\bar{v}) \in E$

$$
(J'(w), \bar{w}) = \alpha \int_{\Omega} h_1(x)|\nabla u|^{\alpha-1}\nabla u\nabla \bar{u} dx + \beta \int_{\Omega} h_2(x)|\nabla v|^{\alpha-1}\nabla v\nabla \bar{v} dx.
$$

Thus $I = J + T$ is weakly continuously differentiable on $E$ and

$$
(I'(w), \bar{w}) = (J'(w), \bar{w}) + (T'(w), \bar{w}), \quad \forall w = (u,v); \quad \bar{w} = (\bar{u},\bar{v}) \in E.
$$

In the proof of the Proposition 2.3 we need the following remarks:

**Remark 2.4.** By similar arguments as those used in the proof of [21, Lemma 2.3] and [10, Lemma 5] we infer that the functional $\wedge : E \to \mathbb{R}$ and operator $\Gamma : E \to E^*$ given by

$$
\wedge(u,v) = \int_{\Omega} |u|^{{\alpha}-1}|v|^{{\beta}-1}uv dx, \quad (u,v) \in E,
$$

and

$$
\langle \Gamma(u,v), (\bar{u},\bar{v}) \rangle = \int_{\Omega} |u|^{{\alpha}-1}|v|^{{\beta}-1}\bar{u}\bar{v} dx + \int_{\Omega} |u|^{{\alpha}-1}|v|^{{\beta}-1}u\bar{v} dx, (u,v); (\bar{u},\bar{v}) \in E,
$$

are compact.
Proof. (i) By the Theorem 2.1 in [16, p. 248] and the Remark 2.4 with some standard arguments we infer that \( T \in C^1(X) \) where \( X = W_0^{1,p} \times W_0^{1,p} \). Moreover since the embedding \( E \rightarrow X \) is continuous, we have \( T \in C^1(E) \) and hence \( T \in C_w^1(E) \) and

\[
(T'(w), \bar{w}) = -\lambda_1 \int_{\Omega} \left( \alpha |u|^\alpha |v|^\beta - 1 \beta \beta u v + \beta |u|^\alpha |v|^\beta \beta \bar{u} \right) dx - \int_{\Omega} \left( \alpha f(x, w) u + \beta g(x, w) v \right) dx + \int_{\Omega} \left( \alpha k_1(x) u + \beta k_2(x) \bar{v} \right) dx, \quad \forall w = (u, v); \bar{w} = (\bar{u}, \bar{v}) \in E.
\]

(ii) By similar arguments used in the proof of [8, Proposition 2.1], we deduce that \( J \in C_w^1(E) \) and (2.5), (2.6) hold true. The proof of Proposition 2.3 is complete. \( \square \)

Remark 2.5. From Proposition 2.3, it implies that the critical points of the functional \( I \) given by (2.1) correspond to the weak solutions of the problem (1.1)

Proposition 2.6. Suppose that the sequence \( \{w_m = (u_m, v_m)\}_m \) converges weakly to \( w_0 = (u_0, v_0) \) in \( X = W_0^{1,p} (\Omega) \times W_0^{1,p} (\Omega) \). Then we have

\[
J(w_0) \leq \liminf_{m \rightarrow +\infty} J(w_m).
\]

Proof. The sequence \( \{w_m = (u_m, v_m)\} \) converges weakly to \( w_0 \in X \). Hence for all bounded \( \Omega' \subset \Omega \), \( \{w_m\} \) is also weakly converging in \( X \). By compactness of the embedding \( W_0^{1,p} (\Omega') \) into \( L^p (\Omega') \), the sequence \( \{w_m\} \) converges strongly in \( L^p (\Omega') \times L^p (\Omega') \). Then the sequences \( \{u_m\} \) and \( \{v_m\} \) converge strongly in \( L^1 (\Omega') \). Applying [16, Theorem 1.6, p9] we deduce that

\[
J(w_0) \leq \liminf_{m \rightarrow +\infty} J(w_m).
\]

The proof of Proposition 2.6 is complete. \( \square \)

Proposition 2.7. Let \( \{w_m = (u_m, v_m)\} \) be a sequence in \( E \) such that:

(i) \( |I(w_m)| \leq c, \ (m = 1, 2, \ldots), \ c \) is positive constant

\[
I'(w_m) \rightarrow 0 \quad \text{in} \ E^* \quad \text{as} \ m \rightarrow +\infty.
\]

(ii) \( \{w_m\} \) converges weakly to \( w_0 = (u_0, v_0) \) in \( X = W_0^{1,p} (\Omega) \times W_0^{1,p} (\Omega) \). Then \( w_0 \in E \) and the sequence \( \{w_m\} \) converges strongly to \( w_0 \) in \( E \).  

Proof. Since \( \{w_m\} \) converges weakly to \( w_0 = (u_0, v_0) \) in \( X \) and the embedding \( W_0^{1,p} \) into \( L^p (\Omega) \) is compact hence the sequences \( \{u_m\} \) and \( \{v_m\} \) converge strongly in \( L^p (\Omega) \) to \( u_0 \) and \( v_0 \), respectively.
By hypothesis (H₁) and (1.7), applying Hölder’s inequality, we obtain
\[
|T(w_m)| \leq \lambda_1 \int_{\Omega} |u_m|^\alpha |v_m|^\beta \ dx + \int_{\Omega} |H(x, u_m, v_m)| \ dx \\
+ \int_{\Omega} (\alpha k_1(x)|u_m| + \beta k_2(x)|v_m|) \ dx \\
\leq \lambda_1 \|u_m\|_{L^p(\Omega)}^\alpha \|v_m\|_{L^p(\Omega)}^\beta + \|\tau\|_{L^{p'}(\Omega)} (\alpha \|u_m\|_{L^p(\Omega)} + \beta \|v_m\|_{L^p(\Omega)}) \\
+ \alpha \|k_1\|_{L^{p'}(\Omega)} \|u_m\|_{L^p(\Omega)} + \beta \|k_2\|_{L^{p'}(\Omega)} \|v_m\|_{L^p(\Omega)}.
\]
Since \( \{u_m\} \) and \( \{v_m\} \) are bounded in \( L^p(\Omega) \), there exists \( M > 0 \) such that:
\[
|T(w_m)| \leq M, \ m = 1, 2, \ldots
\]
Moreover by Proposition 2.6
\[
J(w_0) \leq \liminf_{m \to +\infty} J(w_m) = \liminf_{m \to +\infty} \{I(w_m) - T(w_m)\} \\
\leq \limsup_{m \to +\infty} \{ |I(w_m)| + |T(w_m)| \} \leq C + M < +\infty,
\]
which implies
\[
\int_{\Omega} h_1(x) \nabla u_0 |^p \ dx < +\infty; \int_{\Omega} h_2(x) \nabla v_0 |^p \ dx < +\infty.
\]
Hence \( w_0 = (u_0, v_0) \in E \). Now from (2.4) and hypothesis (H₁) we have:
\[
|(T'(w_m), (w_m - w_0))| \\
\leq \lambda_1 \left\{ \int_{\Omega} \alpha |u_m|^{\alpha-1} |v_m|^\beta |u_m - u_0| \ dx \\
+ \int_{\Omega} \beta |u_m|^{\alpha} |v_m|^{\beta-1} |v_m - v_0| \ dx \\
+ \int_{\Omega} \{ \alpha |f(x, u_m)||u_m - u_0| + \beta |g(x, w_m)||v_m - v_0| \} \ dx \\
+ \int_{\Omega} \{ \alpha k_1(x) |u_m - u_0| + \beta k_2(x) |v_m - v_0| \} \ dx \right\} \\
\leq \lambda_1 \left\{ \alpha \|u_m\|_{L^p(\Omega)}^{\alpha-1} \|v_m\|_{L^p(\Omega)}^\beta \|u_m - u_0\|_{L^p(\Omega)} \\
+ \beta \|u_m\|_{L^p(\Omega)}^{\alpha} \|v_m\|_{L^p(\Omega)}^{\beta-1} \|v_m - v_0\|_{L^p(\Omega)} \right\} \\
+ \{ \|\tau\|_{L^{p'}(\Omega)} (\alpha \|u_m - u_0\|_{L^p(\Omega)} + \beta \|v_m - v_0\|_{L^p(\Omega)}) \\
+ \alpha \|k_1\|_{L^{p'}(\Omega)} \|u_m - u_0\|_{L^p(\Omega)} + \beta \|k_2\|_{L^{p'}(\Omega)} \|v_m - v_0\|_{L^p(\Omega)} \}
\]
Letting \( m \to +\infty \) and remark that
\[
\|u_m - u_0\|_{L^p(\Omega)} \to 0; \quad \|v_m - v_0\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad m \to +\infty,
\]
we deduce that
\[
\lim_{m \to +\infty} (T'(w_m), (w_m - w_0)) = 0.
\]
A generalization of the Landesman-Lazer condition

From this we arrive at
\[
\lim_{m \to +\infty} (J'(w_m), (w_m - w_0)) = \lim_{m \to +\infty} (I'(w_m) - T'(w_m), w_m - w_0) = 0.
\]
Moreover, since \( J \) is convex we have
\[
J(w_0) - J(w_m) \geq (J'(w_m), (w_0 - w_m)).
\]
Letting \( m \to +\infty \) we obtain that
\[
J(w_0) \geq \lim_{m \to +\infty} J(w_m).
\]
On the other hand, by Proposition 2.6 we have
\[
J(w_0) \leq \lim \inf_{m \to +\infty} J(w_m).
\]
This implies that
\[
J(w_0) = \lim_{m \to +\infty} J(w_m).
\]
Next we suppose, by contradiction, that \( \{w_m\} \) does not converge to \( w_0 = (u_0, v_0) \). Then there exists a subsequence \( \{w_{m_k} = (u_{m_k}, v_{m_k})\} \) of \( \{w_m\} \) and \( \epsilon > 0 \) such that
\[
\|w_{m_k} - w_0\|_E \geq \epsilon, \quad k = 1, 2, \ldots.
\]
Recalling the Clarkson’s inequality
\[
\left| \frac{s + t}{2} \right|^p + \left| \frac{s - t}{2} \right|^p \leq \frac{1}{2} (|s|^p + |t|^p), s, t \in \mathbb{R},
\]
we deduce that
\[
\frac{1}{2} J(w_{m_k}) + \frac{1}{2} J(w_0) - J\left(\frac{w_{m_k} + w_0}{2}\right) \geq J\left(\frac{w_{m_k} - w_0}{2}\right), \quad k = 1, 2, \ldots.
\]
Observe that
\[
J\left(\frac{w_{m_k} - w_0}{2}\right) = \alpha \frac{1}{p} 2^p \|u_{m_k} - u_0\|_{E_1}^{p} + \beta \frac{1}{p} 2^p \|v_{m_k} - v_0\|_{E_2}^{p} \\
\geq \frac{1}{p2^p} \min (\alpha, \beta) \|w_{m_k} - w_0\|_E^{p} \geq \frac{\min (\alpha, \beta) \epsilon^{p}}{p2^p} > 0.
\]
Hence
\[
\frac{1}{2} J(w_{m_k}) + \frac{1}{2} J(w_0) - J\left(\frac{w_{m_k} + w_0}{2}\right) \geq \frac{\min (\alpha, \beta) \epsilon^{p}}{p2^p} > 0, \quad k = 1, 2, \ldots.
\]
Letting \( \lim_{k \to +\infty} \inf \) we obtain
\[
J(w_0) - \lim \inf_{k \to +\infty} J\left(\frac{w_{m_k} + w_0}{2}\right) \geq \frac{\min (\alpha, \beta) \epsilon^{p}}{p2^p} > 0.
\]
Again instead of the remark that since \( \left\{ \frac{w_{m_k} + w_0}{2} \right\} \) converges weakly to \( w_0 \) in \( X \), by Proposition 2.6 we have
\[
J(w_0) \leq \lim\inf_{k \to +\infty} J\left( \frac{w_{m_k} + w_0}{2} \right).
\]
Hence we get a contradiction:
\[
0 \geq \min \frac{(\alpha, \beta)}{\frac{2p}{p}} > 0.
\]
Therefore \( \{w_m\} \) converges strongly to \( w_0 \) in \( E \). The Proposition 2.7 is proved.

**Proposition 2.8.** Assume that hypothesis \((H_1)\) and \((H_2)\) are fulfilled. The functional \( I : E \to \mathbb{R} \) given by (2.1) satisfies the Palais-Smale condition on \( E \).

**Proof.** Let \( \{w_m = (u_m, v_m)\} \) be a Palais-Smale sequence in \( E \), i.e:
\[
|I(w_m)| \leq c, \ c \text{ is positive constant.}
\]
(2.8)
\[
I'(w_m) \to 0 \text{ in } E^* \text{ as } m \to +\infty.
\]
(2.9)

First we shall prove that \( \{w_m\} \) is bounded in \( E \). We suppose, by contradiction, that \( \{w_m\} \) is not bounded in \( E \). Without loss of generality we assume that
\[
\|w_m\|_E \to +\infty \text{ as } m \to +\infty.
\]
Let \( \hat{w}_m = \frac{w_m}{\|w_m\|_E} = (\hat{u}_m, \hat{v}_m) \) that is \( \hat{u}_m = \frac{u_m}{\|w_m\|_E} \) and \( \hat{v}_m = \frac{v_m}{\|w_m\|_E} \). Thus \( \hat{w}_m \) is bounded in \( E \), hence \( \hat{w}_m \) is also bounded in \( X = W_0^{1,p} \times W_0^{1,p} \). Then there exists a subsequence \( \{\hat{w}_{m_k} = (\hat{u}_{m_k}, \hat{v}_{m_k})\}_k \) which converges weakly to some \( \hat{w} = (\hat{u}, \hat{v}) \) in \( X \). Since the embedding \( W_0^{1,p}(\Omega) \) into \( L^p(\Omega) \) is compact, the sequences \( \{\hat{u}_{m_k}\} \) and \( \{\hat{v}_{m_k}\} \) converge strongly to \( \hat{u} \) and \( \hat{v} \), respectively, in \( L^p(\Omega) \).

From (2.8) we have
\[
\frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla \hat{u}_{m_k}|^p \, dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla \hat{v}_{m_k}|^p \, dx - \lambda_1 \int_{\Omega} |\hat{u}_{m_k}|^{\alpha-1} |\hat{v}_{m_k}|^{\beta-1} \hat{u}_{m_k} \hat{v}_{m_k} \, dx
\]
(2.10)
\[
- \int_{\Omega} \frac{H(x, u_{m_k})}{\|w_{m_k}\|_E} \, dx + \int_{\Omega} \frac{\alpha k_1 \hat{u}_{m_k} + \beta k_2 \hat{v}_{m_k}}{\|w_{m_k}\|_E} \, dx \leq \frac{c}{\|w_{m_k}\|_E}.
\]
From this, remark that \( h_1(x) \geq 1, h_2(x) \geq 1 \) for a.e \( x \in \Omega \), we get
\[
\lim_{k \to +\infty} \sup \left\{ \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}_{m_k}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}_{m_k}|^p \, dx - \lambda_1 \int_{\Omega} |\hat{u}_{m_k}|^{\alpha-1} |\hat{v}_{m_k}|^{\beta-1} \hat{u}_{m_k} \hat{v}_{m_k} \, dx
\]
(2.11)
\[
- \int_{\Omega} \frac{H(x, u_{m_k})}{\|w_{m_k}\|_E} \, dx + \int_{\Omega} \frac{\alpha k_1 (x) \hat{u}_{m_k} + \beta k_2 (x) \hat{v}_{m_k}}{\|w_{m_k}\|_E} \, dx \right\} \leq 0.
\]
By hypothesis (H₁) on the functions \(f, g, h_i(x), k_i(x), i = 1, 2\), we deduce that
\[
\limsup_{k \to +\infty} \int_{\Omega} \frac{H(x, w_{mk})}{\|w_{mk}\|_E} \, dx = 0, \tag{2.12}
\]
\[
\lim_{k \to +\infty} \int_{\Omega} \frac{\alpha k_1(x) \hat{u}_{mk} + \beta k_2(x) \hat{v}_{mk}}{\|w_{mk}\|_{E}^{p-1}} \, dx = 0. \tag{2.13}
\]
Moreover by Remark 2.4, we infer
\[
\lim_{k \to +\infty} \int_{\Omega} |\hat{u}_{mk}|^{\alpha-1} |\hat{v}_{mk}|^{\beta-1} \hat{u}_{mk} \hat{v}_{mk} \, dx = \int_{\Omega} |\hat{u}|^{\alpha-1} |\hat{v}|^{\beta-1} \hat{u} \hat{v} \, dx. \tag{2.14}
\]
From (2.11) with (2.12), (2.13) and (2.14) we arrive at
\[
\limsup_{k \to +\infty} \left\{ \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}_{mk}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}_{mk}|^p \, dx \right\} \leq \lambda_1 \int_{\Omega} |\hat{u}|^{\alpha-1} |\hat{v}|^{\beta-1} \hat{u} \hat{v} \, dx.
\]
By Proposition 2.6 and the variational characterization of \(\lambda_1\) we get
\[
\lambda_1 \int_{\Omega} |\hat{u}|^{\alpha-1} |\hat{v}|^{\beta-1} \hat{u} \hat{v} \, dx \leq \alpha \int_{\Omega} |\nabla \hat{u}|^p \, dx + \beta \int_{\Omega} |\nabla \hat{v}|^p \, dx
\]
\[
\leq \liminf_{k \to +\infty} \left\{ \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}_{mk}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}_{mk}|^p \, dx \right\}
\]
\[
\leq \limsup_{k \to +\infty} \left\{ \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}_{mk}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}_{mk}|^p \, dx \right\} \leq \lambda_1 \int_{\Omega} |\hat{u}|^{\alpha-1} |\hat{v}|^{\beta-1} \hat{u} \hat{v} \, dx.
\]
Thus these inequalities are indeed equalities and we have
\[
\lim_{k \to +\infty} \left\{ \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}_{mk}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}_{mk}|^p \, dx \right\} = \frac{\alpha}{p} \int_{\Omega} |\nabla \hat{u}|^p \, dx + \frac{\beta}{p} \int_{\Omega} |\nabla \hat{v}|^p \, dx
\]
\[
= \lambda_1 \int_{\Omega} |\hat{u}|^{\alpha-1} |\hat{v}|^{\beta-1} \hat{u} \hat{v} \, dx. \tag{2.15}
\]
We shall prove that \(\hat{u} \neq 0\) and \(\hat{v} \neq 0\).
By contradiction suppose that \(\hat{u} = 0\), thus \(\hat{u}_{mk} \to 0\) in \(L^p(\Omega)\) as \(k \to +\infty\).
Then from the fact that
\[
|\hat{u}_{mk} \hat{v}_{mk}| = \left| \int_{\Omega} |\hat{u}_{mk}|^{\alpha-1} |\hat{v}_{mk}|^{\beta-1} \hat{u}_{mk} \hat{v}_{mk} \, dx \right|
\]
\[
\leq \|\hat{u}_{mk}\|^{\alpha}_{L^p(\Omega)} \|\hat{v}_{mk}\|^{\beta}_{L^p(\Omega)}.
\]
Letting \(k \to +\infty\) since \(\|\hat{u}_{mk}\|_{L^p(\Omega)} \to 0\), we deduce that
\[
\lim_{k \to +\infty} \hat{u}_{mk} \hat{v}_{mk} = 0. \tag{2.16}
\]
From (2.10) taking \(\lim_{k \to +\infty} \sup\) with (2.12), (2.13) and (2.16) we arrive at
(2.17) \[ \limsup_{k \to +\infty} \left\{ \frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla \tilde{u}_{m_k}|^p dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla \tilde{v}_{m_k}|^p dx \right\} = 0. \]

On the other hand, since \( \|\tilde{w}_{m_k}\|_E = 1 \) and
\[
\frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla \tilde{u}_{m_k}|^p dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla \tilde{v}_{m_k}|^p dx \geq \min \left( \frac{\alpha}{p}, \frac{\beta}{p} \right) \|\tilde{w}_{m_k}\|_E = \min \left( \frac{\alpha}{p}, \frac{\beta}{p} \right) > 0,
\]
which contradicts (2.17). Thus \( \tilde{w} \neq 0 \). Similarly we have \( \tilde{v} \neq 0 \).

By again the definition of \( \lambda_1 \) from (2.15) we deduce that \( \tilde{w} = (\tilde{u}, \tilde{v}) = (\varphi_1, \varphi_2) \) or \( \tilde{w} = (\tilde{u}, \tilde{v}) = (-\varphi_1, -\varphi_2) \), where \( (\varphi_1, \varphi_2) \) is eigenpair associated with \( \lambda_1 \) of the problem (1.4).

Next we shall consider following two cases:
Assume that \( \tilde{u}_{m_k} \to \varphi_1, \tilde{v}_{m_k} \to \varphi_2 \) in \( L^p(\Omega) \) as \( k \to +\infty \). Observe that by the variational characterization of \( \lambda_1 \) we have
\[
\frac{\alpha}{p} \int_{\Omega} |\nabla \tilde{u}_{m_k}|^p dx + \frac{\beta}{p} \int_{\Omega} |\nabla \tilde{v}_{m_k}|^p dx - \lambda_1 \int_{\Omega} |u_{m_k}|^{a-1}|v_{m_k}|^{\beta-1} u_{m_k} v_{m_k} dx \geq 0.
\]
From this, note that \( h_1(x) \geq 1, h_2(x) \geq 1 \) a.e \( x \in \Omega \), we have
\[
\frac{\alpha}{p} \int_{\Omega} h_1(x)|\nabla \tilde{u}_{m_k}|^p dx + \frac{\beta}{p} \int_{\Omega} h_2(x)|\nabla \tilde{v}_{m_k}|^p dx - \lambda_1 \int_{\Omega} |u_{m_k}|^{a-1}|v_{m_k}|^{\beta-1} u_{m_k} v_{m_k} dx \geq 0.
\]
Then from (2.8) it implies:
\[
- \int_{\Omega} H(x, u_{m_k}, v_{m_k}) dx + \int_{\Omega} (\alpha k_1(x) u_{m_k} + \beta k_2(x) v_{m_k}) dx \leq c, \quad k = 1, 2, \ldots
\]
After dividing by \( \|w_{m_k}\|_E \) taking \( \lim_{k \to +\infty} \sup \) and remark that
\[
\lim_{k \to +\infty} \int_{\Omega} (\alpha k_1(x) \tilde{u}_{m_k} + \beta k_2(x) \tilde{v}_{m_k}) dx = \int_{\Omega} (\alpha k_1(x) \varphi_1 + \beta k_2(x) \varphi_2) dx,
\]
we arrive at
\[
(2.18) \quad \limsup_{k \to +\infty} \int_{\Omega} \frac{H(x, w_{m_k})}{\|w_{m_k}\|_E} dx \geq \int_{\Omega} (\alpha k_1(x) \varphi_1 + \beta k_2(x) \varphi_2) dx.
\]

We need the following lemma

**Lemma 2.9.** Assume that the hypothesis \((H_1)\) is true. Then
\[
(2.19) \quad \limsup_{k \to +\infty} \int_{\Omega} \frac{H(x, w_{m_k})}{\|w_{m_k}\|_E} dx = \frac{1}{2} \int_{\Omega} (F_1(x) \varphi_1 + G_1(x) \varphi_2) dx,
\]
where \( F_1(x), G_1(x) \) are given by (1.9).

**Proof.** By (1.7), we have
\[
(2.20) \quad H(x, w_{m_k}) = \frac{\alpha}{2} \int_0^{\mu_k} (f(x, s, v_{m_k}) + f(x, s, 0)) ds + \frac{\beta}{2} \int_0^{\nu_k} (g(x, u_{m_k}, t) + g(x, 0, t)) dt.
\]
Set \( l_k = \|w_{mk}\|_E \to +\infty \) as \( k \to +\infty \). Observe that by hypothesis (H_1) on \( f(x, w, g(x, w)) \) we have

\[
\left| \alpha \int_0^{u_{mk}} f(x, s, v_{mk}) ds - \alpha \int_{l_k \psi_1}^{l_k \phi_2} f(x, s, l_k \phi_2) ds \right| \\
\leq \alpha \int_0^{u_{mk}} \left| f(x, s, v_{mk}) - f(x, s, l_k \phi_2) \right| ds + \alpha \int_{l_k \psi_1}^{l_k \phi_2} f(x, s, l_k \phi_2) ds \\
\leq \int_0^{u_{mk}} \frac{\partial f}{\partial s} (x, s, l_k \phi_2 + \delta(v_{mk} - l_k \phi_2)) (v_{mk} - l_k \phi_2) ds \\
+ \alpha \tau(x) \|u_{mk} - l_k \phi_1\| \\
\leq \int_0^{u_{mk}} \beta \frac{\partial g}{\partial s} (x, s, l_k \phi_2 + \delta(v_{mk} - l_k \phi_2)) ds (v_{mk} - l_k \phi_2) \\
+ \alpha \tau(x) \|u_{mk} - l_k \phi_1\| \\
\leq 2\beta \tau(x) |v_{mk} - l_k \phi_2| + \alpha \tau(x) |u_{mk} - l_k \phi_1|, \quad \delta \in (0, 1).
\]

From this and remark that \( \hat{u}_{mk} = \frac{u_{mk}}{l_k}, \hat{v}_{mk} = \frac{v_{mk}}{l_k} \), we get:

\[
\left| \frac{1}{l_k} \int_0^{u_{mk}} f(x, s, v_{mk}) ds - \frac{1}{l_k} \int_{l_k \psi_1}^{l_k \phi_2} f(x, s, l_k \phi_2) ds \right| \\
\leq 2\beta \tau(x) |\hat{v}_{mk} - \varphi_2| + \alpha \tau(x) |\hat{u}_{mk} - \varphi_1|.
\]

Similarly,

\[
(2.22) \quad \left| \frac{1}{l_k} \int_0^{u_{mk}} f(x, s, 0) ds - \frac{1}{l_k} \int_{l_k \psi_1}^{l_k \phi_2} f(x, s, 0) ds \right| \leq \alpha \tau(x) |\hat{u}_{mk} - \varphi_1|.
\]

Combining (2.21) and (2.22) we infer that

\[
\left| \int_{\Omega} \left\{ \frac{1}{l_k} \int_0^{u_{mk}} (f(x, s, v_{mk}) + f(x, s, 0)) ds - \frac{1}{l_k} \int_{l_k \psi_1}^{l_k \phi_2} (f(x, s, l_k \phi_2) + f(x, s, 0)) ds \right\} dx \right| \\
\leq 2\beta \tau(x) \|\hat{v}_{mk} - \varphi_2\|_{L^p(\Omega)} + 2\alpha \tau(x) \|\hat{u}_{mk} - \varphi_1\|_{L^p(\Omega)} + 2\alpha \tau(x) \|\hat{u}_{mk} - \varphi_1\|_{L^p(\Omega)}.
\]

Letting \( k \to +\infty \), since:

\[
\lim_{k \to +\infty} \|\hat{v}_{mk} - \varphi_2\|_{L^2(\Omega)} = 0, \quad \lim_{k \to +\infty} \|\hat{u}_{mk} - \varphi_1\|_{L^2(\Omega)} = 0,
\]

we deduce that

\[
\limsup_{k \to +\infty} \int_{\Omega} \left\{ \frac{1}{l_k} \int_0^{u_{mk}} (f(x, s, v_{mk}) + f(x, s, 0)) ds \right\} dx \\
= \limsup_{k \to +\infty} \int_{\Omega} \left\{ \frac{1}{l_k} \int_{l_k \psi_1}^{l_k \phi_2} (f(x, s, l_k \phi_2) + f(x, s, 0)) ds \right\} dx.
\]
Set $s = y\varphi_1(x)$, $ds = \varphi_1(x)dy$, we get

$$
\int_{l_k}\varphi_1 (f(x, s, l_k\varphi_2) + f(x, s, 0))\,ds = \int_{l_k} (f(x, y\varphi_1, l_k\varphi_2) + f(x, y\varphi_1, 0))\,\varphi_1 dy.
$$

Remark that $l_k = ||w_{mk}||_E \to +\infty$ as $k \to +\infty$, we derive that

$$
\limsup_{k \to +\infty} \int_{\Omega} \left\{ \frac{\alpha}{l_k} \int_{0}^{v_{mk}} (f(x, s, v_{mk}) + f(x, s, 0))\,ds \right\}dx
= \limsup_{k \to +\infty} \int_{\Omega} \left\{ \frac{\alpha}{l_k} \int_{0}^{l_k} (f(x, y\varphi_1, l_k\varphi_2) + f(x, y\varphi_1, 0))\,dy \right\} \varphi_1 dx
$$

(2.23)  \quad = \int_{\Omega} F_1(x)\varphi_1(x)dx.

Similarly, we also derive that

$$
\limsup_{k \to +\infty} \int_{\Omega} \left\{ \frac{\beta}{l_k} \int_{0}^{v_{mk}} (g(x, u_{mk}, t) + g(x, 0, t))\,ds \right\}dx = \int_{\Omega} G_1(x)\varphi_2(x)dx,
$$

where $F_1(x)$ and $G_1(x)$ are given in (1.9). Combining (2.23), (2.24) we obtain:

$$
\limsup_{k \to +\infty} \int_{\Omega} \frac{H(x, w_{mk})}{||w_{mk}||_E}dx = \frac{1}{2} \int_{\Omega} (F_1(x)\varphi_1(x) + G_1(x)\varphi_2(x))\,dx.
$$

Lemma 2.9 is proved. \qed

Now, by (2.19) from (2.18) we obtain

$$
\frac{1}{2} \int_{\Omega} (F_1(x)\varphi_1 + G_1(x)\varphi_2)dx \geq \int_{\Omega} (\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2)dx,
$$

which contradicts (1.10).

If $u_{mk} \to -\varphi_1(x)$, $v_{mk} \to -\varphi_2(x)$ in $L^p(\Omega)$ as $k \to +\infty$, by similar computations as above and remark that in this case:

$$
\limsup_{k \to +\infty} \int_{\Omega} \frac{H(x, w_{mk})}{||w_{mk}||_E}dx = \frac{1}{2} \int_{\Omega} (F_2(x)\varphi_1 + G_2(x)\varphi_2)dx.
$$

Hence from (2.18) we get

$$
-\frac{1}{2} \int_{\Omega} (F_2(x)\varphi_1 + G_2(x)\varphi_2)dx \geq -\int_{\Omega} (\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2)dx,
$$

which gives

$$
\frac{1}{2} \int_{\Omega} (F_2(x)\varphi_1 + G_2(x)\varphi_2)dx \leq \int_{\Omega} (\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2)dx.
$$

Thus we get a contradiction with (1.10).

Hence the Palais-Smale sequence $\{w_m\}$ is bounded in $E$ and it is also bounded in $X$. Then there exists a subsequence $\{w_{mk}\}$ which converges weakly.
to some \( w_0 = (u_0, v_0) \) in \( X \). From Proposition 2.7 we deduce that \( w_0 \in E \) and \( \{w_{m_k}\} \) converges strongly to \( w_0 \) in \( E \). The proof of the Proposition 2.8 is complete.

**Proposition 2.10.** The functional \( I : E \to \mathbb{R} \) given by (2.1) is coercive on \( E \) provided that hypotheses (H1) and (H2) hold.

**Proof.** By contradiction we suppose that \( I \) is not coercive in \( E \). Then it is possible to choose a sequence \( \{w_m = (u_m, v_m)\}_m \) in \( E \) such that

\[
\|w_m\|_E \to +\infty \quad \text{and} \quad I(w_m) \leq c, \; c \text{ is positive constant.}
\]

Let \( \tilde{w}_m = \frac{w_m}{\|w_m\|_E} = (\tilde{u}_m, \tilde{v}_m) \). Hence the sequence \( \{\tilde{w}_m\} \) is bounded in \( E \) and then bounded in \( X = W_0^{1, p}(\Omega) \times W_0^{1, p}(\Omega) \). Therefore it has a subsequence \( \tilde{w}_{m_k} = (\tilde{u}_{m_k}, \tilde{v}_{m_k}) \) which converges weakly in \( X \) and converges strongly in \( L^p(\Omega) \times L^p(\Omega) \). Applying arguments used in the proof of Proposition 2.8, we can prove that \( \tilde{w}_{m_k} \to (\varphi_1, \varphi_2) \) or \( \tilde{w}_{m_k} \to (-\varphi_1, -\varphi_2) \) in \( L^p(\Omega) \times L^p(\Omega) \) as \( k \to +\infty \) where \( (\varphi_1, \varphi_2) \) is eigenpair associated with eigenvalue \( \lambda_1 \) of the problem (1.4). Assume that \( \tilde{w}_{m_k} \to (\varphi_1, \varphi_2) \) in \( L^p(\Omega) \times L^p(\Omega) \) as \( k \to +\infty \). By again the same arguments used in the proof of the Proposition 2.8 we arrive at

\[
\frac{1}{2} \int_{\Omega} (F_1(x)\varphi_1 + G_1(x)\varphi_2) dx \geq \int_{\Omega} (ak_1(x)\varphi_1 + \beta k_2(x)\varphi_2) dx,
\]

which contradicts (1.10). If \( \tilde{w}_m \to (-\varphi_1, -\varphi_2) \) in \( L^p(\Omega) \times L^p(\Omega) \) as \( k \to +\infty \), we get

\[
\frac{1}{2} \int_{\Omega} (F_2(x)\varphi_1 + G_2(x)\varphi_2) dx \leq \int_{\Omega} (ak_1(x)\varphi_1 + \beta k_2(x)\varphi_2) dx.
\]

This is in contradiction with (1.10). Thus \( I \) is coercive on \( E \).

**Proof of Theorem 1.1.** By Propositions 2.8 and Proposition 2.6, applying the Minimum Principle (see Theorem 2.2), we deduce that the functional \( I \) attains its proper infimum at some \( w_0 = (u_0, v_0) \in E \), so that the problem (1.1) has at least a weak solution \( w_0 \in E \). Moreover by hypothesis (H1) on \( f(x, s, t), g(x, s, t), k_1(x), k_2(x) \), it is clear that \( w_0 \) is nontrivial and the proof of Theorem 1.1 is complete.

**Conflict of Interest Statement:** The authors declare that they have no conflict of interest.

**Acknowledgements**

This research supported by the National Foundation for Science and Technology Development of Viet Nam (NAFOSTED under grant number 101.02-2014.03)
REFERENCES


A generalization of the Landesman-Lazer condition

(Bui Quoc Hung) Faculty of Information Technology, Le Quy Don Technical University, 236 Hoang Quoc Viet, Bac Tu Liem, Hanoi, Vietnam.
E-mail address: quochung2806@yahoo.com

(Hoang Quoc Toan) Department of Mathematics, Hanoi University of Science, 334 Nguyen Trai, Thanh Xuan, Ha Noi, Vietnam.
E-mail address: hq.toan@yahoo.com