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ON A P-LAPLACIAN SYSTEM AND A GENERALIZATION OF THE LANDESMAN-LAZER TYPE CONDITION

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Abstract. This article shows the existence of weak solutions of a resonance problem for nonuniformly p-Laplacian system in a bounded domain in $\mathbb{R}^N$. Our arguments are based on the minimum principle and rely on a generalization of the Landesman-Lazer type condition.
Keywords: Semilinear elliptic equation, non-uniform, Landesman-Lazer condition, minimum principle.
MSC(2010): Primary: 35J20, Secondary: 35J60, 58E05.

1. Introduction and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. In the present paper we consider the existence of weak solutions of the following Dirichlet problem at resonance for nonuniformly p-Laplacian system:

(1.1)
$$\begin{cases} \text{-div}(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda_1|u|^{\alpha-1}|v|^{\beta-1}v + f(x,u,v) - k_1(x), & \text{in } \Omega \\ \text{-div}(h_2(x)|\nabla v|^{p-2}\nabla v) = \lambda_1|u|^{\alpha-1}|v|^{\beta-1}u + g(x,u,v) - k_2(x), & \text{in } \Omega \\ u = 0 \ ; \qquad v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$p \geq 2, \ \alpha \geq 1, \ \beta \geq 1, \ \alpha + \beta = p. \tag{1.2}$$

and $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions which will be specified later,

$$h_i(x) \in L^1_{loc}(\Omega), \quad h_i(x) \geq 1, \quad \text{for a.e } x \in \Omega, \ i = 1, 2, \tag{1.3}$$

$$k_i(x) \in L^{p'}(\Omega), p' = \frac{p}{p-1}, \ k_i(x) > 0, \ \text{for a.e } x \in \bar{\Omega}, \ i = 1, 2.$$

λ_1 denotes the first eigenvalue of the problem:

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(1.4)
$$\begin{cases} -\Delta_p u = \lambda|u|^{\alpha-1}|v|^{\beta-1}v, \\ -\Delta_p v = \lambda|u|^{\alpha-1}|v|^{\beta-1}u, \end{cases}$$

and $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $p > 2$, $\alpha > 1$, $\beta > 1$, $\alpha + \beta = p$.

It is well-known that the principle eigenvalue $\lambda_1 = \lambda_1(p)$ of (1.4) is obtained using the Ljusternick-Schnirelmann theory by minimizing the functional

$$J(u,v) = \frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx,$$

on C^1 - manifold:

$$S = \left\{(u,v) \in X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) : \wedge(u,v) = 1\right\},$$

where

$$\wedge(u,v) = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u.vdx,$$

that is $\lambda_1 = \lambda_1(p)$ can be variational characterized as

(1.5)
$$\lambda_1 = \lambda_1(p) = \inf_{\wedge(u,v)>0} \frac{J(u,v)}{\wedge(u,v)} = \inf_{(u,v)\in X: uv>0} \frac{\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx + \frac{\beta}{p}\int_\Omega |\nabla v|^p dx}{\int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uvdx}.$$

Moreover the eigenpair (φ_1, φ_2) associated with λ_1 is componentwise positive and unique (up to multiplication by nonzero scalar) (see [1, Theorem 2.2] and [15, Remark 5.4]).

We firstly make some comments on the problem (1.1). Observe that the existence of weak solutions of (p,q)-Laplacian systems at resonance in bounded domains with Dirichlet boundary condition, was first considered by Zographopoulos in [20]. Later in [10] Kandilakis and Magiropoulos have studied a quasilinear elliptic system with resonance part and nonlinear boundary condition in an unbounded domain by assuming the nonlinearities f and g depending only on variable u or v. In [14], Ou and Tang have considered the same system as in [10] with Dirichlet condition in a bounded domain. In these papers, the existence of weak solutions is obtained by critical point theory under a Landesman-Lazer type condition. At the same time for nonuniformly nonlinear elliptic equations involving p-Laplacian $(p \geq 2)$ at resonance we refer the reader to [12, 13, 18].

In this paper by introducing a generalization of Landesman-Lazer type condition we shall prove an existence result for a p-Laplacian system on resonance in bounded domain with the nonlinearities f and g to be functions depending on both variables u and v.

Note that in [9] we considered system (1.1) in the case $h_1(x) = h_2(x) = 1$ and shows the existence of weak solutions of (1.1) in $W_0^{1,p} \times W_0^{1,p}$. Our arguments are based on the saddle point theorem and rely on a generalization of the Landesman-Lazer type condition.

Recall that due to $h_i(x) \in L^1_{loc}(\Omega),\ i = 1,2$, the problem (1.1) now is nonuniformly in sense that the Euler-Lagrange functional associated to the problem may be infinity at some $w_0 = (u_0, v_0) \in X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Hence we must consider the problem (1.1) in some suitable subspace of $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

As usually $W_0^{1,p}(\Omega)$ denotes the Sobolev space which can be defined as the completion of $C_0^\infty(\Omega)$ under the norm:

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Now we define the following subspaces $E_i,\ i = 1,2$, of $W_0^{1,p}(\Omega)$ by:

$$E_i = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega h_i(x)|\nabla u|^p dx < +\infty\right\},$$

where $h_i(x),\ i = 1,2$, satisfy condition (1.2). E_i can be endowed with the norm

$$\|u\|_{E_i} = \left(\int_\Omega h_i(x)|\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Applying the arguments as those used in the proof of [8, Proposition 1.1] we can prove the following proposition.

Proposition 1.1. *For each $i = 1,2$, E_i is a Banach space and the embeddings E_i into $W_0^{1,p}(\Omega)$ are continuous.*

Proof. It is clear that E_i is a normed space. Let $\{u_m\}$ be a Cauchy sequence in E_i. Then

$$\lim_{m,k\to+\infty} \|u_m - u_k\|^p_{E_i} = \lim_{m,k\to+\infty} \int_\Omega h_i(x)|\nabla u_m - \nabla u_k|^p dx = 0,$$

and $\{\|u_m\|_{E_i}\}$ is bounded. By (1.3) : $\|u_m - u_k\|_{W_0^{1,p}(\Omega)} \le \|u_m - u_k\|_{E_i}$ for $m, k = 1, 2, \ldots$. Hence the sequence $\{u_m\}$ is also a Cauchy sequence in $W_0^{1,p}(\Omega)$ and it converges to some u in $W_0^{1,p}(\Omega)$, i.e:

$$\lim_{m\to+\infty} \int_\Omega |\nabla u_m - \nabla u|^p \, dx = 0.$$

It follows that $\nabla u_m \to \nabla u$ in $L^p(\Omega)$ and there exists a subsequence $\{\nabla u_{m_k}\}$ converging to ∇u a.e $x \in \Omega$. Applying Fatou's lemma we get

$$\int_\Omega h_i(x)|\nabla u|^p dx \le \liminf_{k\to+\infty} \int_\Omega h_i(x) \left|\nabla u_{m_k}\right|^p dx < +\infty$$

Hence $u \in E_i$. Applying again Fatou's lemma we get

$$\begin{aligned}0 &\le \lim_{k\to+\infty}\int_\Omega h_i(x)|\nabla u_{m_k} - \nabla u|^p dx \\ &\le \lim_{k\to+\infty}\left\{\lim_{l\to+\infty}\int_\Omega h_i(x)|\nabla u_{m_k} - \nabla u_{m_l}|^p dx\right\} = 0.\end{aligned}$$

Hence $\{u_{m_k}\}$ converges to u in E_i. From this, it implies the sequence $\{u_m\}$ converges to u in E_i, $i = 1, 2$. Thus E_i is a Banach space and the continuous embedding E_i into $W_0^{1,p}$ holds true. Proposition 1.1 is proved. □

Remark 1.2. Since the embedding $W_0^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact, hence $E_i \hookrightarrow L^p(\Omega)$ compactly.

Set $E = E_1 \times E_2$ and for $w = (u, v) \in E$:

$$\|w\|_E = \left(\|u\|_{E_1}^p + \|v\|_{E_2}^p\right)^{\frac{1}{p}}.$$

Moreover for simplicity of notation denotes by $X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Then we have $\|w\|_X \le \|w\|_E$, $\forall w = (u, v) \in E$.

Definition 1.3. Function $w = (u, v) \in E$ is called a weak solution of the problem (1.1) if and only if

$$\begin{aligned}&\alpha\int_\Omega h_1(x)\nabla u\nabla\bar{u}dx + \beta\int_\Omega h_2(x)\nabla v\nabla\bar{v}dx \\ &\quad - \lambda_1\int_\Omega \left(\alpha|u|^{\alpha-1}|v|^{\beta-1}v\bar{u} + \beta|u|^{\alpha-1}|v|^{\beta-1}u\bar{v}\right)dx \\ &\quad - \int_\Omega (\alpha f(x,u,v)\bar{u} + \beta g(x,u,v)\bar{v})\,dx \\ &\quad + \int_\Omega (\alpha k_1(x)\bar{u} + \beta k_2(x)\bar{v})\,dx = 0, \qquad \forall\bar{w} = (\bar{u}, \bar{v}) \in E.\end{aligned}$$

Let us introduce the following some conditions on nonlinearities of system (1.1):

(H_1)

(i) $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions: $f(x,0,0) = 0$, $g(x,0,0) = 0$.

(ii) There exists function $\tau(x) \in L^{p'}(\Omega)$, $p' = \dfrac{p}{p-1}$ such that:

$|f(x,s,t)| \le \tau(x)$, $|g(x,s,t)| \le \tau(x)$, for a.e $x \in \Omega, (s,t) \in \mathbb{R}^2$.

(iii) For$(s,t) \in \mathbb{R}^2$:

$$\alpha\frac{\partial f(x,s,t)}{\partial t} = \beta\frac{\partial g(x,s,t)}{\partial s} \qquad \text{for a.e } x \in \Omega. \tag{1.6}$$

Denotes, for $(u,v)\in\mathbb{R}^2$

(1.7)

$$H(x,u,v)=\frac{\alpha}{2}\int_0^u(f(x,s,v)+f(x,s,0))\,ds+\frac{\beta}{2}\int_0^v(g(x,u,t)+g(x,0,t))\,dt, \text{ for a.e } x\in\Omega.$$

Remark 1.4. By hypothesis (1.6), from (1.7) with some simple computations we deduce that:

(1.8)

$$\frac{\partial H(x,s,t)}{\partial s}=\alpha f(x,s,t)\ ,\ \frac{\partial H(x,s,t)}{\partial t}=\beta g(x,s,t), \text{ a.e } x\in\Omega, \forall (s,t)\in\mathbb{R}^2.$$

Now we define, for $i,j=1,2$:

(1.9)

$$F_i(x)=\limsup_{\tau\to+\infty}\frac{\alpha}{\tau}\int_0^\tau\left(f\left(x,(-1)^{1+i}y\varphi_1,(-1)^{1+i}\tau\varphi_2\right)+f\left(x,(-1)^{1+i}y\varphi_1,0\right)\right)dy,$$

$$G_j(x)=\limsup_{\tau\to+\infty}\frac{\beta}{\tau}\int_0^\tau\left(g\left(x,(-1)^{1+j}\tau\varphi_1,(-1)^{1+j}y\varphi_2\right)+g\left(x,0,(-1)^{1+j}y\varphi_2\right)\right)dy.$$

Assume that

(H_2)

(1.10)

$$\int_\Omega(F_1(x)\varphi_1(x)+G_1(x)\varphi_2(x))\,dx<2\int_\Omega(\alpha k_1(x)\varphi_1(x)+\beta k_2(x)\varphi_2(x))\,dx$$
$$<\int_\Omega(F_2(x)\varphi_1(x)+G_2(x)\varphi_2(x))\,dx.$$

Remark 1.5. For example, we can take functions $f(x,u,v), g(x,u,v)$ as following:

$$f(x,u,v)=\tau_1(x)\sin\left(\frac{u}{\beta}+\frac{v}{\alpha}\right)+\eta_1(x)\frac{u}{\sqrt{1+u^2}},$$
$$g(x,u,v)=\tau_1(x)\sin\left(\frac{u}{\beta}+\frac{v}{\alpha}\right)+\eta_2(x)\frac{v}{\sqrt{1+v^2}},$$

where $\tau_1(x),\eta_1(x),\eta_2(x)$ are functions in $L^{p'}(\Omega)$ and $\eta_1(x)<0,\eta_2(x)<0$ for $x\in\Omega$.

By some simple computations we get:

$$F_1(x)=2\alpha\eta_1(x), \qquad F_2(x)=-2\alpha\eta_1(x),$$
$$G_1(x)=2\beta\eta_2(x), \qquad G_2(x)=-2\beta\eta_2(x).$$

Therefore, hypothesis (1.10) is satisfied whenever

$$-\eta_1(x)>k_1(x) \quad \text{and} \quad -\eta_2(x)>k_2(x).$$

Our main result is given by the following theorem:

Theorem 1.1. *Assume that the conditions* (H_1) *and* (H_2) *are fulfilled. Then the problem* (1.1) *has at least a nontrivial weak solution in* E.

Proof of Theorem 1.1 is based on variational techniques and the Minimum Principle.

2. Proof of the main result

We define the Euler-Lagrange functional associated to the problem (1.1) by

$$\begin{aligned} I(w) = {} & \frac{\alpha}{p}\int_\Omega h_1(x)|\nabla u|^p dx + \frac{\beta}{p}\int_\Omega h_2(x)|\nabla v|^p dx - \lambda_1 \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uvdx \\ & - \int_\Omega H(x,u,v)dx + \int_\Omega (\alpha k_1(x)u + \beta k_2(x)v)\, dx \\ = {} & J(w) + T(w), \quad \forall w = (u,v) \in E, \end{aligned} \tag{2.1}$$

where

$$J(w) = \frac{\alpha}{p}\int_\Omega h_1(x)|\nabla u|^p dx + \frac{\beta}{p}\int_\Omega h_2(x)|\nabla v|^p dx, \tag{2.2}$$

$$T(w) = -\lambda_1 \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uvdx - \int_\Omega H(x,u,v)dx + \int_\Omega (\alpha k_1(x)u + \beta k_2(x)v)\, dx. \tag{2.3}$$

Firstly we note that due to $h_i(x) \in L^1_{loc}(\Omega)$, $i = 1,2$, in general the functional $J(w)$ may not belong to $C^1(E)$. Therefore we need some modifications in order to apply the critical point theory to our problem.

Definition 2.1. (see [6, Definition 2.1]) Let I be a map from a Banach space X to R. We say that I is weakly continuously differentiable on X if the following conditions are satisfied:

(i) I is continuous on X

(ii) For any $u \in X$ there exists a linear map $I'(u)$ from X into R such that:

$$\lim_{t\to 0}\frac{I(u+tv) - I(u)}{t} = (I'(u), v) \qquad , \forall v \in X.$$

(iii) For any $v \in X$ the map $u \to (I'(u), v)$ is continuous on X.

Denotes by $C^1_w(X)$ the set of weakly continuously differentiable functionals on X. It is clear that $C^1(X) \subset C^1_w(X)$, where we denote by $C^1(X)$ the set of all continuously Fréchet differentiable functionals on X.

Let $I \in C^1_w(X)$ we put:

$$\|I'(u)\| = \text{Sup}\left\{| < I'(u), h > | : h \in X : ||h|| = 1\right\}, \qquad \forall u \in X$$

We say that $I \in C_w^1(X)$ satisfies the Palais-Smale condition on X if any sequence $\{u_m\} \subset X$ for which $\{I(u_m)\}$ is bounded and $\lim_{m\to+\infty} \|I'(u_m)\|_{X^*} = 0$ has a convergent subsequence in X.

Theorem 2.2 (The minimum Principle, see in [12, 13, Theorem 2.3])**.** *Let X be a Banach space and $I \in C_w^1(X)$. Assume that:*

(i) *I is bounded from below, $c = \inf_X I(u)$*

(ii) *I satisfies the Palais-Smale condition on X.*

Then there exists $u_0 \in X$ such that $I(u_0) = c$.

The following proposition concerns the smoothness of the functional $I = J + T$ given by (2.1).

Proposition 2.3. *Assuming hypothesis* (H_1) *and* (H_2) *are fulfilled. We assert that:*

(i) *The functional $T(w), w \in E$ given by* (2.3) *is continuous on E. Moreover, T is weakly continuously differentiable on E and*

$$(T'(w), \bar{w}) = -\lambda_1 \int_\Omega \left(\alpha|u|^{\alpha-1}|v|^{\beta-1}v\bar{u} + \beta|u|^{\alpha-1}|v|^{\beta-1}u\bar{v}\right) dx - \int_\Omega (\alpha f(x,w)\bar{u} + \beta g(x,w)\bar{v})\, dx + \int_\Omega (\alpha k_1(x)\bar{u} + \beta k_2(x)\bar{v})\, dx, \qquad \forall w = (u,v);\ \bar{w} = (\bar{u},\bar{v}) \in E. \tag{2.4}$$

(ii) *The functional $J(w), w \in E$ given by* (2.2) *is weakly continuously differentiable on E and we have: $\forall w = (u,v),\ \bar{w} = (\bar{u},\bar{v}) \in E$*

$$(J'(w), \bar{w}) = \alpha \int_\Omega h_1(x)|\nabla u|^{p-1}\nabla u \nabla \bar{u} dx + \beta \int_\Omega h_2(x)|\nabla v|^{p-1}\nabla v \nabla \bar{v} dx. \tag{2.5}$$

Thus $I = J + T$ is weakly continuously differentiable on E and

$$(I'(w), \bar{w}) = (J'(w), \bar{w}) + (T'(w), \bar{w}), \qquad \forall w = (u,v); \bar{w} = (\bar{u},\bar{v}) \in E. \tag{2.6}$$

In the proof of the Proposition 2.3 we need the following remarks:

Remark 2.4. By similar arguments as those used in the proof of [21, Lemma 2.3] and [10, Lemma 5] we infer that the functional $\wedge : E \to \mathbb{R}$ and operator $\Gamma : E \to E^*$ given by

$$\wedge(u,v) = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uv dx, \qquad (u,v) \in E,$$

and

$$\langle \Gamma(u,v), (\bar{u},\bar{v})\rangle = \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}v\bar{u} dx + \int_\Omega |u|^{\alpha-1}|v|^{\beta-1}u\bar{v} dx, (u,v); (\bar{u},\bar{v}) \in E,$$

are compact.

Proof. (i) By the Theorem C_1 in [16, p. 248] and the Remark 2.4 with some standard arguments we infer that $T \in C^1(X)$ where $X = W_0^{1,p} \times W_0^{1,p}$. Moreover since the embedding $E \to X$ is continuous, we have $T \in C^1(E)$ and hence $T \in C_w^1(E)$ and

$$\begin{aligned}(T'(w), \bar{w}) = &- \lambda_1 \int_\Omega \left(\alpha|u|^{\alpha-1}|v|^{\beta-1}v\bar{u} + \beta|u|^{\alpha-1}|v|^{\beta-1}u\bar{v}\right) dx \\ &- \int_\Omega (\alpha f(x,w)\bar{u} + \beta g(x,w)\bar{v})\, dx \\ &+ \int_\Omega (\alpha k_1(x)\bar{u} + \beta k_2(x)\bar{v})\, dx, \qquad \forall w = (u,v); \bar{w} = (\bar{u}, \bar{v}) \in E.\end{aligned}$$

(ii) By similar arguments used in the proof of [8, Proposition 2.1], we deduce that $J \in C_w^1(E)$ and (2.5), (2.6) hold true. The proof of Proposition 2.3 is complete. □

Remark 2.5. From Proposition 2.3, it implies that the critical points of the functional I given by (2.1) correspond to the weak solutions of the problem (1.1)

Proposition 2.6. *Suppose that the sequence $\{w_m = (u_m, v_m)\}_m$ converges weakly to $w_0 = (u_0, v_0)$ in $X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Then we have*

$$J(w_0) \le \liminf_{m\to+\infty} J(w_m). \tag{2.7}$$

Proof. The sequence $\{w_m = (u_m, v_m)\}$ converges weakly to $w_0 \in X$. Hence for all bounded $\Omega' \subset \Omega$, $\{w_m\}$ is also weakly converging in X. By compactness of the embedding $W_0^{1,p}(\Omega')$ into $L^p(\Omega')$, the sequence $\{w_m\}$ converges strongly in $L^p(\Omega') \times L^p(\Omega')$. Then the sequences $\{u_m\}$ and $\{v_m\}$ converge strongly in $L^1(\Omega')$. Applying [16, Theorem 1.6, p9] we deduce that

$$J(w_0) \le \liminf_{m\to+\infty} J(w_m).$$

The proof of Proposition 2.6 is complete. □

Proposition 2.7. *Let $\{w_m = (u_m, v_m)\}$ be a sequence in E such that:*
(i) $|I(w_m)| \le c$, $(m = 1, 2, \ldots)$, *c is positive constant*
 $I'(w_m) \to 0$ *in* E^* *as* $m \to +\infty$.
(ii) $\{w_m\}$ *converges weakly to* $w_0 = (u_0, v_0)$ *in* $X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.
Then $w_0 \in E$ and the sequence $\{w_m\}$ converges strongly to w_0 in E.

Proof. Since $\{w_m\}$ converges weakly to $w_0 = (u_0, v_0)$ in X and the embedding $W_0^{1,p}$ into $L^p(\Omega)$ is compact hence the sequences $\{u_m\}$ and $\{v_m\}$ converge strongly in $L^p(\Omega)$ to u_0 and v_0, respectively.

By hypothesis (H_1) and (1.7), applying Hölder's inequality, we obtain

$$\begin{aligned}|T(w_m)| &\le \lambda_1 \int_\Omega |u_m|^\alpha |v_m|^\beta dx + \int_\Omega |H(x,u_m,v_m)|dx \\ &\quad + \int_\Omega (\alpha k_1(x)|u_m| + \beta k_2(x)|v_m|)\, dx \\ &\le \lambda_1 \|u_m\|^\alpha_{L^p(\Omega)} \|v_m\|^\beta_{L^p(\Omega)} + \|\tau\|_{L^{p'}(\Omega)} \left(\alpha \|u_m\|_{L^p(\Omega)} + \beta \|v_m\|_{L^p(\Omega)}\right) \\ &\quad + \alpha \|k_1\|_{L^{p'}(\Omega)} \|u_m\|_{L^p(\Omega)} + \beta \|k_2\|_{L^{p'}(\Omega)} \|v_m\|_{L^p(\Omega)}.\end{aligned}$$

Since $\{u_m\}$ and $\{v_m\}$ are bounded in $L^p(\Omega)$, there exists $M > 0$ such that:

$$|T(w_m)| \le M, \ m = 1, 2, \dots .$$

Moreover by Proposition 2.6

$$\begin{aligned}J(w_0) &\le \liminf_{m\to+\infty} J(w_m) = \liminf_{m\to+\infty} \{I(w_m) - T(w_m)\} \\ &\le \limsup_{m\to+\infty} \{|I(w_m)| + |T(w_m)|\} \le C + M < +\infty,\end{aligned}$$

which implies

$$\int_\Omega h_1(x) |\nabla u_0|^p dx < +\infty \ ; \ \int_\Omega h_2(x) |\nabla v_0|^p dx < +\infty.$$

Hence $w_0 = (u_0, v_0) \in E$. Now from (2.4) and hypothesis (H_1) we have:

$$\begin{aligned}&\left|(T'(w_m), (w_m - w_0))\right| \\ &\le \lambda_1 \left\{ \int_\Omega \alpha |u_m|^{\alpha-1} |v_m|^\beta |u_m - u_0| dx \right. \\ &\qquad \left. + \int_\Omega \beta |u_m|^\alpha |v_m|^{\beta-1} |v_m - v_0| dx \right\} \\ &\qquad + \int_\Omega \{\alpha |f(x,w_m)||u_m - u_0| + \beta |g(x,w_m)||v_m - v_0|\}\, dx \\ &\qquad + \int_\Omega \{\alpha k_1(x)|u_m - u_0| + \beta k_2(x)|v_m - v_0|\}\, dx \\ &\le \lambda_1 \left\{ \alpha ||u_m||^{\alpha-1}_{L^p(\Omega)} ||v_m||^\beta_{L^p(\Omega)} ||u_m - u_0||_{L^p(\Omega)} \right. \\ &\qquad \left. + \beta ||u_m||^\alpha_{L^p(\Omega)} ||v_m||^{\beta-1}_{L^p(\Omega)} ||v_m - v_0||_{L^p(\Omega)} \right\} \\ &\qquad + ||\tau||_{L^{p'}(\Omega)} \left(\alpha ||u_m - u_0||_{L^p(\Omega)} + \beta ||v_m - v_0||_{L^p(\Omega)}\right) \\ &\qquad + \alpha \|k_1\|_{L^{p'}(\Omega)} \|u_m - u_0\|_{L^p(\Omega)} + \beta \|k_2\|_{L^{p'}(\Omega)} \|v_m - v_0\|_{L^p(\Omega)}.\end{aligned}$$

Letting $m \to +\infty$ and remark that

$$\|u_m - u_0\|_{L^p(\Omega)} \to 0; \quad \|v_m - v_0\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad m \to +\infty,$$

we deduce that

$$\lim_{m\to+\infty} (T'(w_m), (w_m - w_0)) = 0.$$

From this we arrive at

$$\lim_{m\to+\infty}(J'(w_m),(w_m-w_0))=\lim_{m\to+\infty}(I'(w_m)-T'(w_m),w_m-w_0)=0.$$

Moreover, since J is convex we have

$$J(w_0)-J(w_m)\geq (J'(w_m),(w_0-w_m)).$$

Letting $m\to+\infty$ we obtain that

$$J(w_0)\geq \lim_{m\to+\infty} J(w_m).$$

On the other hand, by Proposition 2.6 we have

$$J(w_0)\leq \liminf_{m\to+\infty} J(w_m).$$

This implies that

$$J(w_0)=\lim_{m\to+\infty} J(w_m).$$

Next we suppose, by contradiction, that $\{w_m\}$ does not converge to $w_0=(u_0,v_0)$. Then there exists a subsequence $\{w_{m_k}=(u_{m_k},v_{m_k})\}_k$ of $\{w_m\}$ and $\epsilon>0$ such that

$$\|w_{m_k}-w_0\|_E\geq\epsilon,\ k=1,2,\dots.$$

Recalling the Clarkson's inequality

$$\left|\frac{s+t}{2}\right|^p+\left|\frac{s-t}{2}\right|^p\leq\frac{1}{2}\left(|s|^p+|t|^p\right)\ ,s,t\in\mathbb{R},$$

we deduce that

$$\frac{1}{2}J(w_{m_k})+\frac{1}{2}J(w_0)-J\left(\frac{w_{m_k}+w_0}{2}\right)\geq J\left(\frac{w_{m_k}-w_0}{2}\right),\ k=1,2,\dots.$$

Observe that

$$\begin{aligned}J\left(\frac{w_{m_k}-w_0}{2}\right)&=\frac{\alpha}{p}\frac{1}{2^p}\|u_{m_k}-u_0\|_{E_1}^p+\frac{\beta}{p}\frac{1}{2^p}\|v_{m_k}-v_0\|_{E_2}^p\\&\geq\frac{1}{p2^p}\min\left(\alpha,\beta\right)\|w_{m_k}-w_0\|_E^p\geq\frac{\min\left(\alpha,\beta\right)}{p}\frac{\epsilon^p}{2^p}>0.\end{aligned}$$

Hence

$$\frac{1}{2}J\left(w_{m_k}\right)+\frac{1}{2}J(w_0)-J\left(\frac{w_{m_k}+w_0}{2}\right)\geq\frac{\min\left(\alpha,\beta\right)}{p}\frac{\epsilon^p}{2^p}>0\ ,\ k=1,2,\dots.$$

Letting $\lim_{k\to+\infty}\inf$ we obtain

$$J(w_0)-\liminf_{k\to+\infty}J\left(\frac{w_{m_k}+w_0}{2}\right)\geq\frac{\min\left(\alpha,\beta\right)}{p}\frac{\epsilon^p}{2^p}>0.$$

Again instead of the remark that since $\left\{\frac{w_{m_k}+w_0}{2}\right\}$ converges weakly to w_0 in X, by Proposition 2.6 we have

$$J(w_0) \leq \liminf_{k\to+\infty} J\left(\frac{w_{m_k}+w_0}{2}\right).$$

Hence we get a contradiction:

$$0 \geq \frac{\min(\alpha,\beta)}{p}\frac{\epsilon^p}{2^p} > 0.$$

Therefore $\{w_m\}$ converges strongly to w_0 in E. The Proposition 2.7 is proved. □

Proposition 2.8. *Assume that hypothesis* (H$_1$) *and* (H$_2$) *are fulfilled. The functional* $I : E \to \mathbb{R}$ *given by* (2.1) *satisfies the Palais-Smale condition on* E.

Proof. Let $\{w_m = (u_m, v_m)\}$ be a Palais-Smale sequence in E, i.e:

$$|I(w_m)| \leq c,\ c \text{ is positive constant.} \tag{2.8}$$

$$I'(w_m) \to 0 \text{ in } E^* \text{ as } m \to +\infty. \tag{2.9}$$

First we shall prove that $\{w_m\}$ is bounded in E. We suppose, by contradiction, that $\{w_m\}$ is not bounded in E. Without loss of generality we assume that

$$\|w_m\|_E \to +\infty \text{ as } m \to +\infty$$

Let $\widehat{w}_m = \frac{w_m}{\|w_m\|_E} = (\widehat{u}_m, \widehat{v}_m)$ that is $\widehat{u}_m = \frac{u_m}{\|w_m\|_E}$ and $\widehat{v}_m = \frac{v_m}{\|w_m\|_E}$. Thus $\widehat{w}_m$ is bounded in E, hence $\widehat{w}_m$ is also bounded in $X = W_0^{1,p} \times W_0^{1,p}$. Then there exists a subsequence $\{\widehat{w}_{m_k} = (\widehat{u}_{m_k}, \widehat{v}_{m_k})\}_k$ which converges weakly to some $\widehat{w} = (\widehat{u}, \widehat{v})$ in X. Since the embedding $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, the sequences $\{\widehat{u}_{m_k}\}$ and $\{\widehat{v}_{m_k}\}$ converge strongly to $\widehat{u}$ and $\widehat{v}$, respectively, in $L^p(\Omega)$.

From (2.8) we have

$$\begin{aligned}\frac{\alpha}{p}\int_\Omega h_1(x)|\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega h_2(x)\,|\nabla\widehat{v}_{m_k}|^p\,dx - \lambda_1\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}\,|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx \\ -\int_\Omega \frac{H(x,w_{m_k})}{\|w_{m_k}\|_E^p}dx + \int_\Omega \frac{\alpha k_1\widehat{u}_{m_k} + \beta k_2\widehat{v}_{m_k}}{\|w_{m_k}\|_E^{p-1}}dx \leq \frac{c}{\|w_{m_k}\|_E^p}.\end{aligned} \tag{2.10}$$

From this, remark that $h_1(x) \geq 1, h_2(x) \geq 1$ for a.e $x \in \Omega$, we get

$$\begin{aligned}\limsup_{k\to+\infty}\Bigg\{\frac{\alpha}{p}\int_\Omega |\nabla\widehat{u}_{m_k}|^p dx + \frac{\beta}{p}\int_\Omega |\nabla\widehat{v}_{m_k}|^p dx - \lambda_1\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx \\ -\int_\Omega \frac{H(x,w_{m_k})}{\|w_{m_k}\|_E^p}dx + \int_\Omega \frac{\alpha k_1(x)\widehat{u}_{m_k} + \beta k_2(x)\widehat{v}_{m_k}}{||w_{m_k}||_E^{p-1}}dx\Bigg\} \leq 0.\end{aligned} \tag{2.11}$$

By hypothesis (H_1) on the functions $f, g, h_i(x), k_i(x)$, $i=1,2$, we deduce that

$$\limsup_{k\to+\infty}\int_\Omega \frac{H(x,w_{m_k})}{\|w_{m_k}\|_E^p}dx=0, \tag{2.12}$$

$$\lim_{k\to+\infty}\int_\Omega \frac{\alpha k_1(x)\widehat{u}_{m_k}+\beta k_2(x)\widehat{v}_{m_k}}{\|w_{m_k}\|_E^{p-1}}dx=0. \tag{2.13}$$

Moreover by Remark 2.4, we infer

$$\lim_{k\to+\infty}\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}|\widehat{v}_{m_k}|^{\beta-1}\widehat{u}_{m_k}\widehat{v}_{m_k}dx=\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}\widehat{v}dx. \tag{2.14}$$

From (2.11) with (2.12), (2.13) and (2.14) we arrive at

$$\limsup_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^pdx\right\}\le\lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}\widehat{v}dx.$$

By Proposition 2.6 and the variational characterization of λ_1 we get

$$\begin{aligned}\lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}\widehat{v}dx &\le \frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}|^pdx\\ &\le \liminf_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^pdx\right\}\\ &\le \limsup_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^pdx\right\}\le\lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}\widehat{v}dx.\end{aligned}$$

Thus theses inequalities are indeed equalities and we have

$$\begin{aligned}\lim_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^pdx\right\}&=\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}|^pdx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}|^pdx\\ &=\lambda_1\int_\Omega |\widehat{u}|^{\alpha-1}|\widehat{v}|^{\beta-1}\widehat{u}\widehat{v}dx.\end{aligned} \tag{2.15}$$

We shall prove that $\widehat{u}\neq 0$ and $\widehat{v}\neq 0$.

By contradiction suppose that $\widehat{u}=0$, thus $\widehat{u}_{m_k}\to 0$ in $L^p(\Omega)$ as $k\to+\infty$. Then from the fact that

$$\begin{aligned}|\wedge(\widehat{u}_{m_k},\widehat{v}_{m_k})| &= \left|\int_\Omega |\widehat{u}_{m_k}|^{\alpha-1}\,|\widehat{v}_{m_k}|^{\beta-1}\,\widehat{u}_{m_k}\widehat{v}_{m_k}dx\right|\\ &\le \|\widehat{u}_{m_k}\|_{L^p(\Omega)}^{\alpha}\|\widehat{v}_{m_k}\|_{L^p(\Omega)}^{\beta}.\end{aligned}$$

Letting $k\to+\infty$ since $\|\widehat{u}_{m_k}\|_{L^p(\Omega)}\to 0$, we deduce that

$$\lim_{k\to+\infty}\wedge(\widehat{u}_{m_k},\widehat{v}_{m_k})=0. \tag{2.16}$$

From (2.10) taking $\lim_{k\to+\infty}\sup$ with (2.12), (2.13) and (2.16) we arrive at

$$\limsup_{k\to+\infty}\left\{\frac{\alpha}{p}\int_\Omega h_1(x)|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega h_2(x)|\nabla\widehat{v}_{m_k}|^p dx\right\}=0. \tag{2.17}$$

On the other hand, since $\|\widehat{w}_{m_k}\|_E=1$ and

$$\frac{\alpha}{p}\int_\Omega h_1(x)|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega h_2(x)|\nabla\widehat{v}_{m_k}|^p dx\geq\min\left(\frac{\alpha}{p},\frac{\beta}{p}\right)\|\widehat{w}_{m_k}\|_E=\min\left(\frac{\alpha}{p},\frac{\beta}{p}\right)>0,$$

which contradicts (2.17). Thus $\widehat{u}\neq 0$. Similary we have $\widehat{v}\neq 0$.
By again the definition of λ_1 from (2.15) we deduce that $\widehat{w}=(\widehat{u},\widehat{v})=(\varphi_1,\varphi_2)$ or $\widehat{w}=(\widehat{u},\widehat{v})=(-\varphi_1,-\varphi_2)$, where (φ_1,φ_2) is eigenpair associated with λ_1 of the problem (1.4).

Next we shall consider following two cases:
Assume that $\widehat{u}_{m_k}\to\varphi_1,\widehat{v}_{m_k}\to\varphi_2$ in $L^p(\Omega)$ as $k\to+\infty$. Observe that by the variational characterization of λ_1 we have

$$\frac{\alpha}{p}\int_\Omega|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega|\nabla\widehat{v}_{m_k}|^p dx-\lambda_1\int_\Omega|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}v_{m_k}dx\geq 0.$$

From this, note that $h_1(x)\geq 1$, $h_2(x)\geq 1$ a.e $x\in\Omega$, we have

$$\frac{\alpha}{p}\int_\Omega h_1(x)|\nabla\widehat{u}_{m_k}|^p dx+\frac{\beta}{p}\int_\Omega h_2(x)|\nabla\widehat{v}_{m_k}|^p dx-\lambda_1\int_\Omega|u_{m_k}|^{\alpha-1}|v_{m_k}|^{\beta-1}u_{m_k}v_{m_k}dx\geq 0.$$

Then from (2.8) it implies:

$$-\int_\Omega H(x,u_{m_k},v_{m_k})dx+\int_\Omega\left(\alpha k_1(x)u_{m_k}+\beta k_2(x)v_{m_k}\right)dx\leq c,\ k=1,2,\ldots.$$

After dividing by $\|w_{m_k}\|_E$ taking $\lim_{k\to+\infty}\sup$ and remark that

$$\lim_{k\to+\infty}\int_\Omega\left(\alpha k_1(x)\widehat{u}_{m_k}+\beta k_2(x)\widehat{v}_{m_k}\right)dx=\int_\Omega\left(\alpha k_1(x)\varphi_1+\beta k_2(x)\varphi_2\right)dx,$$

we arrive at

$$\limsup_{k\to+\infty}\int_\Omega\frac{H(x,w_{m_k})}{\|w_{m_k}\|_E}dx\geq\int_\Omega\left(\alpha k_1(x)\varphi_1+\beta k_2(x)\varphi_2\right)dx. \tag{2.18}$$

We need the following lemma

Lemma 2.9. *Assume that the hypothesis* (H_1) *is true. Then*

$$\limsup_{k\to+\infty}\int_\Omega\frac{H(x,w_{m_k})}{\|w_{m_k}\|_E}dx=\frac{1}{2}\int_\Omega\left(F_1(x)\varphi_1+G_1(x)\varphi_2\right)dx, \tag{2.19}$$

where $F_1(x)$, $G_1(x)$ *are given by* (1.9).

Proof. By (1.7), we have

$$H(x,w_{m_k})=\frac{\alpha}{2}\int_0^{u_{m_k}}\left(f(x,s,v_{m_k})+f(x,s,0)\right)ds+\frac{\beta}{2}\int_0^{v_{m_k}}\left(g(x,u_{m_k},t)+g(x,0,t)\right)dt. \tag{2.20}$$

Set $l_k = \|w_{m_k}\|_E \to +\infty$ as $k \to +\infty$. Observe that by hypothesiss (H$_1$) on $f(x,w), g(x,w)$ we have

$$\begin{aligned}
&\left|\alpha \int_0^{u_{m_k}} f(x,s,v_{m_k})ds - \alpha \int_0^{l_k\varphi_1} f(x,s,l_k\varphi_2)ds\right| \\
&\qquad \le \alpha \left|\int_0^{u_{m_k}} \left(f(x,s,v_{m_k}) - f(x,s,l_k\varphi_2)\right) ds\right| + \alpha \left|\int_{l_k\varphi_1}^{u_{m_k}} f(x,s,l_k\varphi_2)ds\right| \\
&\qquad \le \left|\int_0^{u_{m_k}} \alpha \frac{\partial f}{\partial t}\left(x,s,l_k\varphi_2 + \delta(v_{m_k} - l_k\varphi_2)\right)(v_{m_k} - l_k\varphi_2)\, ds\right| \\
&\qquad\quad + \alpha\tau(x)\left|u_{m_k} - l_k\varphi_1\right| \\
&\qquad \le \left|\int_0^{u_{m_k}} \beta \frac{\partial g}{\partial s}\left(x,s,l_k\varphi_2 + \delta(v_{m_k} - l_k\varphi_2)\right) ds\, (v_{m_k} - l_k\varphi_2)\right| \\
&\qquad\quad + \alpha\tau(x)\left|u_{m_k} - l_k\varphi_1\right| \\
&\qquad \le 2\beta\tau(x)|v_{m_k} - l_k\varphi_2| + \alpha\tau(x)|u_{m_k} - l_k\varphi_1|\ ,\ \delta \in (0,1).
\end{aligned}$$

From this and remark that $\widehat{u}_{m_k} = \frac{u_{m_k}}{l_k}$, $\widehat{v}_{m_k} = \frac{v_{m_k}}{l_k}$, we get:

$$\begin{aligned}
&\left|\alpha \frac{1}{l_k}\int_0^{u_{m_k}} f(x,s,v_{m_k})ds - \alpha \frac{1}{l_k}\int_0^{l_k\varphi_1} f(x,s,l_k\varphi_2)ds\right| \\
&\qquad \le 2\beta\tau(x)\left|\widehat{v}_{m_k} - \varphi_2\right| + \alpha\tau(x)\left|\widehat{u}_{m_k} - \varphi_1\right|. \qquad (2.21)
\end{aligned}$$

Similarly,

$$\left|\frac{\alpha}{l_k}\int_0^{u_{m_k}} f(x,s,0)ds - \frac{\alpha}{l_k}\int_0^{l_k\varphi_1} f(x,s,0)ds\right| \le \alpha\tau(x)\left|\widehat{u}_{m_k} - \varphi_1\right|. \qquad (2.22)$$

Combining (2.21) and (2.22) we infer that

$$\begin{aligned}
&\left|\int_\Omega \left\{\frac{\alpha}{l_k}\int_0^{u_{m_k}} \left(f(x,s,v_{m_k}) + f(x,s,0)\right) ds - \frac{\alpha}{l_k}\int_0^{l_k\varphi_1} \left(f(x,s,l_k\varphi_2) + f(x,s,0)\right) ds\right\} dx\right| \\
&\quad \le \int_\Omega \left\{2\beta\tau(x)\left|(\widehat{v}_{m_k} - \varphi_2)\right| + 2\alpha\tau(x)\left|\widehat{u}_{m_k} - \varphi_1\right|\right\} dx \\
&\quad \le 2\beta\|\tau(x)\|_{L^{p'}(\Omega)}\|\widehat{v}_{m_k} - \varphi_2\|_{L^p(\Omega)} + 2\alpha\|\tau(x)\|_{L^{p'}(\Omega)}\|\widehat{u}_{m_k} - \varphi_1\|_{L^p(\Omega)}.
\end{aligned}$$

Letting $k \to +\infty$, since:

$$\lim_{k\to+\infty} \|\widehat{v}_{m_k} - \varphi_2\|_{L^2(\Omega)} = 0\ ,\ \lim_{k\to+\infty} \|\widehat{u}_{m_k} - \varphi_1\|_{L^2(\Omega)} = 0,$$

we deduce that

$$\begin{aligned}
&\limsup_{k\to+\infty} \int_\Omega \left\{\frac{\alpha}{l_k}\int_0^{u_{m_k}} \left(f(x,s,v_{m_k}) + f(x,s,0)\right) ds\right\} dx \\
&\qquad = \limsup_{k\to+\infty} \int_\Omega \left\{\frac{\alpha}{l_k}\int_0^{l_k\varphi_1} \left(f(x,s,l_k\varphi_2) + f(x,s,0)\right) ds\right\} dx.
\end{aligned}$$

Set $s = y\varphi_1(x)$, $ds = \varphi_1(x)dy$, we get

$$\int_0^{l_k\varphi_1} \left(f(x,s,l_k\varphi_2) + f(x,s,0)\right) ds = \int_0^{l_k} \left(f\left(x, y\varphi_1, l_k\varphi_2\right) + f(x, y\varphi_1, 0)\right) \varphi_1 dy.$$

Remark that $l_k = ||w_{m_k}||_E \to +\infty$ as $k \to +\infty$, we derive that

$$\begin{aligned} &\limsup_{k\to+\infty} \int_\Omega \left\{ \frac{\alpha}{l_k} \int_0^{u_{m_k}} \left(f(x,s,v_{m_k}) + f(x,s,0)\right) ds \right\} dx \\ &\qquad = \limsup_{k\to+\infty} \int_\Omega \left\{ \frac{\alpha}{l_k} \int_0^{l_k} \left(f(x,y\varphi_1,l_k\varphi_2) + f(x,y\varphi_1,0)\right) dy \right\} \varphi_1 dx \\ &\qquad = \int_\Omega F_1(x)\varphi_1(x)dx. \end{aligned} \tag{2.23}$$

Similarly, we also derive that

$$\limsup_{k\to+\infty} \int_\Omega \left\{ \frac{\beta}{l_k} \int_0^{v_{m_k}} \left(g(x,u_{m_k},t) + g(x,0,t)\right) ds \right\} dx = \int_\Omega G_1(x)\varphi_2(x)dx, \tag{2.24}$$

where $F_1(x)$ and $G_1(x)$ are given in (1.9). Combining (2.23), (2.24) we obtain:

$$\limsup_{k\to+\infty} \int_\Omega \frac{H(x,w_{m_k})}{\|w_{m_k}\|_E} dx = \frac{1}{2} \int_\Omega \left(F_1(x)\varphi_1(x) + G_1(x)\varphi_2(x)\right) dx. \tag{2.25}$$

Lemma 2.9 is proved. □

Now, by (2.19) from (2.18) we obtain

$$\frac{1}{2} \int_\Omega (F_1(x)\varphi_1 + G_1(x)\varphi_2) dx \geq \int_\Omega \left(\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2\right) dx,$$

which contradicts (1.10).

If $\widehat{u}_{m_k} \to -\varphi_1(x)$,$\widehat{v}_{m_k} \to -\varphi_2(x)$ in $L^p(\Omega)$ as $k \to +\infty$, by similar computations as above and remark that in this case:

$$\limsup_{k\to+\infty} \int_\Omega \frac{H(x,w_{m_k})}{\|w_{m_k}\|_E} dx = -\frac{1}{2} \int_\Omega \left(F_2(x)\varphi_1 + G_2(x)\varphi_2\right) dx.$$

Hence from (2.18) we get

$$-\frac{1}{2} \int_\Omega \left(F_2(x)\varphi_1 + G_2(x)\varphi_2\right) dx \geq -\int_\Omega \left(\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2\right) dx,$$

which gives

$$\frac{1}{2} \int_\Omega \left(F_2(x)\varphi_1 + G_2(x)\varphi_2\right) dx \leq \int_\Omega \left(\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2\right) dx.$$

Thus we get a contradiction with (1.10).

Hence the Palais-Smale sequence $\{w_m\}$ is bounded in E and it is also bounded in X. Then there exists a subsequence $\{w_{m_k}\}$ which converges weakly to some $w_0 = (u_0, v_0)$ in X. From Proposition 2.7 we deduce that $w_0 \in E$ and $\{w_{m_k}\}$ converges strongly to w_0 in E. The proof of the Proposition 2.8 is complete. □

Proposition 2.10. *The functional* $I : E \to \mathbb{R}$ *given by* (2.1) *is coercive on* E *provided that hypotheses* (H_1) *and* (H_2) *hold.*

Proof. By contradiction we suppose that I is not coercive in E. Then it is possible to choose a sequence $\{w_m = (u_m, v_m)\}_m$ in E such that

$$\|w_m\|_E \to +\infty \text{ and } I(w_m) \leq c, \ c \text{ is positive constant.}$$

Let $\widehat{w}_m = \frac{w_m}{\|w_m\|_E} = (\widehat{u}_m, \widehat{v}_m)$. Hence the sequence $\{\widehat{w}_m\}$ is bounded in E and then bounded in $X = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. Therefore it has a subsequence $\widehat{w}_{m_k} = (\widehat{u}_{m_k}, \widehat{v}_{m_k})$ which converges weakly in X and converges strongly in $L^p(\Omega) \times L^p(\Omega)$. Applying arguments used in the proof of Proposition 2.8, we can proof that $\widehat{w}_{m_k} \to (\varphi_1, \varphi_2)$ or $\widehat{w}_{m_k} \to (-\varphi_1, -\varphi_2)$ in $L^p(\Omega) \times L^p(\Omega)$ as $k \to +\infty$ where (φ_1, φ_2) is eigenpair associated with eigenvalue λ_1 of the problem (1.4). Assume that $\widehat{w}_{m_k} \to (\varphi_1, \varphi_2)$ in $L^p(\Omega) \times L^p(\Omega)$ as $k \to +\infty$. By again the same arguments used in the proof of the Proposition 2.8 we arrive at

$$\frac{1}{2}\int_\Omega (F_1(x)\varphi_1 + G_1(x)\varphi_2)dx \geq \int_\Omega (\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2)\, dx,$$

which contradicts (1.10). If $\widehat{w}_m \to (-\varphi_1, -\varphi_2)$ in $L^p(\Omega) \times L^p(\Omega)$ as $k \to +\infty$, we get

$$\frac{1}{2}\int_\Omega (F_2(x)\varphi_1 + G_2(x)\varphi_2)\, dx \leq \int_\Omega (\alpha k_1(x)\varphi_1 + \beta k_2(x)\varphi_2)\, dx.$$

This is in contradiction with (1.10). Thus I is coercive on E. □

Proof of Theorem 1.1. By Propositions 2.8 and Proposition 2.6, applying the Minimum Principle (see Theorem 2.2), we deduce that the functional I attains its proper infimum at some $w_0 = (u_0, v_0) \in E$, so that the problem (1.1) has at least a weak solution $w_0 \in E$. Moreover by hypothesis (H_1) on $f(x, s, t), g(x, s, t), k_1(x), k_2(x)$, it is clear that w_0 is nontrivial and the proof of Theorem 1.1 is complete. □

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