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Author(s):

T. Petik and H. Özdemir

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SOME RANK EQUALITIES FOR FINITELY MANY TRIPOTENT MATRICES

T. PETİK* AND H. ÖZDEMİR

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Abstract. A rank equality is established for the sum of finitely many tripotent matrices via elementary block matrix operations. Moreover, by using this equality and Theorems 8 and 10 in [Chen M. and et al. On the open problem related to rank equalities for the sum of finitely many idempotent matrices and its applications, *The Scientific World Journal* 2014 (2014), Article ID 702413, 7 pages.], some other rank equalities for tripotent matrices are given. Furthermore, we obtain several rank equalities related to some special types of matrices, some of which are available in the literature, from the results established.
Keywords: Rank, elementary block matrix operations, idempotent matrix, tripotent matrix.
MSC(2010): 15A03.

1. Introduction and preliminaries

Let $\mathbb{C}_n$ and $\mathbb{C}_{n\times m}$ be the sets of all complex matrices of dimensions $n \times n$ and $n\times m$, respectively. The symbol $\mathbb{Z}^+$ is used to denote the set of all positive integers. I and $\mathbf{0}$ stand for the identity and zero matrices of suitable sizes, respectively. For $A \in \mathbb{C}_{n\times m}$, the rank and the transpose of A are denoted by $\mathrm{rk}(A)$ and A^T, respectively. And also, the trace of the matrix $A \in \mathbb{C}_n$ is denoted by $\mathrm{tr}(A)$. Moreover, $A \oplus B$ denotes the direct sum of the matrices A and B.

Recall that a matrix $A \in \mathbb{C}_n$ is said to be a *tripotent matrix* if $A^3 = A$. If $A^3 = A$ and $A^2 \neq \pm A$, then the matrix A becomes an *essentially tripotent matrix*. If a tripotent matrix is nonsingular, i.e., $A^2 = I$, then A is called an *involutive matrix*. Also, if $A^2 = \lambda A$ for some $\lambda \in \mathbb{C}$, the matrix A is said to be a *scalar-potent matrix determined by* λ. In addition, a matrix $A \in \mathbb{C}_n$ is

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*Corresponding author.

called a *generalized quadratic matrix with respect to an idempotent matrix* P if there exist $\alpha, \beta \in \mathbb{C}$ such that

$$(A - \alpha P)(A - \beta P) = \mathbf{0} \quad \text{and} \quad AP = PA = A. \tag{1.1}$$

From now on, we shall say that the matrix A satisfying the conditions in (1.1) is a *generalized* $\{\alpha, \beta\}$*-quadratic matrix with respect to the idempotent matrix* P. Also, it is noteworthy that the set of generalized quadratic matrices covers the sets of idempotent, involutive, and tripotent matrices (see, e.g., [6] and [11]).

Many studies on rank equalities related to special types of matrices have been done, recently (see, e.g., [8–10, 12]). For example, it was obtained some rank equalities on the sum of two idempotent or involutive matrices in [8]. Then, the authors of [8] also gave a new rank formula for idempotent matrices in [9]. And then, they established a rank equality on the sum of three idempotent matrices in [10]. Also, it was given some rank equalities related to some combinations of idempotent matrices in [12].

Quite recently, Chen M. et al. gave a more general rank equality on the sum of finitely many idempotent matrices in [2]. The aims of the present paper are, first, to generalize the main result in the just mentioned work to the tripotent matrices, and then to derive some rank equalities for finitely many tripotent matrices using some related rank equalities in [2].

Now, let us recall the following known properties about the trace and the rank of matrices (see, e.g., [3]).

Theorem 1.1. *Let* $A, B \in \mathbb{C}_n$*. Then the following statements are true.*

(i) $\operatorname{tr}(c_1 A + c_2 B) = c_1 \operatorname{tr}(A) + c_2 \operatorname{tr}(B)$ *for* $c_1, c_2 \in \mathbb{C}$.
(ii) *If* A *is idempotent, then* $\operatorname{rk}(A) = \operatorname{tr}(A)$.
(iii) *If* A *is tripotent, then* $\operatorname{rk}(A) = \operatorname{tr}(A^2) = \operatorname{rk}(A^2)$.

We want to close this section giving a lemma in the literature which will be used to derive the result in Theorem 2.1.

Lemma 1.2 ([2, Lemma 4])**.** *Let* $A_1, \ldots, A_k \in \mathbb{C}_{n\times m}$*. Then*

$$\operatorname{rk}\begin{bmatrix} A_1 & \mathbf{0} & \cdots & \mathbf{0} & A_1 \\ \mathbf{0} & A_2 & \cdots & \mathbf{0} & A_2 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \cdots & A_k & A_k \\ A_1 & A_2 & \cdots & A_k & \mathbf{0} \end{bmatrix} = \sum_{i=1}^{k} \operatorname{rk}(A_i) + \operatorname{rk}\left(\sum_{i=1}^{k} A_i\right)$$

for any $k \in \mathbb{Z}^+$.

This lemma was proved in [2] by using the method of [5, Theorem 9].

2. A Rank equality for the sum of finitely many tripotent matrices and some other rank equalities for finitely many tripotent matrices

Throughout the work we shall use the matrix representations

$$W(T_1,T_2,\ldots,T_k)=\begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_k \\ 2T_2+T_2^2T_1-T_2T_1^2 & \mathbf{0} & T_2^2T_3 & \cdots & T_2^2T_k \\ 2T_3+T_3^2T_1-T_3T_1^2 & T_3^2T_2 & \mathbf{0} & \cdots & T_3^2T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2T_{k-1}+T_{k-1}^2T_1-T_{k-1}T_1^2 & T_{k-1}^2T_2 & T_{k-1}^2T_3 & \cdots & T_{k-1}^2T_k \\ 2T_k+T_k^2T_1-T_kT_1^2 & T_k^2T_2 & T_k^2T_3 & \cdots & \mathbf{0} \end{bmatrix}$$

and

$$G(T_1,T_2,\ldots,T_k)=\begin{bmatrix} \frac{1}{2}T_1 & T_2 & T_3 & \cdots & T_{k-1} & T_k \\ T_2 & \mathbf{0} & T_2T_3 & \cdots & T_2T_{k-1} & T_2T_k \\ T_3 & T_3T_2 & \mathbf{0} & \cdots & T_3T_{k-1} & T_3T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{k-1} & T_{k-1}T_2 & T_{k-1}T_3 & \cdots & \mathbf{0} & T_{k-1}T_k \\ T_k & T_kT_2 & T_kT_3 & \cdots & T_kT_{k-1} & \mathbf{0} \end{bmatrix}$$

for $T_1,T_2,\ldots,T_k\in\mathbb{C}_n$.

The following theorem establishes a relation between the rank of the sum of finitely many tripotent matrices and the sum of ranks of the same tripotent matrices.

Theorem 2.1. *Let $T_1,T_2,\ldots,T_k\in\mathbb{C}_n$ be finitely many tripotent matrices with $k\in\mathbb{Z}^+$. Then we have the rank equality*

$$\operatorname{rk}\left(\sum_{i=1}^{k}T_i\right)=\operatorname{rk}\left(W(T_1,\ldots,T_k)\right)-\sum_{i=2}^{k}\operatorname{rk}\left(T_i\right).$$

Proof. Let us define the matrix

$$T=\begin{bmatrix} T_1 & \mathbf{0} & \ldots & \mathbf{0} & T_1 \\ \mathbf{0} & T_2 & \ldots & \mathbf{0} & T_2 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & T_k & T_k \\ T_1 & T_2 & \ldots & T_k & \mathbf{0} \end{bmatrix},$$

and partition it as

$$T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & \mathbf{0} \end{bmatrix},$$

where

$$T_{11}=T_1\oplus T_2\oplus\ldots\oplus T_k, \tag{2.1}$$

$$T_{12} = \begin{bmatrix} T_1^T & T_2^T & \dots & T_k^T \end{bmatrix}^T, \tag{2.2}$$

and

$$T_{21} = \begin{bmatrix} T_1 & T_2 & \cdots & T_k \end{bmatrix}. \tag{2.3}$$

It is easy to see that

$$T_{11} + W_1T_{21} = \begin{bmatrix} T_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -T_2^2T_1 & \mathbf{0} & -T_2^2T_3 & \cdots & -T_2^2T_k \\ -T_3^2T_1 & -T_3^2T_2 & \mathbf{0} & \cdots & -T_3^2T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -T_k^2T_1 & -T_k^2T_2 & -T_k^2T_3 & \cdots & \mathbf{0} \end{bmatrix}, \tag{2.4}$$

where $W_1 = \begin{bmatrix} \mathbf{0} & -(T_2^2)^T & \cdots & -(T_k^2)^T \end{bmatrix}^T$. From now on, we shall denote the matrix $T_{11} + W_1T_{21}$ by Q. Now let us define the matrices W_2, W_3, and W_4 as

$$W_2 = \begin{bmatrix} -\frac{1}{2}I & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \tag{2.5}$$

$$W_3 = \begin{bmatrix} T_1^2 & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \tag{2.6}$$

and

$$W_4 = \begin{bmatrix} -\frac{1}{2}(T_1^2)^T & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}^T. \tag{2.7}$$

From (2.2), (2.4), and (2.6), we get

$$Q + T_{12}W_3 = \begin{bmatrix} 2T_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ T_2T_1^2 - T_2^2T_1 & \mathbf{0} & -T_2^2T_3 & \cdots & -T_2^2T_k \\ T_3T_1^2 - T_3^2T_1 & -T_3^2T_2 & \mathbf{0} & \cdots & -T_3^2T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_kT_1^2 - T_k^2T_1 & -T_k^2T_2 & -T_k^2T_3 & \cdots & \mathbf{0} \end{bmatrix}. \tag{2.8}$$

So, the equalities (2.2), (2.7), and (2.8) imply the equality

$$(Q + T_{12}W_3)W_4 + T_{12} = \begin{bmatrix} \mathbf{0} \\ T_2 + \frac{1}{2}(T_2^2T_1 - T_2T_1^2) \\ \vdots \\ T_k + \frac{1}{2}(T_k^2T_1 - T_kT_1^2) \end{bmatrix}. \tag{2.9}$$

On the other hand, taking into account the equalities (2.3), (2.5), and (2.8) yields the equality

$$W_2(Q + T_{12}W_3) + T_{21} = \begin{bmatrix} \mathbf{0} & T_2 & \cdots & T_k \end{bmatrix}. \tag{2.10}$$

Thus, from the equalities (2.2), (2.5), (2.7), and (2.10), we can write

$$(W_2(Q+T_{12}W_3)+T_{21})\,W_4+W_2T_{12}=-\frac{1}{2}T_1. \tag{2.11}$$

It is easily seen from the equalities (2.8)-(2.11) that

(2.12)
$$\begin{bmatrix} Q+T_{12}W_3 & (Q+T_{12}W_3)W_4+T_{12} \\ W_2(Q+T_{12}W_3)+T_{21} & (W_2Q+T_{21}+W_2T_{12}W_3)W_4+W_2T_{12} \end{bmatrix}$$

$$=\begin{bmatrix} 2T_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ T_2T_1^2-T_2^2T_1 & \mathbf{0} & -T_2^2T_3 & \cdots & -T_2^2T_k & T_2+\frac{1}{2}(T_2^2T_1-T_2T_1^2) \\ T_3T_1^2-T_3^2T_1 & -T_3^2T_2 & \mathbf{0} & \cdots & -T_3^2T_k & T_3+\frac{1}{2}(T_3^2T_1-T_3T_1^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T_kT_1^2-T_k^2T_1 & -T_k^2T_2 & -T_k^2T_3 & \cdots & \mathbf{0} & T_k+\frac{1}{2}(T_k^2T_1-T_kT_1^2) \\ \mathbf{0} & T_2 & T_3 & \cdots & T_k & -\frac{1}{2}T_1 \end{bmatrix}.$$

Observe that
(2.13)
$$\begin{bmatrix} I & \mathbf{0} \\ W_2 & I \end{bmatrix}\begin{bmatrix} I & W_1 \\ \mathbf{0} & I \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & \mathbf{0} \end{bmatrix}\begin{bmatrix} I & \mathbf{0} \\ W_3 & I \end{bmatrix}\begin{bmatrix} I & W_4 \\ \mathbf{0} & I \end{bmatrix}$$
$$=\begin{bmatrix} Q+T_{12}W_3 & (Q+T_{12}W_3)W_4+T_{12} \\ W_2(Q+T_{12}W_3)+T_{21} & (W_2Q+T_{21}+W_2T_{12}W_3)W_4+W_2T_{12} \end{bmatrix}.$$

Thus, we have

$$\operatorname{rk}(T)=\operatorname{rk}\begin{bmatrix} 2T_1 & \mathbf{0} \\ K & L \end{bmatrix} \tag{2.14}$$

in view of the equalities (2.12) and (2.13), where

$$K=\begin{bmatrix} T_2T_1^2-T_2^2T_1 \\ T_3T_1^2-T_3^2T_1 \\ \vdots \\ T_kT_1^2-T_k^2T_1 \\ \mathbf{0} \end{bmatrix} \text{ and } L=\begin{bmatrix} \mathbf{0} & -T_2^2T_3 & \cdots & -T_2^2T_k & T_2+\frac{1}{2}(T_2^2T_1-T_2T_1^2) \\ -T_3^2T_2 & \mathbf{0} & \cdots & -T_3^2T_k & T_3+\frac{1}{2}(T_3^2T_1-T_3T_1^2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -T_k^2T_2 & -T_k^2T_3 & \cdots & \mathbf{0} & T_k+\frac{1}{2}(T_k^2T_1-T_kT_1^2) \\ T_2 & T_3 & \cdots & T_k & -\frac{1}{2}T_1 \end{bmatrix}.$$

Let us, first, define the matrix

$$W_5=\begin{bmatrix} \frac{1}{2}(T_2^2-T_2T_1) \\ \vdots \\ \frac{1}{2}(T_k^2-T_kT_1) \\ \\ \mathbf{0} \end{bmatrix}. \tag{2.15}$$

Then, premultiplying the matrix $\begin{bmatrix} 2T_1 & \mathbf{0} \\ K & L \end{bmatrix}$ by the nonsingular matrix $\begin{bmatrix} I & \mathbf{0} \\ W_5 & I \end{bmatrix}$ yields the matrix $2T_1 \oplus L$. Hence we have

$$\mathrm{rk}\begin{bmatrix} 2T_1 & \mathbf{0} \\ K & L \end{bmatrix} = \mathrm{rk}(2T_1 \oplus L),$$

and therefore,

$$\mathrm{rk}(T) = \mathrm{rk}(T_1) + \mathrm{rk}(L) \tag{2.16}$$

on account of (2.14). Now, let us partition the matrix L as $\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -\frac{1}{2}T_1 \end{bmatrix}$, where

$$L_{11} = \begin{bmatrix} \mathbf{0} & -T_2^2T_3 & \cdots & -T_2^2T_{k-1} & -T_2^2T_k \\ -T_3^2T_2 & \mathbf{0} & \cdots & -T_3^2T_{k-1} & -T_3^2T_k \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ -T_{k-1}^2T_2 & -T_{k-1}^2T_3 & \cdots & \mathbf{0} & -T_{k-1}^2T_k \\ -T_k^2T_2 & -T_k^2T_3 & \cdots & -T_k^2T_{k-1} & \mathbf{0} \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} T_2 + \frac{1}{2}(T_2^2T_1 - T_2T_1^2) \\ T_3 + \frac{1}{2}(T_3^2T_1 - T_3T_1^2) \\ \vdots \\ T_{k-1} + \frac{1}{2}(T_{k-1}^2T_1 - T_{k-1}T_1^2) \\ T_k + \frac{1}{2}(T_k^2T_1 - T_kT_1^2) \end{bmatrix},$$

and

$$L_{21} = \begin{bmatrix} T_2 & T_3 & \cdots & T_{k-1} & T_k \end{bmatrix}.$$

Then we can write

$$\begin{bmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{bmatrix}\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -\frac{1}{2}T_1 \end{bmatrix}\begin{bmatrix} \mathbf{0} & I \\ -2I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} T_1 & L_{21} \\ 2L_{12} & -L_{11} \end{bmatrix}.$$

Hence we get the equality

$$\mathrm{rk}(L) = \mathrm{rk}\begin{bmatrix} L_{11} & 2L_{12} \\ L_{21} & -\frac{1}{2}T_1 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} T_1 & L_{21} \\ 2L_{12} & -L_{11} \end{bmatrix}.$$

So, in view of (2.16), we get

$$\mathrm{rk}(T) = \mathrm{rk}(T_1) + \mathrm{rk}\left(W(T_1, \ldots, T_k)\right),$$

i.e.,

$$\mathrm{rk}\left(\sum_{i=1}^{k} T_i\right) = \mathrm{rk}\left(W(T_1, \ldots, T_k)\right) - \sum_{i=2}^{k} \mathrm{rk}(T_i)$$

using Lemma 1.2. Thus, it is completed the proof of the theorem. □

Taking $k = 2$, first, in Theorem 2.1, we get the following two results.

Corollary 2.2. *Let T_1 and T_2 be two tripotent matrices. Then we have the rank equality*

$$\mathrm{rk}(T_1 + T_2) = \mathrm{rk}\begin{bmatrix} T_1 & T_2 \\ 2T_2 + T_2^2T_1 - T_2T_1^2 & \mathbf{0} \end{bmatrix} - \mathrm{rk}(T_2),$$

and hence, if the equality $T_2^2T_1 = T_2T_1^2$ also holds, then we have

$$\mathrm{rk}(T_1 + T_2) = \mathrm{rk}\begin{bmatrix} T_1 & T_2 \\ T_2 & \mathbf{0} \end{bmatrix} - \mathrm{rk}(T_2).$$

If T_1 and T_2 are idempotent, then we have already the equality $T_2^2T_1 - T_2T_1^2 = \mathbf{0}$. So, we get the following known rank equality.

Corollary 2.3 ([10, Theorem 2.1])**.** *Let T_1 and T_2 be two idempotent matrices. Then we have the rank equality*

$$\mathrm{rk}(T_1 + T_2) = \mathrm{rk}\begin{bmatrix} T_1 & T_2 \\ T_2 & \mathbf{0} \end{bmatrix} - \mathrm{rk}(T_2).$$

Taking $k = 3$, secondly, in Theorem 2.1, we get the following result.

Corollary 2.4. *Let T_1, T_2, and T_3 be three tripotent matrices. Then we have*

$$\mathrm{rk}(T_1 + T_2 + T_3) = \mathrm{rk}\begin{bmatrix} T_1 & T_2 & T_3 \\ 2T_2 + T_2^2T_1 - T_2T_1^2 & \mathbf{0} & T_2^2T_3 \\ 2T_3 + T_3^2T_1 - T_3T_1^2 & T_3^2T_2 & \mathbf{0} \end{bmatrix} - \mathrm{rk}(T_2) - \mathrm{rk}(T_3),$$

and hence, if T_1, T_2, and T_3 are idempotent matrices, then we have the rank equality, which is mentioned in [10],

$$\mathrm{rk}(T_1 + T_2 + T_3) = \mathrm{rk}\begin{bmatrix} \frac{1}{2}T_1 & T_2 & T_3 \\ T_2 & \mathbf{0} & T_2T_3 \\ T_3 & T_3T_2 & \mathbf{0} \end{bmatrix} - \mathrm{rk}(T_2) - \mathrm{rk}(T_3).$$

Observe that in general, Theorem 2.1 covers [2, Theorem 6]. In fact, if the matrices $T_1, T_2, \ldots, T_k$ are idempotents, then we have

$$W(T_1, T_2, \ldots, T_k) = \begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_k \\ 2T_2 & \mathbf{0} & T_2T_3 & \cdots & T_2T_k \\ 2T_3 & T_3T_2 & \mathbf{0} & \cdots & T_3T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2T_k & T_kT_2 & T_kT_3 & \cdots & \mathbf{0} \end{bmatrix}$$

using the fact that every idempotent matrix is also tripotent, and hence we get

$$\mathrm{rk}\left(W(T_1, T_2, \cdots, T_k)\right) = \mathrm{rk}\begin{bmatrix} \frac{1}{2}T_1 & T_2 & T_3 & \ldots & T_k \\ T_2 & \mathbf{0} & T_2T_3 & \cdots & T_2T_k \\ T_3 & T_3T_2 & \mathbf{0} & \cdots & T_3T_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_k & T_kT_2 & T_kT_3 & \cdots & \mathbf{0} \end{bmatrix}.$$

Thus, [2, Theorem 6] is obtained via Theorem 2.1.

We want to give a numerical example to exemplify Theorem 2.1.

Example 2.5. Let the tripotent matrices T_1, T_2, T_3, and T_4 be given by

$$T_1 = \begin{bmatrix} -1 & -\frac{4}{5} & -\frac{32}{5} & \frac{34}{5} \\ 0 & -\frac{1}{5} & -\frac{8}{5} & \frac{6}{5} \\ 0 & -\frac{12}{5} & -\frac{11}{5} & \frac{12}{5} \\ 0 & -\frac{12}{5} & -\frac{16}{5} & \frac{17}{5} \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & \frac{6}{5} & \frac{8}{5} & -\frac{6}{5} \\ 0 & \frac{6}{5} & \frac{3}{5} & -\frac{1}{5} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} \frac{6}{7} & \frac{3}{7} & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 0 & 0 & 0 \\ \frac{2}{7} & \frac{1}{7} & -\frac{3}{7} & \frac{2}{7} \\ -\frac{10}{7} & -\frac{5}{7} & \frac{8}{7} & -\frac{3}{7} \end{bmatrix}, \quad \text{and} \quad T_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

After simple numerical computations, it follows that $\mathrm{rk}(T_1) = 4$, $\mathrm{rk}(T_2) = 3$, $\mathrm{rk}(T_3) = 2$, and $\mathrm{rk}(T_4) = 1$. So, we have $\sum\limits_{i=2}^{4} \mathrm{rk}(T_i) = 6$. Let us denote the matrix $T_1+T_2+T_3+T_4$ by T. It is obvious that $T = \begin{bmatrix} \frac{6}{7} & \frac{36}{35} & -\frac{227}{35} & \frac{219}{35} \\ 0 & -\frac{3}{5} & -\frac{9}{5} & \frac{8}{5} \\ \frac{2}{7} & -\frac{72}{35} & -\frac{36}{35} & \frac{52}{35} \\ -\frac{10}{7} & \frac{3}{35} & -\frac{51}{35} & \frac{97}{35} \end{bmatrix}$ and $\mathrm{rk}(T) = 4$. On the other hand, for the matrices T_1, T_2, T_3, and T_4, it follows that $W(T_1, T_2, T_3, T_4)$ is equal to the matrix

$$\begin{bmatrix}
-1 & -\frac{4}{5} & -\frac{32}{5} & \frac{34}{5} & 1 & \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{6}{7} & \frac{3}{7} & -\frac{2}{7} & -\frac{1}{7} & 0 & 1 & 0 & 0 \\
0 & -\frac{1}{5} & -\frac{8}{5} & \frac{6}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -\frac{12}{5} & -\frac{11}{5} & \frac{12}{5} & 0 & \frac{6}{5} & \frac{8}{5} & -\frac{6}{5} & \frac{2}{7} & \frac{1}{7} & -\frac{3}{7} & \frac{2}{7} & 0 & -1 & 0 & 0 \\
0 & -\frac{12}{5} & -\frac{16}{5} & \frac{17}{5} & 0 & \frac{6}{5} & \frac{3}{5} & -\frac{1}{5} & -\frac{10}{7} & -\frac{5}{7} & \frac{8}{7} & -\frac{3}{7} & 0 & 2 & 0 & 0 \\
0 & 0 & -6 & 6 & 0 & 0 & 0 & 0 & \frac{52}{35} & \frac{26}{35} & -\frac{29}{35} & \frac{3}{35} & 0 & -\frac{2}{5} & 0 & 0 \\
0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & -\frac{22}{35} & -\frac{11}{35} & \frac{19}{35} & -\frac{8}{35} & 0 & \frac{2}{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{76}{35} & \frac{38}{35} & -\frac{72}{35} & \frac{34}{35} & 0 & -\frac{26}{5} & 0 & 0 \\
0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & \frac{16}{35} & \frac{8}{35} & -\frac{17}{35} & \frac{9}{35} & 0 & -\frac{11}{5} & 0 & 0 \\
0 & \frac{24}{35} & -\frac{188}{35} & \frac{176}{35} & \frac{6}{7} & \frac{3}{35} & -\frac{16}{35} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{7} & \frac{2}{35} & \frac{8}{5} & -\frac{62}{35} & -\frac{2}{7} & -\frac{1}{35} & \frac{17}{35} & -\frac{2}{5} & 0 & 0 & 0 & 0 & 0 & -\frac{8}{7} & 0 & 0 \\
-\frac{8}{7} & -\frac{4}{5} & \frac{76}{35} & -\frac{52}{35} & -\frac{2}{7} & -\frac{1}{35} & -\frac{18}{35} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & \frac{13}{7} & 0 & 0 \\
0 & \frac{6}{5} & \frac{8}{5} & -\frac{6}{5} & 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{6}{5} & -\frac{8}{5} & \frac{6}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{6}{5} & -\frac{8}{5} & -\frac{8}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12}{5} & \frac{16}{5} & \frac{16}{5} & 0 & -\frac{6}{5} & \frac{2}{5} & \frac{-4}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Therefore we get $\mathrm{rk}(W) = 10$. So, we have

$$\mathrm{rk}\left(\sum_{i=1}^{4} T_i\right) = \mathrm{rk}\left(W(T_1, T_2, T_3, T_4)\right) - \sum_{i=2}^{4} \mathrm{rk}(T_i)$$

as stated in Theorem 2.1.

Now, we shall investigate some other rank equalities for finitely many tripotent matrices using [2, Theorems 8 and 10].

It is well known that if M is any tripotent matrix, then there exist two disjoint idempotent matrices M_1 and M_2 such that $M = M_1 - M_2$. Also, M_1 and M_2 are unique, and $M_1 = \frac{1}{2}(M^2+M)$ and $M_2 = \frac{1}{2}(M^2-M)$ (see, e.g., [7]).

Now, let $T_1, \ldots, T_k \in \mathbb{C}_n$ be tripotent matrices. So, there exist disjoint idempotent matrices X_i and Y_i such that $T_i = X_i - Y_i$, $i = 1, \ldots, k$. Let us take $Q_m = X_{\frac{m+1}{2}}$ if m is an odd number, $Q_m = -Y_{\frac{m}{2}}$ if m is an even number, where $m = 1, \ldots, 2k$. It is obvious that if m is an odd number, then Q_m is a scalar-potent matrix determined by $\lambda_m = 1$, if m is an even number, then Q_m is a scalar-potent matrix determined by $\lambda_m = -1$. So, by [2, Theorem 8], we can write

$$\operatorname{rk}\left(\sum_{m=1}^{2k}\left(\prod_{n\neq m}^{2k}\lambda_n\right)Q_m\right) = \operatorname{rk}\left(G\left(\tfrac{1}{\lambda_1}Q_1, Q_2, \ldots, Q_{2k}\right)\right) - \sum_{m=2}^{2k}\operatorname{rk}(Q_m).$$

From this, we get

$$\begin{aligned}&\operatorname{rk}(\lambda_2\lambda_3\cdots\lambda_{2k}Q_1 + \lambda_1\lambda_3\ldots\lambda_{2k}Q_2 + \cdots \lambda_1\lambda_2\cdots\lambda_{2k-1}Q_{2k}) = \\ &\operatorname{rk}\left(G\left(\frac{1}{\lambda_1}Q_1, Q_2, \ldots, Q_{2k}\right)\right) - \sum_{m=2}^{2k}\operatorname{rk}(Q_m).\end{aligned} \tag{2.17}$$

Since $\lambda_m = 1$ if m is an odd number, and $\lambda_m = -1$ if m is an even number, the coefficients of every term Q_m, where m is an odd number, in the matrix in the left of (2.17) are equal to -1, the coefficients of every term Q_m, where m is an even number, in the matrix in the left of (2.17) are equal to 1. So, we get from (2.17),

$$\begin{aligned}&\operatorname{rk}(Q_1 - Q_2 + Q_3 - Q_4 + \cdots + Q_{2k-1} - Q_{2k}) \\ &= \operatorname{rk}\left(G(Q_1, Q_2, \ldots, Q_{2k})\right) - \sum_{m=2}^{2k}\operatorname{rk}(Q_m).\end{aligned} \tag{2.18}$$

On the other hand, we know that $X_i = \frac{1}{2}(T_i^2 + T_i)$ and $Y_i = \frac{1}{2}(T_i^2 - T_i)$, $i = 1, \ldots, k$. Hence we have $\operatorname{rk}(X_i) = \operatorname{tr}(X_i)$ and $\operatorname{rk}(Y_i) = \operatorname{tr}(Y_i)$ by Theorem 1.1(ii). Moreover, by Theorem 1.1(i), we get $\operatorname{tr}\left(\frac{1}{2}(T_i^2 \pm T_i)\right) = \frac{1}{2}\operatorname{tr}(T_i^2) \pm \frac{1}{2}\operatorname{tr}(T_i)$, $i = 1, \ldots, k$. Also, since T_i, $i = 1, \ldots, k$, is tripotent, we have $\operatorname{rk}(T_i) = \operatorname{tr}(T_i^2)$ by Theorem 1.1(iii). So, we get

$$\sum_{m=2}^{2k}\operatorname{rk}(Q_m) = \frac{1}{2}\operatorname{rk}(T_1) - \frac{1}{2}\operatorname{tr}(T_1) + \sum_{i=2}^{k}\operatorname{rk}(T_i). \tag{2.19}$$

Also, it is easy to see that

$$\operatorname{rk}(Q_1 - Q_2 + Q_3 - Q_4 + \cdots + Q_{2k-1} - Q_{2k}) = \operatorname{rk}\left(\sum_{i=1}^{k}{T_i}^2\right). \tag{2.20}$$

On the other hand, the matrix

$$G(Q_1,\ldots,Q_{2k}) = G\left(\frac{1}{2}({T_1}^2+T_1),\ -\frac{1}{2}({T_1}^2-T_1),\ \ldots\ ,\ \frac{1}{2}({T_k}^2+T_k),\ -\frac{1}{2}({T_k}^2-T_k)\right)$$

is equal to the matrix

$$\frac{1}{4}\begin{bmatrix} T_1^2+T_1 & -2(T_1^2-T_1) & 2(T_2^2+T_2) & \cdots & 2(T_k^2+T_k) & -2(T_k^2-T_k) \\ -2(T_1^2-T_1) & \mathbf{0} & -(T_1^2-T_1)(T_2^2+T_2) & \cdots & -(T_1^2-T_1)(T_k^2+T_k) & (T_1^2-T_1)(T_k^2-T_k) \\ 2(T_2^2+T_2) & -(T_2^2+T_2)(T_1^2-T_1) & \mathbf{0} & \cdots & (T_2^2+T_2)(T_k^2+T_k) & -(T_2^2+T_2)(T_k^2-T_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(T_k^2+T_k) & -(T_k^2+T_k)(T_1^2-T_1) & (T_k^2+T_k)(T_2^2+T_2) & \cdots & \mathbf{0} & -(T_k^2+T_k)(T_k^2-T_k) \\ -2(T_k^2-T_k) & (T_k^2-T_k)(T_1^2-T_1) & -(T_k^2-T_k)(T_2^2+T_2) & \cdots & -(T_k^2-T_k)(T_k^2+T_k) & \mathbf{0} \end{bmatrix}.$$

Let us denote the last matrix by $\frac{1}{4}V(T_1,\ldots,T_k)$. So, in view of (2.18)-(2.20), we get

$$\operatorname{rk}\left(\sum_{i=1}^{k}{T_i}^2\right) = \operatorname{rk}\left(V(T_1,\ldots,T_k)\right) - \frac{1}{2}\operatorname{rk}(T_1) + \frac{1}{2}\operatorname{tr}(T_1) - \sum_{i=2}^{k}\operatorname{rk}(T_i).$$

Thus, we have proved the following theorem.

Theorem 2.6. *Let T_1, T_2, ... ,T_k be tripotent matrices. Then*

$$\operatorname{rk}\left(\sum_{i=1}^{k}{T_i}^2\right) + \sum_{i=2}^{k}\operatorname{rk}(T_i) = \operatorname{rk}\left(V(T_1,\ldots,T_k)\right) - \frac{1}{2}\operatorname{rk}(T_1) + \frac{1}{2}\operatorname{tr}(T_1).$$

Considering this theorem together with Theorem 2.1, we get the following result.

Corollary 2.7. *Let T_1, T_2, ... ,T_k be tripotent matrices. Then*

$$\operatorname{rk}\left(\sum_{i=1}^{k}{T_i}^2\right) - \operatorname{rk}\left(\sum_{i=1}^{k}T_i\right) = \operatorname{rk}\left(V(T_1,\ldots,T_k)\right) - \operatorname{rk}\left(W(T_1,\ldots,T_k)\right) - \frac{1}{2}\operatorname{rk}(T_1) + \frac{1}{2}\operatorname{tr}(T_1).$$

Notice that if a matrix M is tripotent, then the matrix M^2 is idempotent, and according to Theorem 1.1 (iii), we have $\operatorname{rk}(M) = \operatorname{rk}(M^2)$. Theorefore, by [2, Theorem 6], we get

$$\operatorname{rk}\left(\sum_{i=1}^{k}{T_i}^2\right) + \sum_{i=2}^{k}\operatorname{rk}(T_i) = \operatorname{rk}\left(G(T_1^2,\ldots,T_k^2)\right) \tag{2.21}$$

under the assumptions. Considering this fact together with Theorem 2.6, we can give the following corollary.

Corollary 2.8. *Let T_1, T_2, ... ,T_k be tripotent matrices. Then*
$\operatorname{rk}\left(G(T_1^2,\ldots,T_k^2)\right) + \frac{1}{2}\operatorname{rk}(T_1) = \operatorname{rk}\left(V(T_1,\ldots,T_k)\right) + \frac{1}{2}\operatorname{tr}(T_1)$.

Benítez et al. have stated in [1, Remark 2.1] that if $T_1,\ T_2 \in \mathbb{C}_n$ are commuting tripotent matrices, then $T_1^2 + T_2^2$ is nonsingular if and only if

$$\operatorname{rk}(T_1) + \operatorname{rk}(T_2) = n + \operatorname{rk}(T_1 T_2). \tag{2.22}$$

Now, let $T_1, T_2 \in \mathbb{C}_n$ be commuting tripotent matrices. It can be written

$$\operatorname{rk}(T_1^2 + T_2^2) + \operatorname{rk}(T_2) = \operatorname{rk}\left(G(T_1^2, T_2^2)\right) = \operatorname{rk}\begin{bmatrix} \frac{1}{2}T_1^2 & T_2^2 \\ T_2^2 & \mathbf{0} \end{bmatrix} \tag{2.23}$$

from (2.21). If $T_1^2 + T_2^2$ is nonsingular, then we get

$$\operatorname{rk}(T_2) = \operatorname{rk}\begin{bmatrix} \frac{1}{2}T_1^2 & T_2^2 \\ T_2^2 & \mathbf{0} \end{bmatrix} - n \tag{2.24}$$

from (2.23). So, we obtain from (2.22) and (2.24),

$$\operatorname{rk}\begin{bmatrix} \frac{1}{2}T_1^2 & T_2^2 \\ T_2^2 & \mathbf{0} \end{bmatrix} - n = n + \operatorname{rk}(T_1 T_2) - \operatorname{rk}(T_1),$$

and therefore,

$$\operatorname{rk}(T_1 T_2) - \operatorname{rk}(T_1) + 2n = \operatorname{rk}\begin{bmatrix} \frac{1}{2}T_1^2 & T_2^2 \\ T_2^2 & \mathbf{0} \end{bmatrix}.$$

Thus, we have the following theorem.

Theorem 2.9. *Let $T_1,\ T_2 \in \mathbb{C}_n$ be commuting tripotent matrices. Then, $T_1^2 + T_2^2$ is nonsingular if and only if*

$$\operatorname{rk}(T_1 T_2) - \operatorname{rk}(T_1) + 2n = \operatorname{rk}\begin{bmatrix} \frac{1}{2}T_1^2 & T_2^2 \\ T_2^2 & \mathbf{0} \end{bmatrix}.$$

In [2, Theorem 10], the authors obtained that if T_i, $i = 1, \ldots, k$ are generalized $\{\alpha_i, \beta_i\}$-quadratic matrices with respect to the idempotent matrices P_i such that $\alpha_i \neq \beta_i$, then

$$\begin{aligned} &\operatorname{rk}\left(\sum_{i=1}^{k}\left(\prod_{j\neq i}(\beta_j - \alpha_j)\right)(T_i - \alpha_i P_i)\right) + \sum_{i=2}^{k}\frac{1}{\beta_i - \alpha_i}\operatorname{tr}(T_i - \alpha_i P_i) \\ &= \operatorname{rk}\left(G\left(\frac{1}{\beta_1 - \alpha_1}(T_1 - \alpha_1 P_1), T_2 - \alpha_2 P_2, \ldots, T_k - \alpha_k P_k\right)\right). \end{aligned} \tag{2.25}$$

On the other hand, as we have pointed out in introduction, the set of generalized quadratic matrices covers the set of tripotent matrices. More clearly, an essentially tripotent matrix M can be considered as a generalized $\{1, -1\}$-quadratic matrix with respect to M^2. So, if it is taken $\alpha_i = 1$ and $\beta_i = -1$, and $P_i = T_i^2$ in (2.25), then we get

$$\begin{aligned}\text{rk}\left(\sum_{i=1}^{k} T_i - \sum_{i=1}^{k} {T_i}^2\right) - \frac{1}{2}\sum_{i=2}^{k}\text{tr}(T_i - {T_i}^2) \\ = \text{rk}\left(G\left(-\frac{1}{2}(T_1 - {T_1}^2),\ T_2 - {T_2}^2, \dots, T_k - {T_k}^2\right)\right).\end{aligned} \tag{2.26}$$

It is obvious that the matrix $G(-\frac{1}{2}(T_1 - {T_1}^2),\ T_2 - {T_2}^2, \dots, T_k - {T_k}^2)$ turns to the matrix

$$\begin{bmatrix} T_2 - T_2^2 - \frac{1}{8}\left(2I + \sum\limits_{i=2}^{k}(T_i - T_i^2)\right)(T_1 - T_1^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & T_2 - T_2^2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T_3 - T_3^2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & T_k - T_k^2 \end{bmatrix} \tag{2.27}$$

by elementary block matrix operations. Note that the matrix in (2.27) is obtained, first, premultiplying the matrix

$$G(-\frac{1}{2}(T_1 - {T_1}^2),\ T_2 - {T_2}^2, \dots, T_k - {T_k}^2)$$

by C_2C_1, and then, postmultiplying the matrix

$$C_2C_1G(-\frac{1}{2}(T_1 - {T_1}^2),\ T_2 - {T_2}^2, \dots, T_k - {T_k}^2)$$

by D_1D_2, where

$$C_1 = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -(T_2 - T_2^2) & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -(T_3 - T_3^2) & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(T_{k-1} - T_{k-1}^2) & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ -(T_k - T_k^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix} \in \mathbb{C}_{nk},$$

$$C_2 = \begin{bmatrix} I & -\frac{1}{2}I & -\frac{1}{2}I & \dots & -\frac{1}{2}I & -\frac{1}{2}I \\ \mathbf{0} & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix} \in \mathbb{C}_{nk},$$

$$D_1 = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}(T_2 - T_2^2) & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}(T_3 - T_3^2) & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{2}(T_{k-1} - T_{k-1}^2) & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ -\frac{1}{2}(T_k - T_k^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix} \in \mathbb{C}_{nk},$$

and

$$D_2 = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{8}(T_1 - T_1^2) & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ -\frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix} \in \mathbb{C}_{nk}.$$

It is obvious that the matrices

$$C_2C_1 = \begin{bmatrix} I + \frac{1}{2}\sum\limits_{i=2}^{k}(T_i - T_i^2) & -\frac{1}{2}I & -\frac{1}{2}I & \dots & -\frac{1}{2}I & -\frac{1}{2}I \\ -(T_2 - T_2^2) & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -(T_3 - T_3^2) & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(T_{k-1} - T_{k-1}^2) & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ -(T_k - T_k^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix}$$

and

$$D_1D_2 = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}(T_2 - T_2^2) - \frac{1}{8}(T_1 - T_1^2) & I & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}(T_3 - T_3^2) - \frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & I & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{2}(T_{k-1} - T_{k-1}^2) - \frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & \mathbf{0} & \dots & I & \mathbf{0} \\ -\frac{1}{2}(T_k - T_k^2) - \frac{1}{8}(T_1 - T_1^2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I \end{bmatrix}$$

are nonsingular. So, we get

(2.28)

$$\begin{aligned} &\operatorname{rk}\left(G(-\frac{1}{2}(T_1 - {T_1}^2),\ T_2 - {T_2}^2, \dots, T_k - {T_k}^2)\right) \\ &= \operatorname{rk}\left(8(T_2 - T_2^2) - \left(2I + \sum_{i=2}^{k}(T_i - T_i^2)\right)(T_1 - T_1^2)\right) + \sum_{i=2}^{k}\operatorname{rk}(T_i - T_i^2) \end{aligned}$$

due to the fact that the rank of a block diagonal matrix is equal to the sum of the ranks of the matrices on diagonal (see, e.g., [4]). Thus, from the equalities (2.26) and (2.28), it follows that

$$\begin{aligned}&\operatorname{rk}\left(\sum_{i=1}^{k} T_i - \sum_{i=1}^{k} {T_i}^2\right)\\ &= \operatorname{rk}\left(8(T_2^2 - T_2) + \left(-2I + \sum_{i=2}^{k}(T_i^2 - T_i)\right)(T_1^2 - T_1)\right),\end{aligned}$$

since $\frac{1}{2}(T_i^2 - T_i)$, $i = 1, \cdots, k$, is idempotent (and therefore, its rank is equal to its trace.) So, we have proved the following theorem.

Theorem 2.10. *Let T_1, T_2, ... ,T_k be essentially tripotent matrices. Then*

$$\begin{aligned}&\operatorname{rk}\left(\sum_{i=1}^{k} ({T_i}^2 - T_i)\right)\\ &= \operatorname{rk}\left(8(T_2^2 - T_2) + \left(-2I + \sum_{i=2}^{k}(T_i^2 - T_i)\right)(T_1^2 - T_1)\right).\end{aligned}$$

Recall that a square matrix M is called a *skew-idempotent matrix* if $M^2 = -M$. Also, skew-idempotent matrices are tripotent. It is clear that Theorem 2.10 is trivial for idempotent and skew-idempotent matrices. So, Theorem 2.10 is valid for all tripotent matrices. Also, considering this theorem, it can be derived some rank equalities for some special types of matrices. For example, if T_1 and T_2 are involutive matrices, then, by using Theorem 2.10, we get

$$\operatorname{rk}(2I - T_1 - T_2) = \operatorname{rk}\left(8(I - T_2) - (I + T_2)(I - T_1)\right),$$

since involutive matrices are also tripotent.

Now, if T_1 and T_2 are skew idempotent matrices, then we get

$$\operatorname{rk}(T_1 + T_2) = \operatorname{rk}(T_2T_1 + T_1 - 4T_2), \tag{2.29}$$

by Theorem 2.10.

On the other hand, Zuo stated in [13, Theorem 2.1] that if P and Q are idempotent matrices and $c \neq a + b$ with $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$, then

$$\operatorname{rk}(aP + bQ - cPQ) = \operatorname{rk}(P + Q). \tag{2.30}$$

Now, let A and B be two skew-idempotent matrices. So, the matrices $-A$ and $-B$ are idempotent matrices. Thus, from (2.30), it can be written

$$\operatorname{rk}(A + B) = \operatorname{rk}(aA + bB + cAB) \tag{2.31}$$

with $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$. Notice that the rank equality in (2.29) is a special case of the equality in (2.31). To see this, it is sufficient to take $A = T_2$, $B = T_1$, $a = -4$, $b = 1$, and $c = 1$.

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(Tuğba Petik) Sakarya University, Department of Mathematics, Sakarya, 54187, Turkey.

E-mail address: tpetik@sakarya.edu.tr

(Halim Özdemir) Sakarya University, Department of Mathematics, Sakarya, 54187, Turkey.

E-mail address: hozdemir@sakarya.edu.tr