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ON THREE-DIMENSIONAL $N(k)$-PARACONTACT METRIC MANIFOLDS AND RICCI SOLITONS

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Abstract. The aim of this paper is to characterize 3-dimensional $N(k)$-paracontact metric manifolds satisfying certain curvature conditions. We prove that a 3-dimensional $N(k)$-paracontact metric manifold M admits a Ricci soliton whose potential vector field is the Reeb vector field ξ if and only if the manifold is a paraSasaki-Einstein manifold. Several consequences of this result are discussed. Finally, an illustrative example is constructed.
Keywords: Ricci semisymmetric, cyclic parallel Ricci tensor, η-parallel Ricci tensor, Ricci soliton, Einstein manifold.
MSC(2010): Primary: 53B30; Secondary: 53C15, 53C25.

1. Introduction

The study of nullity distribution on paracontact geometry is one among the most interesting topics in modern contact geometry. Kaneyuki and Williams [25] initiated the study of paracontact geometry. Since then many authors [1, 2, 6, 13, 18, 39] contribute to the study of paracontact geometry. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [40]. The importance of paracontact geometry comes from the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. More recently, Cappelletti-Montano et al [9] introduced a new type of paracontact geometry, so-called paracontact metric (k, μ)-spaces, where k and μ are some real constants. Martin-Molina [28, 29] obtained some classification theorems on paracontact metric (k, μ)-spaces and constructed some examples.

The conformal curvature tensor C is invariant under conformal transformations and vanishes identically for 3-dimensional manifolds. Using this fact, several authors [15–17, 24] studied various types of 3-dimensional manifolds.

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A pseudo-Riemannian manifold is called semisymmetric (respectively, Ricci semisymmetric) if $R(X,Y)\cdot R = 0$ (respectively, $R(X,Y)\cdot S = 0$) [35], where $R(X,Y)$ is considered as a field of linear operators acting on R (respectively, S).

An algebraic curvature tensor field R on a pseudo-Riemannian manifold (M,g) is said to be harmonic [31] if $(\mathrm{div}R)(X,Y,Z) = 0$, for any vector fields $X,Y,Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M and 'div' denotes the divergence operator with respect to the metric g. On the other hand $\mathrm{div}R = 0$ holds in a pseudo-Riemannian manifold if and only if the Ricci tensor is of Codazzi type [30], that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{1.1}$$

where ∇ is the Levi-Civita connection.

A Ricci soliton is a natural generalization of an Einstein metric [3]. In a manifold M a Ricci soliton is a triplet (g,V,λ), with g, a Riemannian metric, V a vector field (called the potential vector field) and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_V$ is the Lie derivative with respect to V and S is the Ricci tensor of type $(0,2)$. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive, respectively. The compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [22]) and also in dimension 3 (Ivey [23]). It is well known [32] that a Ricci soliton on a compact manifold is a gradient Ricci soliton. For more details, we refer to Chow and Knopf [12].

Sharma [34] has initiated the study of Ricci solitons in K-contact manifolds. Recently, Yildiz et al [38] studied Ricci solitons in 3-dimensional f-Kenmotsu manifolds. Also Ricci solitons have been studied by several authors such as Cho [10, 11], Tripathi [36], De and Matsuyama [14], Ghosh [19, 20], Turan et al [37] and many others.

The paper is organized as follows: In Section 2, we give some basic results of $N(k)$-paracontact metric manifolds. Section 3 is devoted to study Ricci semisymmetric $N(k)$-paracontact metric manifolds. Sections 4 and 5 respectively deal with 3-dimensional $N(k)$-paracontact metric manifolds satisfying $\mathrm{div}R = 0$ and cyclic parallel Ricci tensor. In the next section we study η-parallel Ricci tensor on 3-dimensional $N(k)$-paracontact metric manifolds. Section 7 is devoted to study Ricci soliton on 3-dimensional $N(k)$-paracontact metric manifolds. Several consequences of this result are discussed. Finally, an illustrative example is constructed.

2. **Preliminaries**

A smooth manifold M^{2n+1} has an almost paracontact structure (ϕ, ξ, η) if it admits a $(1,1)$-type tensor field ϕ, a vector field ξ (called the Reeb vector field) and a 1-form η satisfying the following conditions [25]
(i) $\phi^2 X = X - \eta(X)\xi$,
(ii) $\phi(\xi) = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1$,
(iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to eigenvalues 1 and -1, respectively, have same dimension n.
An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, is called *almost paracontact metric manifold* and (ϕ, ξ, η, g) is said to be an *almost paracontact metric structure*.

An almost paracontact structure is said to be *normal* [40] if and only if the $(1,2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. An almost paracontact structure is said to be a *paracontact* structure if $g(X, \phi Y) = d\eta(X, Y)$ [40]. Any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis [40], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, \ldots, E_n, \phi E_1, \phi E_2, \ldots, \phi E_n\}$, where $\xi, E_1, E_2, \ldots, E_n$ are space-like vector fields and then, by (2.1) vector fields $\phi E_1, \phi E_2, \ldots, \phi E_n$ are time-like. For a three dimensional almost paracontact metric manifold, any (local) pseudo-orthonormal basis of $\ker(\eta)$ determines a ϕ-basis, up to sign. If $\{e, e_3\}$ is a (local) pseudo-orthonormal basis of $\ker(\eta)$, with e_3, time-like, so by (2.1) vector field $\phi e_2 \in \ker(\eta)$ is time-like and orthogonal to e_2. Therefore, $\phi e_2 = \pm e_3$ and $\{\xi, e_2, \pm e_3\}$ is a ϕ-basis [5]. In a paracontact metric manifold one can easily define a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$ satisfying (see [8, 40])

$$\phi h + h\phi = 0, \quad h\xi = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.3}$$

for all $X \in \chi(M)$. Clearly h vanishes identically if and only if ξ is a Killing vector field and then (ϕ, ξ, η, g) is said to be *K-paracontact structure*. An almost paracontact metric manifold is said to be *para-Sasakian* manifold if and only if (see [40])

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.4}$$

holds, for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.5}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.5) does not imply that the paracontact manifold is para-Sasakian. It is well known that every para-Sasakian manifold is K-paracontact, but the converse is not always true, as it is shown in three dimensional case [4].

According to Cappelletti-Montano and Di Terlizzi [8] we give the definition of paracontact metric (k,μ)-manifolds.

Definition 2.1. A paracontact metric manifold is said to be a *paracontact* (k,μ)*-manifold* if the curvature tensor R satisfies

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$ and real constants k, μ.

In particular, if $\mu = 0$, then the paracontact metric (k,μ)-manifold is called an $N(k)$-paracontact metric manifold. Thus for an $N(k)$-paracontact metric manifold we have

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{2.7}$$

for all $X, Y \in \chi(M)$.

In an $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$, the following relations hold (see [33])

$$QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi, \tag{2.8}$$

$$S(X,Y) = (\frac{r}{2} - k)g(X,Y) + (3k - \frac{r}{2})\eta(X)\eta(Y), \tag{2.9}$$

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} - 2k)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad +(3k - \frac{r}{2})\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2k\eta(X), \tag{2.11}$$

where Q, S, R and r are the Ricci operator, Ricci tensor, curvature tensor and the scalar curvature respectively. From (2.10) it follows that

$$R(\xi, X)Y = k\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.12}$$

Also using (2.3) we have

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y), \tag{2.13}$$

for all $X, Y \in \chi(M)$. Immediately from (2.10) we have the following:

Proposition 2.2. *A* 3*-dimensional* $N(k)$*-paracontact metric manifold is a manifold of constant curvature* k *if and only if the scalar curvature* $r = 6k$.

We recall a result due to Cappelletti-Montano et al ([9, p. 686]).

Lemma 2.3. *Any paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$.*

Though any paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$, it always admits some compatible Einstein metrics [7].

3. Ricci semisymmetric $N(k)$-paracontact metric manifolds

In this section we discuss about Ricci semisymmetric $N(k)$-paracontact metric manifolds. Hence

$$R(X, Y) \cdot S = 0.$$

This is equivalent to

$$(R(X, Y) \cdot S)(U, V) = 0, \tag{3.1}$$

for any $X, Y, U, V \in \chi(M)$.
From (3.1), we have

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \tag{3.2}$$

Substituting $X = U = \xi$ in (3.2) we obtain

$$S(R(\xi, Y)\xi, V) + S(\xi, R(\xi, Y)V) = 0. \tag{3.3}$$

Using (2.11) we have from (3.3)

$$S(R(\xi, Y)\xi, V) + 2k\eta(R(\xi, Y)V) = 0. \tag{3.4}$$

Taking (2.12) and (3.4) into account it follows that

$$kS(Y, V) - 2k^2 g(Y, V) = 0. \tag{3.5}$$

From (3.5) we get

$$k\{S(Y, V) - 2kg(Y, V)\} = 0. \tag{3.6}$$

Suppose $k = 0$, then from Lemma 2.3 we get that the manifold is an Einstein manifold.
Also, if $k \neq 0$, then it follows from (3.6) that the manifold is an Einstein manifold. Conversely, if the manifold is an Einstein manifold, then obviously $R \cdot S = 0$.
Thus we have the following:

Theorem 3.1. *A 3-dimensional $N(k)$-paracontact metric manifold is Ricci semisymmetric if and only if the manifold is an Einstein manifold.*

From $S(Y, V) = 2kg(Y, V)$, it immediately follows that $r = 6k$. Taking account of Proposition 2.2 and the above theorem we can state the following:

Corollary 3.2. *A 3-dimensional $N(k)$-paracontact metric manifold is Ricci semisymmetric if and only if the manifold is of constant curvature k.*

Again Ricci symmetry ($\nabla S = 0$) implies Ricci semisymmetric ($R \cdot S = 0$), thus we have the following:

Corollary 3.3. *A 3-dimensional $N(k)$-paracontact metric manifold is Ricci symmetric if and only if the manifold is of constant curvature k.*

4. 3-dimensional $N(k)$-paracontact metric manifolds with harmonic curvature tensor

This section is devoted to the study of 3-dimensional $N(k)$-paracontact metric manifolds with harmonic curvature tensor. Then $\mathrm{div} R = 0$, which implies

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{4.1}$$

Taking the covariant derivative of (2.9) along an arbitrary vector field Y and making use of (2.3) and (2.13) we obtain

$$\begin{aligned}(\nabla_Y S)(X, Z) &= \frac{dr(Y)}{2}\{g(X, Z) - \eta(X)\eta(Z)\} + (3k - \frac{r}{2})\{g(Y, \phi X)\eta(Z) \\ &\quad -g(hY, \phi X)\eta(Z) - \eta(X)g(\phi Y, Z) + \eta(X)g(\phi hY, Z)\}.\end{aligned} \tag{4.2}$$

Interchanging X and Y in (4.2) yields

$$\begin{aligned}(\nabla_X S)(Y, Z) &= \frac{dr(X)}{2}\{g(Y, Z) - \eta(Y)\eta(Z)\} + (3k - \frac{r}{2})\{g(X, \phi Y)\eta(Z) \\ &\quad -g(hX, \phi Y)\eta(Z) - \eta(Y)g(\phi X, Z) + \eta(Y)g(\phi hX, Z)\}.\end{aligned} \tag{4.3}$$

Applying (4.2) and (4.3) in (4.1) we have

$$\begin{aligned}&(3k - \frac{r}{2})\{2g(X, \phi Y)\eta(Z) - g(\phi X, Z)\eta(Y) \\ &+g(\phi hX, Z)\eta(Y) + g(\phi Y, Z)\eta(X) - g(\phi hY, Z)\eta(X)\} = 0.\end{aligned} \tag{4.4}$$

Replacing Y by ϕY in (4.4) yields

$$\begin{aligned}&(3k - \frac{r}{2})\{2g(X, Y)\eta(Z) + g(Y, Z)\eta(X) \\ &+g(hY, Z)\eta(X) - 3\eta(X)\eta(Y)\eta(Z)\} = 0.\end{aligned} \tag{4.5}$$

Let $\{e_i\}$, $i = 1, 2, 3$ be a ϕ-basis of the tangent space at each point of the manifold. Then putting $Y = Z = e_i$ in (4.5) and taking summation over i, $1 \leq i \leq 3$, we get

$$(3k - \frac{r}{2})\eta(X) = 0. \tag{4.6}$$

This gives $r = 6k$ (since $\eta(X) \neq 0$), which implies by Proposition 2.2 that the manifold is of constant curvature k.

This leads to the following:

Theorem 4.1. *If a 3-dimensional $N(k)$-paracontact metric manifold is of harmonic curvature, then the manifold is of constant curvature k.*

5. Cyclic parallel Ricci tensor

In this section we study cyclic parallel Ricci tensor in 3-dimensional $N(k)$-paracontact metric manifolds. Suppose that the manifold M has cyclic parallel Ricci tensor [21], then the Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{5.1}$$

Taking the covariant derivative of (2.9) along arbitrary vector field X and making use of (2.3) and (2.13) we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \frac{dr(X)}{2}\{g(Y,Z)-\eta(Y)\eta(Z)\} + (3k-\frac{r}{2})\{g(X,\phi Y)\eta(Z)\\ &\quad -g(hX,\phi Y)\eta(Z) - \eta(Y)g(\phi X,Z) + \eta(Y)g(\phi hX,Z)\}.\end{aligned} \tag{5.2}$$

Similarly, we have

$$\begin{aligned}(\nabla_Y S)(Z,X) &= \frac{dr(Y)}{2}\{g(Z,X)-\eta(Z)\eta(X)\} + (3k-\frac{r}{2})\{g(Y,\phi Z)\eta(X)\\ &\quad -g(hY,\phi Z)\eta(X) - \eta(Z)g(\phi Y,X) + \eta(Z)g(\phi hY,X)\},\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}(\nabla_Z S)(X,Y) &= \frac{dr(Z)}{2}\{g(X,Y)-\eta(X)\eta(Y)\} + (3k-\frac{r}{2})\{g(Z,\phi X)\eta(Y)\\ &\quad -g(hZ,\phi X)\eta(Y) - \eta(X)g(\phi Z,Y) + \eta(X)g(\phi hZ,Y)\}.\end{aligned} \tag{5.4}$$

Also it is known [26] that Cartan hypersurfaces are manifolds, with non-parallel Ricci tensor, satisfying (5.1). From (5.1), it follows that r = constant. Using (5.2)-(5.4) in (5.1) we obtain

$$(3k-\frac{r}{2})\{g(\phi hY,Z)\eta(X) + g(\phi hX,Z)\eta(Y) + g(\phi hY,X)\eta(Z)\} = 0. \tag{5.5}$$

Replacing X by ϕX in (5.5) gives

$$(3k-\frac{r}{2})\{g(hX,Z)\eta(Y) + g(hX,Y)\eta(Z)\} = 0. \tag{5.6}$$

Substituting $Z=\xi$ in (5.6) and using the last part of (2.2) we get

$$(3k-\frac{r}{2})g(hX,Y) = 0. \tag{5.7}$$

Thus we get either, $r = 6k$, or, $h = 0$.
By the above discussions and Proposition 2.2 we get the following:

Theorem 5.1. *If an $N(k)$-paracontact metric manifold M^3 admits cyclic parallel Ricci tensor, then either the manifold is of constant curvature k, or para-Sasakian.*

6. η-parallel Ricci tensor

Definition 6.1. An $N(k)$-paracontact metric manifold M^3 is called η-parallel if its Ricci tensor satisfies

$$(\nabla_Z S)(\phi X, \phi Y) = 0, \tag{6.1}$$

for all vector fields X, Y and $Z \in \chi(M)$.

The notion of η-parallel Ricci tensor for Sasakian manifolds was introduced by Kon [27].

Let us assume that the Ricci tensor of an $N(k)$-paracontact metric manifold is η-parallel. Then (6.1) holds.
From (2.9) we can easily get

$$S(\phi X, \phi Y) = (\frac{r}{2} - k)g(\phi X, \phi Y). \tag{6.2}$$

Taking covariant derivative of (6.2) along any arbitrary vector field Z we obtain

$$\begin{aligned}(\nabla_Z S)(\phi X, \phi Y) &= \frac{dr(Z)}{2}\{-g(X,Y) + \eta(X)\eta(Y)\} \\ &\quad +(\frac{r}{2} - k)\{(\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\}.\end{aligned} \tag{6.3}$$

Applying (2.13) in the above equation gives

$$\begin{aligned}(\nabla_Z S)(\phi X, \phi Y) &= \frac{dr(Z)}{2}\{-g(X,Y) + \eta(X)\eta(Y)\} + (\frac{r}{2} - k)\{g(Z,\phi X)\eta(Y) \\ &\quad -g(hZ,\phi X)\eta(Y) + g(Z,\phi Y)\eta(X) - g(hZ,\phi Y)\eta(X)\}.\end{aligned} \tag{6.4}$$

In view of (6.1) and (6.4) we have

$$\begin{aligned}&\frac{dr(Z)}{2}\{-g(X,Y) + \eta(X)\eta(Y)\} + (\frac{r}{2} - k)\{g(Z,\phi X)\eta(Y) \\ &-g(hZ,\phi X)\eta(Y) + g(Z,\phi Y)\eta(X) - g(hZ,\phi Y)\eta(X)\} = 0.\end{aligned} \tag{6.5}$$

Putting $X = Y = e_i$ and taking summation over i, $1 \leq i \leq 3$, we obtain

$$dr(Z) = 0, \tag{6.6}$$

from which it follows that $r=$ constant. Using this fact we have from (6.5)

$$\begin{aligned}&(\frac{r}{2} - k)\{g(Z,\phi X)\eta(Y) - g(hZ,\phi X)\eta(Y) \\ &+g(Z,\phi Y)\eta(X) - g(hZ,\phi Y)\eta(X)\} = 0.\end{aligned} \tag{6.7}$$

Replacing X by ξ in the above equation yields

$$(\frac{r}{2} - k)\{g(Z,\phi Y) - g(hZ,\phi Y)\} = 0. \tag{6.8}$$

Interchanging Y and Z in (6.8) we have

$$(\frac{r}{2} - k)\{g(Y,\phi Z) - g(hY,\phi Z)\} = 0. \tag{6.9}$$

Subtracting the above two equations we get

$$(\frac{r}{2} - k)g(Z, \phi Y) = 0. \tag{6.10}$$

It follows from (6.10) that $r = 2k$.

Conversely, if $r = 2k$, one can easily find from (6.4) that

$$(\nabla_Z S)(\phi X, \phi Y) = 0,$$

for all vector fields X, Y and $Z \in \chi(M)$.

Hence we can state the following:

Theorem 6.2. *An $N(k)$-paracontact metric manifold of dimension 3 is η-parallel if and only if the scalar curvature $r = 2k$.*

7. Ricci solitons

Suppose that a 3-dimensional $N(k)$-paracontact metric manifold admits a Ricci soliton whose potential vector field is the Reeb vector field ξ. Then from (1.2) we get

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{7.1}$$

Taking into account of (2.3) the above equation implies

$$g(\phi hX, Y) + S(X, Y) + \lambda g(X, Y) = 0. \tag{7.2}$$

Replacing Y by ξ in the above equation gives

$$(\lambda + 2k)\eta(X) = 0. \tag{7.3}$$

Putting $X = \xi$ in (7.3) to get

$$\lambda = -2k. \tag{7.4}$$

Thus, (7.2) and (7.4) together gives

$$S(X, Y) = 2kg(X, Y) - g(\phi hX, Y). \tag{7.5}$$

Replace X by ϕX in (7.5) to get

$$S(\phi X, Y) = 2kg(\phi X, Y) + g(hX, Y). \tag{7.6}$$

Also from (2.9) we obtain

$$S(\phi X, Y) = (\frac{r}{2} - k)g(\phi X, Y). \tag{7.7}$$

Equating the right hand sides of (7.6) and (7.7) we get

$$g(hX, Y) = (\frac{r}{2} - 3k)g(\phi X, Y). \tag{7.8}$$

Replacing X and Y in (7.8) yields

$$g(hY, X) = (\frac{r}{2} - 3k)g(\phi Y, X). \tag{7.9}$$

Adding (7.8) and (7.9) we have $g(hX,Y)=0$, which gives

$$h=0. \tag{7.10}$$

Now $h=0$ holds if and only if ξ is a Killing vector field and thus M is a K-paracontact metric manifold. Then equation (1.2) yields that M is Einstein. Also in dimension 3, a K-paracontact metric manifold is a para-Sasakian manifold. Thus M is a paraSasaki-Einstein manifold. The converse is trivial.

Thus we can state the following:

Theorem 7.1. *A 3-dimensional $N(k)$-paracontact metric manifold admits a Ricci soliton whose potential vector field is the Reeb vector field ξ if and only if the manifold is a paraSasaki-Einstein.*

Remark 7.2. [5, Theorem 3.3] is a particular case of Theorem 7.1.

Corollary 7.3. *If a conformally flat $N(k)$-paracontact metric manifold admits a Ricci soliton, then the manifold is a paraSasaki-Einstein.*

8. Example of a 3-dimensional $N(k)$-paracontact metric manifold

In this section we construct an example of a 3-dimensional $N(k)$-paracontact metric manifold such that $k=-1$ and $h\neq 0$. We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3, z\neq 0\}$, where (x,y,z) are the standard coordinates of $\mathbb{R}^3$.

The vector fields

$$e_1=\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-2y\frac{\partial}{\partial z},\ e_2=\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold M. We define the pseudo-Riemannian metric g as follows

$$g(e_1,e_2)=g(e_3,e_3)=1 \text{ and } g(e_i,e_j)=0,\ \text{otherwise} .$$

We obtain

$$[e_1,e_2]=2e_3,\ [e_1,e_3]=-e_2,\ [e_2,e_3]=0.$$

We consider $\eta=2ydx+dz$ and satisfying $\eta(e_1)=0=\eta(e_2)$, $\eta(e_3)=1$. Let ϕ be the $(1,1)$-tensor field defined by $\phi e_1=e_1,\ \phi e_2=-e_2,\ \phi e_3=0$. Then we have

$$\begin{aligned} d\eta(e_1,e_2)&=g(e_1,\phi e_2),\\ d\eta(e_1,e_3)&=g(e_1,\phi e_3),\\ d\eta(e_2,e_3)&=g(e_2,\phi e_3).\end{aligned}$$

Thus for $e_3=\xi$, the structure (ϕ,ξ,η,g) is a paracontact metric structure on M with $he_1=e_2,\ he_2=he_3=0$.

Using the well known Koszul's folmula we have the following:

$$\nabla_{e_1}e_1 = e_3,\ \nabla_{e_1}e_2 = e_3,\ \nabla_{e_1}e_3 = -e_1 - e_2,$$
$$\nabla_{e_2}e_1 = -e_3,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = e_2,$$
$$\nabla_{e_3}e_1 = -e_1,\ \nabla_{e_3}e_2 = e_2,\ \nabla_{e_3}e_3 = 0.$$

By the above results we can easily obtain the components of the curvature tensor as follows:

$$R(e_1,e_2)e_1 = 3e_1,\ R(e_1,e_2)e_2 = -3e_2,\ R(e_1,e_2)e_3 = 0,$$
$$R(e_1,e_3)e_1 = -2e_3,\ R(e_1,e_3)e_2 = e_3,\ R(e_1,e_3)e_3 = 2e_2 - e_1,$$
$$R(e_2,e_3)e_1 = e_3,\ R(e_2,e_3)e_2 = 0,\ R(e_2,e_3)e_3 = -e_2.$$

We conclude that the manifold is an $N(k)$-paracontact metric manifold with $k = -1$. Also from the above expressions it is not hard to see that the scalar curvature r of the manifold is -2. Therefore we obtain $r = 2k$, where $k = -1$. Thus Theorem 6.2 is verified.

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