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Author(s):

M. Davoudian and O. Ghayour

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THE LENGTHS OF ARTINIAN MODULES WITH COUNTABLE NOETHERIAN DIMENSIONS

M. DAVOUDIAN* AND O. GHAYOUR

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Abstract. It is shown that if M is an Artinian module over a ring R, then M has Noetherian dimension α, where α is a countable ordinal number, if and only if $\omega^{\alpha} + 2 \leq l(M) \leq \omega^{\alpha+1}$, where $l(M)$ is the length of M, *i.e.*, the least ordinal number such that the interval $[0,\ l(M))$ cannot be embedded in the lattice of all submodules of M.
Keywords: Artinian module, Noetherian dimension, atomic module, the length of an R-module.
MSC(2010): Primary: 16P60; Secondary: 16P20, 6P40.

1. Introduction

Lemonnier [20] has introduced the concept of deviation (respectively, codeviation) of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R gives the concept of Krull dimension, see [9, 10] and [22] (respectively, the concept of dual Krull dimension of M. The dual Krull dimension in [6, 7, 11, 12, 14–17] and [18] is called Noetherian dimension and in [5] is called N-dimension. This dimension is called Krull dimension in [23]. The name of dual Krull dimension is also used by some authors, see [2, 3] and [1]). The Noetherian dimension of an R-module M is denoted by n-$\dim M$ and by k-$\dim M$ we denote the Krull dimension of M.

If an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-atomic if n-$\dim M = \alpha$ and n-$\dim N < \alpha$ for all proper submodules N of M. An R-module M is called atomic if M is α-atomic for some ordinal α (note, atomic modules are also called dual critical, N-critical and conotable module by some authors, see [3, 5] and [21], respectively). Bass [4] showed that if R is a commutative Noetherian ring with countable classical Krull dimension α, then $\omega^{\alpha} \leq O(R)$, where $O(R)$ is the supremum of all the

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*Corresponding author.

ordinal lengths of well-ordered descending chains of ideals of R (he also has shown that all these ordinal lengths are countable). Later, the use of ordinal length of a module received some attention, in the context of Krull dimension (respectively, Noetherian dimension) by some authors, see [8,17]. In [17], the aforementioned result of Bass, is extended to modules with Noetherian dimension, moreover their results implies this interesting fact that in a module with countable Noetherian dimension every submodule is countably generated, see [17, Crollary 1.8]. Consequently, over locally Noetherian rings, modules with countable Noetherian dimension are countably generated, see [17, Theorem 2.11]. Recall that Artinian modules over commutative rings have finite Noetherian dimension, hence they are all countably generated. But these modules over non-commutative rings may have any ordinal number as their Noetherian dimension, see [16,17] and [6]. The reader is also reminded that ω^{ω} is an upper bound for the lengths of all chains in Artinian modules over commutative rings, see [17, Remark 2.6]. The length of a module M, $l(M)$, plays a basic role in all previous results. In [17, Corollary 1.4] it is observed that if M is a module with $n\text{-dim}\, M = \alpha$, then $l(M) \leq \omega^{\alpha+1}$. Motivated by the latter fact we are interested to connect the countability of the Noetherian dimension of an Artinian module M (note, Artinian modules always have Noetherian dimension) to the boundedness of its length. We show that for an Artinian module M, $n\text{-dim}\, M = \alpha$ if and only if $\omega^{\alpha} + 2 \leq l(M) \leq \omega^{\alpha+1}$, where α is a countable ordinal number. We also observe that an Artinian module M is α-atomic if and only if $l(M) = \omega^{\alpha} + 2$, where α is a countable ordinal number. Throughout this paper R will always denote an associative ring with a non-zero identity and M a unital R-module. The notation $N \subseteq M$ (respectively, $N \subset M$) means that N is a submodule (respectively, proper submodule) of M.

2. **Preliminaries**

In this section we recall some useful facts about the concepts of Noetherian dimension and the length of an R-module M.

First, we recall the following definition from [17].

Definition 2.1. If $M_0 \subset M_1 \subset \dots \subset M_{\alpha} \subset M_{\alpha+1} \subset \dots$ is an ascending chain of submodules of a module M, then the length of this chain is λ if λ is the least ordinal such that for each M_{β} we have $\beta < \lambda$. We define $l(M)$ to be the least ordinal such that given any ascending chain of length λ in M, then $\lambda < l(M)$. This means that $l(M)$ is the least ordinal such that $[0,\ l(M))$ cannot be embedded in the lattice of submodules of M. Clearly, if $O(M)$ denotes the supremum of all lengths of ascending chains in M, then $O(M) \leq l(M) \leq O(M) + 1$. We also observe that for a limit ordinal α, we cannot have $l(M) = \alpha + 1$, for otherwise $l(M) > \alpha$ means that M has an ascending chain of submodules $M_0 \subset M_1 \subset \dots \subset M_{\beta} \subset \cdots$, $\beta < \alpha$, of length α, i.e., $M_0 \subset M_1 \subset \dots \subset M_{\beta} \subset \dots \subset M_{\alpha}$, where $M_{\alpha} = \cup_{\beta<\alpha} M_{\beta}$, is of length $\alpha + 1$

and therefore $l(M) > \alpha + 1$, which is absurd. It should also be noted that M has an ascending chain of maximum length if and only if $l(M)$ is a successor ordinal. Finally, if M is Noetherian, then $l(M) \leq \omega$; and if $l(M)$ is finite, then M has a composition series.

The following lemma, which is evident by definition, is needed (note, if $l(M) \leq \omega$, then M is Noetherian).

Lemma 2.2. *Let M be an R-module. If n-$\dim M = 1$, then $l(M) > \omega$.*

The next result is well-known, see [15, Proposition 1.4].

Lemma 2.3. *Let M be an R-module. If each proper submodule N of M has Noetherian dimension, then so does M and n-$\dim M \leq \sup\{n\text{-}\dim N + 1 : N \subset M\}$.*

We should remind the reader that by a quotient finite dimensional module M we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension. Let us recall the following well-known and important result, due to Lemonnier, see [21, Theorem 2.4] and [1, Proposition 2.2].

Proposition 2.4. *The following statements are equivalent for any R-module M and any ordinal $\alpha \geq 0$.*

(1) *n-$\dim M \leq \alpha$;*
(2) *M is quotient finite dimensional and for any $N \subset P \subseteq M$, there exists X with $N \subseteq X \subset P$ with n-$\dim \frac{P}{X} \leq \alpha$.*

3. The length of Artinian modules with countable Noetherian dimension

In this section we investigate the relationship between the Noetherian dimension of an Artinian module M with countable Noetherian dimension and the length of M. Let M be an Artinian module, we show that n-$\dim M = \alpha$ if and only if $\omega^{\alpha} + 2 \leq l(M) \leq \omega^{\alpha+1}$, where α is a countable ordinal number. We also observe that an Artinian module M is α-atomic if and only if $l(M) = \omega^{\alpha} + 2$, where α is a countable ordinal number.

We begin with the following lemma, whose proof is given for the sake of completeness.

Lemma 3.1. *Let M be an R-module and N be any submodule of M with $l(N) > \gamma$, $l(\frac{M}{N}) > \alpha$. If γ is a limit ordinal, then $l(M) > \gamma + \alpha$.*

Proof. Let $N_0 \subset N_1 \subset ... \subset N_\gamma$ be a chain in N (note, since $l(N) > \gamma$ such a chain exists). Now we consider $\frac{M}{N_\gamma}$, since $l(\frac{M}{N}) > \alpha$ we infer that $l(\frac{M}{N_\gamma}) > \alpha$.

Now we consider two cases for α. First, let α be a limit ordinal, hence there must exist a chain

$$\frac{P_0}{N_\gamma} \subset \frac{P_1}{N_\gamma} \subset \frac{P_2}{N_\gamma} \subset ... \subset \frac{P_\alpha}{N_\gamma}$$

in $\frac{M}{N_\gamma}$, by definition. Thus we have the chain

$$N_0 \subset N_1 \subset ... \subset N_\gamma \subset P_0 \subset P_1 \subset ... \subset P_\alpha$$

of submodules of M. This shows that $l(M) \geq \gamma + \alpha + 2 > \gamma + \alpha$ and we are done. Now let α be a non-limit ordinal, i.e., $\alpha = \beta + 1$. This shows that there exists a chain

$$\frac{P_0}{N_\gamma} \subset \frac{P_1}{N_\gamma} \subset ... \subset \frac{P_\beta}{N_\gamma}$$

in $\frac{M}{N_\gamma}$. Consequently, in this case we also have $l(M) \geq \gamma + \beta + 2 = \gamma + \alpha + 1 > \gamma + \alpha$ and we are through. □

The following lemma improves [17, Lemma 1.5], in case M is an Artinian module.

Lemma 3.2. *Let N be an Artinian R-module. If n-$\dim N = \alpha$, then $l(N) > \omega^\alpha$, where α is a countable ordinal number.*

Proof. The proof is by induction on α. For $\alpha = 0$ it is trivial. Let us first prove it for $\alpha = 1$. If $\alpha = 1$, then by Lemma 2.2, we get $l(N) > \omega$. Now let n-$\dim N = \alpha$. We consider two cases:

Case 1: $\alpha = \beta + 1$. In this case there must exist a chain of submodules of M, namely, $N_1 \subset N_2 \subset N_3 \subset ...$ such that n-$\dim \frac{N_{i+1}}{N_i} = \beta$ for each i. Thus by induction hypothesis $l(\frac{N_{i+1}}{N_i}) > \omega^\beta$ for each i. Now by [17, Lemma 1.2], we have $l(N) > \omega^{\beta+1} = \omega^\alpha$ and we are through.

Case 2: Let α be a limit ordinal and assume that it is the supremum of ordinals $\alpha_1, \alpha_2, \ldots$. Now we apply the proof of [17, Proposition 1.12] from line 7 onwards to obtain an infinite chain

$$N_1 \subset N_2 \subset N_3 \subset ...$$

such that each $\frac{N_i}{N_{i-1}}$ is α_i-atomic, i.e., n-$\dim \frac{N_{i+1}}{N_i} = \alpha_i$ for each i. Thus by induction hypothesis $l(\frac{N_{i+1}}{N_i}) > \omega^{\alpha_i}$ for each i. Consequently, in view of [17, Lemma 1.2], we infer that $l(N) > \sum \omega^{\alpha_i} \geq \omega^\alpha$ and we are done. □

Next, we present our main result of this paper, which gives the value of Noetherian dimension of an Artinian module in relation to its length, where this dimension is countable.

Theorem 3.3. *Let M be an Artinian module. Then n-$\dim M = \alpha$ if and only if $\omega^\alpha + 2 \leq l(M) \leq \omega^{\alpha+1}$, where α is a countable ordinal number.*

Proof. Let n-dim $M = \alpha$, where α is a countable ordinal number. Then by Lemma 3.2, we have $l(M) \geq \omega^{\alpha} + 2$ (note, for a limit ordinal α, $l(M) \neq \alpha + 1$). But by [17, Corollary 1.4], we also have $l(M) \leq \omega^{\alpha+1}$ and we are through. Conversely, let $\omega^{\alpha} + 2 \leq l(M) \leq \omega^{\alpha+1}$. We are to show that n-dim $M = \alpha$. To this end, it suffices to prove that M satisfies part (2) of Proposition 2.4. Hence it must be shown that for any $N \subset P \subseteq M$, $Q = \frac{P}{N}$ has a nonzero factor module with Noetherian dimension less than or equal to α. If $l(Q) < \omega^{\alpha}$, then by Lemma 3.2 and [17, Lemma 1.5], n-dim $Q < \alpha$ and we are through. Now suppose that $l(Q) > \omega^{\alpha}$, hence we may begin with $l(Q) = \omega^{\alpha} + 2$ (note, we always have $l(Q) \neq \omega^{\alpha} + 1$ by definition). It follows that for each proper submodule N of Q, $l(N) \leq \omega^{\alpha}$, by definition. We infer that for each proper submodule N of Q, n-dim $N < \alpha$, by Lemma 3.2 and [17, Lemma 1.5]. Hence in view of Lemma 2.3, n-dim $Q \leq \sup\{n\text{-dim}\, N + 1 : N \subset Q\} \leq \alpha$ and we are done. Now suppose that $\omega^{\alpha} + 2 < l(Q)$. It is clear that $l(Q) \leq \omega^{\alpha+1}$ (note, we always have $l(Q) \leq l(M)$) . Now suppose that $l(Q) < \omega^{\alpha+1}$. Hence $l(Q) = \omega^{\alpha} n + \gamma$, where n is an integer number and γ is an ordinal number. We now consider two cases.

Case 1: $\gamma \neq 0$. In this case Q has a chain of submodules

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_{\omega^{\alpha} n} = \cup_{i<\omega^{\alpha} n} Q_i \subset Q_{\alpha_1} \subset Q_{\alpha_2} \subset \cdots \subset Q_{\alpha_i} \subset \cdots,$$

where $\alpha_i = \omega^{\alpha} n + i$ and $i = 1, 2, 3, \ldots$. It follows that $l(\frac{Q}{Q_{\omega^{\alpha} n}}) \leq \gamma$, by Lemma 3.1 (note, if $l(\frac{Q}{Q_{\omega^{\alpha} n}}) > \gamma$, then $l(Q) > \omega^{\alpha} n + \gamma$). In view of Lemma 3.2 and [17, Lemma 1.5] and the fact that $l(\frac{Q}{Q_{\omega^{\alpha} n}}) \leq \gamma < \omega^{\alpha}$ we infer that n-dim $\frac{Q}{Q_{\omega^{\alpha} n}} < \alpha$ and we are through.

Case 2: $\gamma = 0$, therefore $l(Q) = \omega^{\alpha} n$. In this case Q has a chain of submodules

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_{\omega^{\alpha}(n-1)} = \cup_{i<\omega^{\alpha}(n-1)} Q_i \subset Q_{\alpha_1} \subset Q_{\alpha_2} \subset \cdots \subset Q_{\alpha_i} \subset \cdots,$$

where $\alpha_i = \omega^{\alpha}(n-1) + i$ and $i = 1, 2, 3, \ldots$. It follows that $l(\frac{Q}{Q_{\omega^{\alpha}(n-1)}}) \leq \omega^{\alpha}$, by Lemma 3.1, (note, if $l(\frac{Q}{Q_{\omega^{\alpha}(n-1)}}) > \omega^{\alpha}$, then $l(Q) > \omega^{\alpha} n$). In view of Lemma 3.2 and [17, Lemma 1.5] and the fact that $l(\frac{Q}{Q_{\omega^{\alpha}(n-1)}}) \leq \omega^{\alpha}$ we infer that n-dim $\frac{Q}{Q_{\omega^{\alpha}(n-1)}} < \alpha$ and we are through. Finally, let $l(Q) = \omega^{\alpha+1}$. It follows that Q has a chain of submodules

$$Q_0 \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset Q_{\omega^{\alpha}} = \cup_{i<\omega^{\alpha}} Q_i \subset Q_{\omega^{\alpha}+1} \subset Q_{\omega^{\alpha}+2} \subset \cdots.$$

Put $N_1 = Q_{\omega^{\alpha}}$, if n-dim $\frac{Q}{N_1} \leq \alpha$ we are done. Now let n-dim $\frac{Q}{N_1} > \alpha$. In view of [17, Lemma 1.5], this implies that $l(\frac{Q}{N_1}) > \omega^{\alpha}$. It is clear that $l(\frac{Q}{N_1}) \leq \omega^{\alpha+1}$ (note, we always have $l(\frac{Q}{N_1}) \leq l(Q) \leq l(M)$). If $l(\frac{Q}{N_1}) < \omega^{\alpha+1}$, then $l(\frac{Q}{N_1}) \leq \omega^{\alpha} n + \gamma$, where $0 \leq \gamma < \omega^{\alpha}$ and n is an integer number. In this case by what we have already shown (note, *i.e.*, by cases 1, 2, and the proof before the cases)

$\frac{Q}{N_1}$ and therefore Q has a nonzero quotient module with Noetherian dimension less than or equal to α and we are through. Now suppose that $l(\frac{Q}{N_1}) = \omega^{\alpha+1}$. Hence $\frac{Q}{N_1}$ has a chain of submodules

$$\frac{Q'_0}{N_1} \subset \frac{Q'_1}{N_1} \subset \frac{Q'_2}{N_1} \subset \cdots \subset \cup_{i<\omega^\alpha} \frac{Q'_i}{N_1} = \frac{Q'_{\omega^\alpha}}{N_1} \subset \frac{Q'_{\omega^\alpha+1}}{N_1} \subset \cdots .$$

Put $\frac{Q'_{\omega^\alpha}}{N_1} = \frac{N_2}{N_1}$. If $l(\frac{Q}{N_2}) < \omega^{\alpha+1}$, then by the argument we have just given $\frac{Q}{N_2}$ and therefore Q has a nonzero quotient module with Noetherian dimension less than or equal to α and we are done. Otherwise $l(\frac{Q}{N_2}) = \omega^{\alpha+1}$. By continuing this method there exists an integer i, such that $l(\frac{Q}{N_i}) < \omega^{\alpha+1}$ and hence $\frac{Q}{N_i}$ and therefore Q has a nonzero quotient module with Noetherian dimension less than or equal to α and we are done. Otherwise Q has the infinite ascending chain of submodules

$$N_0 \subset N_1 \subset N_2 \subset \cdots$$

and by definition we have $l(\frac{N_{i+1}}{N_i}) > \omega^\alpha$ for each i (note, we have $Q_0 \subset Q_1 \subset Q_3 \subset \cdots \subset Q_{\omega^\alpha} = \cup_{i<\omega^\alpha} Q_i \subset Q_{\omega^\alpha+1} \subset Q_{\omega^\alpha+2} \subset \cdots$, and $N_1 = Q_{\omega^\alpha}$, and we also have $\frac{Q'_0}{N_i} \subset \frac{Q'_1}{N_i} \subset \frac{Q'_2}{N_i} \subset \cdots \subset \cup_{j<\omega^\alpha} \frac{Q'_j}{N_i} = \frac{Q'_{\omega^\alpha}}{N_i} \subset \frac{Q'_{\omega^\alpha+1}}{N_i} \subset \cdots$ and $\frac{N_{i+1}}{N_i} = \frac{Q'_{\omega^\alpha}}{N_i}$). In this case by [17, Lemma 1.2], we have $l(M) > \omega^\alpha + \omega^\alpha + \omega^\alpha + \cdots = \omega^\alpha \omega = \omega^{\alpha+1}$, which is a contradiction. □

In view of the previous theorem and Lemma 3.2 and [17, Proposition 1.11, Lemma 1.5], now we have the following result which improves [17, Proposition 1.12], (in fact, it proves the converse of this proposition, too).

Corollary 3.4. *Let M be an Artinian module and let α be a countable ordinal number. Then M is an α-atomic module if and only if $l(M) = \omega^\alpha + 2$.*

Corollary 3.5. *Let M be a right S-module, where S is an algebra over commutative ring R (e.g., $n \times n$ over R or infinite row matrices over R). If M is Artinian as an R-module, then for some natural numbers $p \leq q$, $\omega^p + 2 \leq l(M_S) \leq l(M_R) \leq \omega^{q+1}$. In particular, if S is the ring of $n \times n$-matrices over R, then M_R is Artinian if and only if M_S is Artinian and $n\text{-dim}\, M_S = n\text{-dim}\, M_R < \infty$.*

Proof. Clearly, M_S is Artinian too, $n\text{-dim}\, M_S \leq n\text{-dim}\, M_R$ and $l(M_S) \leq l(M_R)$. But $n\text{-dim}\, M_R$ is finite, q say, see [18, 23]. Hence $n\text{-dim}\, M_S$ is finite too, p say. Now in view of Theorem 3.3, we are done. The last part is evident by noticing that S in this case is finite R-algebra and invoke a nice result of Lemmonier [21], which shows $k\text{-dim}\, M_R = k\text{-dim}\, M_S$ and $n\text{-dim}\, M_R = n\text{-dim}\, M_S$. □

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(Maryam Davoudian) Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.
E-mail address: m.davoudian@scm.ac.ir

(Omid Ghayour) Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.
E-mail address: o.ghayour@yahoo.com