ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 6, pp. 1629–1644

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 6, pp. 1629–1644
Online ISSN: 1735-8515

ON DERIVATIONS AND BIDERIVATIONS OF TRIVIAL EXTENSIONS AND TRIANGULAR MATRIX RINGS

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(Communicated by Omid Ali S. Karamzadeh)

Abstract. Triangular matrix rings are examples of trivial extensions. In this article we determine the structure of derivations and biderivations of the trivial extensions, and thereby we describe the derivations and biderivations of the upper triangular matrix rings. Some related results are also obtained.
Keywords: Trivial extension, triangular matrix ring, derivation, biderivation.
MSC(2010): Primary: 16S50; Secondary: 16W25.

1. Introduction

Let R be a ring with identity and $Z(R)$ be the center of R. For each $x, y \in R$, denote the commutator of x, y by $[x, y] = xy - yx$. Let M be a unitary R-bimodule. An additive mapping $d : R \to M$ is said to be a *derivation* if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. Let $a \in R$. The mapping $I_a : R \to R$ given by $I_a(x) = [x, a]$ is easily seen to be a derivation of R. I_a is called the *inner derivation* induced by a. The biadditive mapping $d : R \times R \to M$ is called a *biderivation* if it is a derivation in each component; that is,

$$d(xy, z) = d(x, z)y + xd(y, z) \quad \text{and} \quad d(x, yz) = d(x, y)z + yd(x, z) \tag{1.1}$$

are fulfilled for all x, y, z in R. In the sequel, we use (1.1) without explicit mention.

A mapping $d : R \times R \to R$ is said to be an *extremal biderivation* if $d(x, y) = [x, [y, a]]$ for all $x, y \in R$, where $a \in R$ is such that $[a, [R, R]] = 0$.

Let R and M be as above. The *trivial extension* $T(R, M)$ of R by M is defined to be

$$T(R, M) = \{(r, m) : r \in R, m \in M\}.$$

Article electronically published on 30 November, 2017.
Received: 13 April 2014, Accepted: 3 August 2016.

It is easy to see that $T(R,M)$ with the componentwise addition and the multiplication given by

$$(r,m)(r',m') = (rr', rm' + mr') \qquad (r,r' \in R; m,m' \in M)$$

is a ring with the multiplicative identity $(1,0)$. It is useful to note that $T(R,M)$ has also the matrix representation

$$T(R,M) = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R, m \in M \right\}$$

with the usual addition and multiplication of matrices.

Trivial extensions have been extensively studied in algebra and analysis (see, for instance, [1, 6, 8, 10, 11]).

Let R and S be rings with identity, M be a unitary (R,S)-bimodule, and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the upper triangular matrix ring determined by R, S and M with the usual addition and multiplication of matrices. For the special case when $R = S$ (so that M is an R-bimodule), the matrix representation of $T(R,M)$ shows that it is a subring of $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. In particular, $T(R,R)$ is a subring of the upper triangular matrix ring $T_2(R)$.

Many authors have studied T in several directions. Characterizing its automorphisms, derivations and biderivations are a few among all. The readers interested in the structure of automorphisms and derivations of T may refer to [2–5, 9], and the readers interested in biderivations of T may refer to [3, 7, 12] and the references therein.

Let T be as above. Then one can easily verify that M can be made into a unitary $R \times S$-bimodule via the scalar multiplications given by

$$(1.2) \qquad (r,s)m = rm \quad \text{and} \quad m(r,s) = ms \qquad ((r,s) \in R \times S, m \in M).$$

Hence, $T(R \times S, M)$ is the trivial extension of $R \times S$ by M. Now, it is straightforward to show that the mapping

$$(1.3) \qquad T \to T(R \times S, M) \quad \text{given by} \quad \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mapsto ((r,s),m)$$

is a ring isomorphism. Therefore, upon the (natural) identification above, the upper triangular matrix rings are examples of trivial extensions. Since it turns out that the trivial extensions are easier to work with, one can prove a property for an upper triangular matrix ring via proving the same property for the corresponding trivial extension. In this paper, first we determine the structure of derivations and biderivations of $T(R,M)$ (Theorems 2.1 and 2.6), and then, following the procedure explained above, we determine the structure of derivations and biderivations of T (Theorems 2.4 and 2.11). Some other related results are also established.

In the sequel, unless there is a doubt of ambiguity, the zero elements of the rings and modules, zero subrings, zero submodules and zero functions are all denoted by 0. As usual, E_{ij} stands for the standard matrix unit.

2. Main results and proofs

Let R be a ring with identity, M be unitary R-bimodule and $T(R, M)$ be the trivial extension of R by M. Our first result describes the derivations of $T(R, M)$.

Theorem 2.1. *Let d be a derivation of the trivial extension $T(R, M)$. Then there exists*

(i) *a derivation δ of R,*
(ii) *a derivation $\gamma : R \to M$,*
(iii) *a bimodule homomorphism $f : M \to R$ satisfying*

$$mf(m') + f(m)m' = 0 \quad \textit{for all } m, m' \in M, \tag{2.1}$$

(iv) *and there exists a biadditive mapping g on M satisfying*

$$g(rm) = rg(m) + \delta(r)m \quad \textit{and} \quad g(mr) = g(m)r + m\delta(r), \tag{2.2}$$

for all $r \in R, m \in M$, such that

$$d(r, m) = (\delta(r) + f(m), \gamma(r) + g(m)) \quad \textit{for all } r \in R, m \in M.$$

Proof. Let $r \in R$ and set $d(r, 0) = (\delta(r), \gamma(r))$. Since d is additive, so are δ and γ. For any $r, r' \in R$ we have

$$\begin{aligned}
(\delta(rr'), \gamma(rr')) &= d(rr', 0) = d((r, 0)(r', 0)) \\
&= d(r, 0)(r', 0) + (r, 0)d(r', 0) \\
&= (\delta(r), \gamma(r))(r', 0) + (r, 0)(\delta(r'), \gamma(r')) \\
&= (\delta(r)r', \gamma(r)r') + (r\delta(r'), r\gamma(r')) \\
&= (\delta(r)r' + r\delta(r'), \gamma(r)r' + r\gamma(r')).
\end{aligned}$$

Therefore, the mappings $\delta : R \to R$ and $\gamma : R \to M$ are derivations. This proves (i) and (ii). To prove (iii) and (iv), let $m \in M$ be arbitrary and set $d(0, m) = (f(m), g(m))$. Obviously, f and g are additive. Let $r \in R$. Using (i) and (ii), we have

$$\begin{aligned}
(f(rm), g(rm)) &= d(0, rm) = d((r, 0)(0, m)) \\
&= d(r, 0)(0, m) + (r, 0)d(0, m) \\
&= (\delta(r), \gamma(r))(0, m) + (r, 0)(f(m), g(m)) \\
&= (0, \delta(r)m) + (rf(m), rg(m)) \\
&= (rf(m), rg(m) + \delta(r)m).
\end{aligned}$$

Replacing rm by mr in the above computation, we find also that

$$(f(mr), g(mr)) = (f(m)r, g(m)r + m\delta(r)) \quad \text{for all } r \in R, m \in M.$$

Thus, f is a bimodule homomorphism and g satisfies (2.2). To prove (2.1), let m, m' be in M. Then applying d to $(0,m)(0,m') = (0,0)$, we have

$$\begin{aligned}(0,0) &= d((0,m)(0,m'))\\ &= (f(m),g(m))(0,m') + (0,m)(f(m),g(m'))\\ &= (0,f(m)m') + (0,mf(m'))\\ &= (0,f(m)m' + mf(m')).\end{aligned}$$

Since d is additive, the last conclusion is obvious. □

Now, it is not hard to verify:

Corollary 2.2. *Let d and $T(R,M)$ be as above. Then d can be decomposed into the sum of three derivations D, Γ and F of $T(R,M)$ given by*

$$D(r,m) = (\delta(r), g(m)), \quad \Gamma(r,m) = (0,\gamma(r)), \quad F(r,m) = (f(m),0).$$

For the special case when $M = R$, the result is more interesting. The derivation D decomposes into the sum of two special derivations, and the derivation F takes a simple form:

Corollary 2.3. *In Corollary 2.2, assume that $M = R$. Then:*

(i) *D is decomposed into the sum of derivations D_1, D_2, where D_1 is induced by the derivation δ, and D_2 is the restriction of the inner derivation $I_{bE_{12}}$ of $T_2(R)$ to the subring $T(R,R)$ for some $b \in Z(R)$.*

(ii) *There exists a central element $a \in R$ such that $2a = 0$, and for every $(r,s) \in R \times R$, we have $F(r,s) = (as,0)$.*

In particular, if R is 2-torsionfree, then $F = 0$, and for every (r,s) in $R \times S$, we have

$$\begin{aligned}d(r,s) &= (\delta(r),\delta(s)) + (0,bs) + (0,\gamma(r))\\ &= D_1(r,s) + D_2(r,s) + \Gamma(r,s)\end{aligned}$$

Proof. (i) Put $g(1) = b$, and let $r \in R$. According to (2.2), we have

$$g(r) = g(r.1) = rg(1) + \delta(r) = rb + \delta(r);$$

$$g(r) = g(1.r) = g(1)r + \delta(r) = br + \delta(r).$$

These equations imply that $b \in Z(R)$ and that $g(r) = br + \delta(r)$. Therefore, for every $(r,s) \in R \times R$, we have

$$\begin{aligned}D(r,s) &= (\delta(r), g(s)) = (\delta(r), bs + \delta(s))\\ &= (\delta(r),\delta(s)) + (0,bs).\end{aligned}$$

Now, the mappings D_1, D_2 defined by

$$D_1(r,s) = (\delta(r),\delta(s)) \quad \text{and} \quad D_2(r,s) = (0,bs)$$

are easily seen to be derivations of $T(R,R)$. Note that D_1 is determined by applying δ to the entries of (r,s). On the other hand, since $b \in Z(R)$, for every $X = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \in T(R,R) \subseteq T_2(R)$, we have

$$I_{bE_{12}}(X) = XI_{bE_{12}} - I_{bE_{12}}X = bsE_{12}.$$

That is, D_2 is the restriction of the inner derivation $I_{bE_{12}}$ of $T_2(R)$ to the subring $T(R,R)$.
(ii) Put $f(1) = a$. In view of (2.1), we have $0 = 1.f(1) + f(1).1 = 2a$. Hence, since f is a bimodule homomorphism on R, for any $r \in R$, we have

$$ra = rf(1) = f(r.1) = f(1.r) = f(1)r = ar,$$

so that $a \in Z(R)$. Therefore, for any $(r,s) \in R \times R$, we have $F(r,s) = (sa, 0)$.

To prove the particular case, notice that from $2f(1) = 0$ and the torsion assumption on R it follows that $f(1) = 0$. So, for any $r \in R$, we get $f(r) = f(r.1) = rf(1) = 0$. □

Now, upon the identification given in (1.3) and using Theorem 2.1, we can (re)determine the structure of the derivations of the upper triangular matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ (see also [4, Theorem 2.2.1]):

Theorem 2.4. *Let d be a derivation of the upper triangular matrix ring T. Then there are derivations δ_1 of R, δ_2 of S, an additive mapping g on M satisfying*

(2.3) $g(rm) = rg(m) + \delta_1(r)m$ *and* $g(mr) = g(m)r + m\delta_2(r)$ $(r \in R, m \in M)$,

and an element $m^* \in M$, such that, for every $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in T$, we have

$$d\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_1(r) & rm^* - m^*s + g(m) \\ 0 & \delta_2(s) \end{pmatrix}.$$

Proof. Let us identify T with $T(R \times S, M)$. By Theorem 2.1, there are derivations δ of $R \times S$ and $\gamma : R \times S \to M$ such that, for every $(r,s) \in R \times S$, we have

$$d((r,s),0) = (\delta(r,s), \gamma(r,s)).$$

First, we claim that there are derivations δ_1 of R and δ_2 of S such that

$$\delta(r,s) = (\delta_1(r), \delta_2(s)) \quad \text{for all } r \in R, s \in S. \tag{2.4}$$

To see this, let $\delta(1,0) = (a,b)$. Then, from

$$\begin{aligned} (a,b) &= \delta((1,0)^2) = \delta(1,0)(1,0) + (1,0)\delta(1,0) \\ &= (a,0) + (a,0) = (2a,0) \end{aligned}$$

it follows that $\delta(1,0) = (0,0)$. Similarly, $\delta(0,1) = (0,0)$. Now let r be in R and assume that $\delta(r,0) = (\delta_1(r), \alpha(r))$. Since $\delta(1,0) = (0,0)$, we have

$$\begin{aligned}(\delta_1(r), \alpha(r)) &= \delta(r,0) = \delta((r,0)(1,0)) \\ &= \delta(r,0)(1,0) + (r,0)\delta(1,0) \\ &= (\delta_1(r), 0).\end{aligned}$$

Hence $\alpha(r) = 0$. Likewise, there exists a mapping δ_2 on S such that for every $s \in S, \delta(0,s) = (0, \delta_2(s))$. Consequently, for every $r \in R, s \in S$, we have

$$d(r,s) = d(r,0) + d(0,s) = (\delta_1(r), \delta_2(s)).$$

To see that δ_1, δ_2 are derivations, let (r', s') be also in $R \times S$. Then

$$\begin{aligned}(\delta_1(rr'), \delta_2(ss')) &= \delta(rr', ss') = \delta((r,s)(r',s')) \\ &= \delta(r,s)(r',s') + (r,s)\delta(r',s') \\ &= (\delta_1(r), \delta_2(s))(r',s') + (r,s)(\delta_1(r'), \delta_2(s')) \\ &= (\delta_1(r)r' + r\delta(r'), \delta_2(s)s' + s\delta_2(s')).\end{aligned}$$

Since δ is additive, so are δ_1, δ_2. Therefore, δ_1, δ_2 are derivations.

Now, we claim that there exists $m^* \in M$ such that

$$\gamma(r,s) = rm^* - m^*s \quad \text{for all } r \in R, s \in S. \tag{2.5}$$

Since γ is a derivation, we have $\gamma(1,1) = 0$. Put $\gamma(1,0) = m^*$. Then, in view of (1.2) and noting that $\gamma(0,1) = -\gamma(1,0) = -m^*$, for every $r \in R, s \in S$, we have

$$\gamma(r,0) = \gamma((r,0)(1,0)) = \gamma(r,0)(1,0) + (r,0)\gamma(1,0) = (r,0)m^* = rm^*;$$

$$\gamma(0,s) = \gamma((0,1)(0,s)) = \gamma(0,1)(0,s) + (0,1)\gamma(0,s) = -m^*(0,s) = -m^*s.$$

Hence, $\gamma(r,s) = \gamma(r,0) + \gamma(0,s) = rm^* - m^*s$, proving (2.5).

Now, by Theorem 2.1, there exists a bimodule homomorphism $f : M \to R \times S$, and an additive mapping $g : M \to M$ that satisfies the conditions in (2.3). First, we prove that $f = 0$: Let m be in M and assume that $f(m) = (u,v)$. Using (1.2) twice, we have

$$(u,v) = f(m) = f(m(0,1)) = f(m)(0,1) = (u,v)(0,1) = (0,v);$$

$$(u,v) = f(m) = f((1,0)m) = (1,0)f(m) = (1,0)(u,v) = (u,0).$$

The above equations imply of course that $u = v = 0$.

To show that g satisfies (6), let $r \in R$ and $m \in M$ be arbitrary. Noting that here M is an $R \times S$-bimodule, using (1.2), (2.3) and (2.4), we have

$$\begin{aligned}g(rm) &= g((r,0)m) = (r,0)g(m) + \delta(r,0)m \\ &= rg(m) + (\delta_1(r), \delta_2(0))m \\ &= rg(m) + \delta_1(r)m;\end{aligned}$$

$$\begin{aligned} g(mr) &= g(m(0,r)) = g(m)(0,r) + m\delta(0,r) \\ &= g(m)r + m(\delta_1(0), \delta_2(r)) \\ &= g(m)r + m\delta_2(r). \end{aligned}$$

Finally, by Theorem 2.1, the identification described in (1.3), and noting that $f = 0$, for any $r \in R,\ s \in S$ and $m \in M$, we have

$$\begin{aligned} d((r,s),m) &= (\delta(r,s) + f(m), \gamma(r,s) + g(m)) \\ &= ((\delta_1(r), \delta_2(s)), rm^* - m^*s + g(m)) \\ &= \begin{pmatrix} \delta_1(r) & rm^* - m^*s + g(m) \\ 0 & \delta_2(s) \end{pmatrix}. \end{aligned}$$

□

Remark 2.5. Let d and T be as above. Define $M^* = m^*E_{12} = d(E_{11})$, and let I_{M^*} be the inner derivation of T induced by M^*. Then for every $X = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ in T, we have

$$I_{M^*}(X) = Xm^*E_{12} - m^*E_{12}X = (rm^* - m^*s)E_{12}.$$

Therefore, the derivation d can be decomposed into the sum $d = \Delta + I_{M^*}$, where (the derivation) Δ is defined by

$$\Delta \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \delta_1(r) & g(m) \\ 0 & \delta_2(s) \end{pmatrix}.$$

Let R be a ring with identity. Recall that a biadditive mapping $d : R \times R \to R$ is said to be a biderivation of R if d is a derivation in each argument. Our next aim is to describe the biderivations of the trivial extension $T(R, M)$.

Theorem 2.6. *Let d be a biderivation of the trivial extension $T(R, M)$. Then there exist*

(i) *biderivations δ of R and $\gamma : R \times R \to M$,*

(ii) *a biadditive mapping $\alpha : R \times M \to R$ which is a derivation in the first coordinate and a bimodule homomorphism in the second coordinate,*

(iii) *a biadditive mapping $\beta : R \times M \to M$ which is a derivation in the first coordinate, and*

$$\beta(r, r'm) = r'\beta(r, m) + \delta(r, r')m, \tag{2.6}$$

$$\beta(r, mr') = \beta(r, m)r' + m\delta(r, r') \tag{2.7}$$

for all $r, r' \in R$ and $m \in M$,

(iv) *a biadditve mapping $\theta : R \times M \to R$ which is a derivation in the first coordinate and a bimodule homomorphism in the second coordinate,*

(v) *a biadditive mapping $\eta : R \times M \to M$ which is a derivation in the first coordinate, and*

$$\eta(r, r'm) = r'\eta(r, m) + \delta(r, r')m,$$

$$\eta(r, mr') = \eta(r, m)r' + m\delta(r, r')$$

for all $r, r' \in R$ and $m \in M$,

(vi) *a biadditive mapping $f : M \times M \to R$ which is a bimodule homomorphism in each coordinate, and finally,*

(vii) *there exists a biadditive mapping $g : M \times M \to M$ for which*

$$g(rm, m') = rg(m, m') + \alpha(r, m')m, \tag{2.8}$$

$$g(mr, m') = g(m, m')r + m\alpha(r, m'), \tag{2.9}$$

$$g(m, rm') = rg(m, m') + \theta(r, m)m', \tag{2.10}$$

$$g(m, m'r) = g(m, m')r + m'\theta(r, m), \tag{2.11}$$

are fulfilled for all $r, r' \in R$ and $m, m' \in M$, such that

$$d((r, m), (r', m')) = (\delta(r, r') + \alpha(r, m') + \theta(r', m) + f(m, m'),$$
$$\gamma(r, r') + \beta(r, m') + \eta(r', m) + g(m, m')).$$

Proof. To prove the conclusions involving biderivations, we deal only with the first coordinates. Let $r, r' \in R$, and let $d((r, 0), (r', 0)) = (\delta(r, r'), \gamma(r, r'))$. Since d is biadditive, so are δ and γ. For any $r_1, r_2, r' \in R$, we have

$$\begin{aligned}(\delta(r_1r_2, r'), \gamma(r_1r_2, r')) &= d((r_1r_2, 0), (r', 0)) = d((r_1, 0)(r_2, 0), (r', 0)) \\ &= (r_1, 0)d((r_2, 0), (r', 0)) + d((r_1, 0), (r', 0))(r_2, 0) \\ &= (r_1, 0)(\delta(r_2, r'), \gamma(r_2, r')) \\ &\quad + (\delta(r_1, r'), \gamma(r_1, r'))(r_2, 0) \\ &= (r_1\delta(r_2, r') + \delta(r_1, r')r_2, r_1\gamma(r_2, r') + \gamma(r_1, r')r_2).\end{aligned}$$

Therefore, δ and γ are derivations in the first coordinate. This proves (i).

To prove (ii) and (iii), let $r \in R, m \in M$ be arbitrary, and set

$$d((r, 0), (0, m)) = (\alpha(r, m), \beta(r, m)) \in R \times M.$$

Biadditivity of α and β follow from d. Let r' be also in R. Then, from

$$\begin{aligned}(\alpha(rr', m), \beta(rr', m)) &= d((rr', 0), (0, m)) = d((r, 0)(r', 0), (0, m)) \\ &= (r, 0)d((r', 0), (0, m)) + d((r, 0), (0, m))(r', 0) \\ &= (r, 0)(\alpha(r', m), \beta(r', m)) + (\alpha(r, m), \beta(r, m))(r', 0) \\ &= (r\alpha(r', m) + \alpha(r, m)r', r\beta(r', m) + \beta(r, m)r')\end{aligned}$$

it follows that α, β are derivations in the first coordinate. Moreover, using (i), we have

$$\begin{aligned}(\alpha(r, r'm), \beta(r, r'm)) &= d((r,0),(0,r'm)) = d((r,0),(r',0)(0,m)) \\ &= (r',0)(d(r,0),(0,m)) + d((r,0),(r',0))(0,m) \\ &= (r',0)(\alpha(r,m),\beta(r,m)) + (\delta(r,r'),\gamma(r,r'))(0,m) \\ &= (r'\alpha(r,m), r'\beta(r,m)) + (0, \delta(r,r')m) \\ &= (r'\alpha(r,m), r'\beta(r,m) + \delta(r,r')m).\end{aligned}$$

Hence, α is a left R-homomorphism in the second coordinate, and β satisfies (2.6). Similarly, one shows that α is also a right R-homomorphism in the second coordinate, and β satisfies (2.7).

The proofs of the existence of the mappings $\theta : R \times M \to R$ and $\eta : R \times M \to M$ satisfying the properties given in (iv) and (v) are similar to those of α and β in (ii) and (iii), hence suppressed.

To prove (vi) and (vii), let $m, m' \in M$ be arbitrary, and assume

$$d((0,m),(0,m')) = (f(m,m'), g(m,m')) \in R \times M.$$

Obviously, f and g are biadditive. Let r be in R. Then, from

$$\begin{aligned}(f(rm,m'), g(rm,m')) &= d((0,rm)(0,m')) = d((r,0)(0,m),(0,m')) \\ &= (r,0)(d(0,m),(0,m')) + d((r,0),(0,m'))(0,m) \\ &= (r,0)(f(m,m'),g(m,m')) \\ &\quad + (\alpha(r,m'),\beta(r,m'))(0,m) \\ &= (rf(m,m'), rg(m,m') + \alpha(r,m')m)\end{aligned}$$

it follows that $f(rm,m') = rf(m,m')$ and $g(rm,m') = rg(m,m') + \alpha(r,m')m$. In similar fashions, we can show that f is a right R-homomorphism in the first coordinate, and an R-bimodule homomorphism in the second coordinate; moreover, g satisfies the properties (2.9)-(2.11). Since d is biadditive, the last conclusion is now obvious. □

Remark 2.7. Let d and $T(R,M)$ be as above. Then the mappings D, Γ, F and $G : T(R,M) \times T(R,M) \to T(R,M)$ defined by

$$\begin{aligned}D((r,m),(r',m')) &= (\delta(r,r'), \beta(r,m') + \eta(r',m)), \\ \Gamma((r,m),(r',m')) &= (0, \gamma(r,r')), \\ F((r,m),(r',m')) &= (f(m,m'), 0), \\ G((r,m),(r',m')) &= (\alpha(r,m') + \theta(r',m), g(m,m')),\end{aligned}$$

are easily seen to be biderivations of $T(R,M)$, and $d = D+\Gamma+F+G$. (Compare with Theorem 2.11.)

Corollary 2.8. *In Theorem 2.6, assume that the ring R is 2-torsionfree and $M = R$. Then the mappings f, α and θ are identically zero, and for all $r, s \in R$, we have*

$$\beta(r,s) = s\beta(r,1) + \delta(r,s) = \beta(r,1)s + \delta(r,s), \quad \textit{and} \quad \beta(r,1) \in Z(R); \tag{2.12}$$

$$\eta(r,s) = s\eta(r,1) + \delta(r,s) = \eta(r,1)s + \delta(r,s), \quad \textit{and} \quad \eta(r,1) \in Z(R). \tag{2.13}$$

Moreover, there exists an element $a \in Z(R)$ such that, for every $r, s \in R$, we have

$$g(r,s) = ars, \quad \textit{and} \quad [r,s]a = 0. \tag{2.14}$$

Hence, $d = D + \Gamma + G$, in which, for every $(r,s), (r',s')$ in $R \times R$, we have

$$D((r,s),(r',s')) = (\delta(r,r'), s'\beta(r,1) + s\eta(r',1) + \delta(r,s') + \delta(s,r')),$$

$$\Gamma((r,s),(r',s')) = (0, \gamma(r,r')), \quad \textit{and} \quad G((r,s),(r',s')) = (0, ass').$$

Proof. Let $r, r_1, r_2 \in R$. Then $(0, r_1)(0, r_2) = (0,0)$, so that

$$\begin{aligned}(0,0) &= d((0,r_1)(0,r_2),(0,r)) \\ &= (0,r_1)d((0,r_2),(0,r)) + d((0,r_1),(o,r))(0,r_2) \\ &= (0,r_1)(f(r_2,r), g(r_2,r)) + (f(r_1,r), g(r_1,r))(0,r_2) \\ &= (0, r_1 f(r_2,r)) + (0, f(r_1,r)r_2) \\ &= (0, r_1 f(r_2,r) + f(r_1,r)r_2).\end{aligned}$$

Substituting $r = r_1 = r_2 = 1$ in the above equation, and using the torsion assumption on R, we get $f(1,1) = 0$, and thus

$$f(r,s) = rsf(1,1) = 0 \quad \text{for all } \ r, s \in R.$$

To see that $\alpha = 0$, let $r, r_1, r_2 \in R$. Noting that $(0, r_1)(0, r_2) = (0,0)$, we have

$$\begin{aligned}(0,0) &= d((r,0),(0,r_1)(0,r_2)) \\ &= (0,r_1)d((r,0),(0,r_2)) + d((r,0),(0,r_1))(0,r_2) \\ &= (0,r_1)(\alpha(r,r_2), \beta(r,r_2)) + (\alpha(r,r_1), \beta(r,r_1))(0,r_2) \\ &= (0, r_1\alpha(r,r_2)) + (0, \alpha(r,r_1)r_2) \\ &= (0, r_1\alpha(r,r_2) + \alpha(r,r_1)r_2).\end{aligned}$$

Substituting $r_1 = r_2 = 1$ in the above equation, leads to $2\alpha(r,1) = 0$, and thus $\alpha(r,1) = 0$. Hence, using the fact that α is a bimodule homomorphism in the second coordinate, for every $r, s \in R$, we have

$$\alpha(r,s) = \alpha(r, s.1) = s\alpha(r,1) = 0.$$

A similar argument shows that $\theta = 0$.

Next, we prove (2.14). Relations (2.8)-(2.11) and the facts that α and θ

are zero mappings show that the mapping $g : R \times R \to R$ is merely a bimodule homomorphism in each component. Therefore, for any $r \in R$, we have $rg(1,1) = g(r,1) = g(1,1)r$, so that $a := g(1,1) \in Z(R)$. Accordingly, for every $r, s \in R$, we have

$$rsa = rsg(1,1) = g(r,s) = srg(1,1) = sra,$$

implying that $g(r,s) = ars$ and $[r,s]a = 0$.

Now we prove (2.12). Let r, s be in R. In view of (2.6) and (2.7), we have

$$\beta(r,s) = \beta(r, s.1) = s\beta(r,1) + \delta(r,s);$$

$$\beta(r,s) = \beta(r, 1.s) = \beta(r,1)s + \delta(r,s).$$

These equations imply that $\beta(r,1)s = s\beta(r,1)$, whence $\beta(r,1) \in Z(R)$, as desired. The proof of (2.13) is similar, hence omitted. The remaining parts are easily verified. □

For the upper triangular ring T, let us identify $T \times T$ with $T(R \times S, M) \times T(R \times S, M)$. Using Theorem 2.6 and Remark 2.7, we can determine the structure of the biderivations of T. First, let us apply Theorem 2.6 to the case when the trivial extension $T(R, M)$ is replaced by $T(R \times S, M)$, and see what happen to the mappings $\delta, \gamma, f, \alpha, \theta, \beta, \eta$ and g:

Theorem 2.9. *Let d be a biderivation of the trivial extension $T(R \times S, M)$. Then:*

(i) *There exist biderivations δ_1 of R and δ_2 of S such that, for every $r, r' \in R$ and $s, s' \in S$, we have*

$$\delta((r,s),(r',s')) = (\delta_1(r,r'), \delta_2(s,s')). \tag{2.15}$$

(ii) *There exists an element $m^* \in M$ such that, for every $r, r' \in R$ and $s, s' \in S$, we have*

$$\gamma((r,s),(r',s')) = rr'm^* + m^*ss' - rm^*s' - r'm^*s, \tag{2.16}$$

$$[r,r']m^* = 0 = m^*[s,s']. \tag{2.17}$$

(iii) *The mappings f, α and θ are identically zero.*

(iv) *There exists an (R,S)-bimodule homomorphism h on M such that, for every $(r,s) \in R \times S, r' \in R$ and $m \in M$, we have*

$$\beta((r,s),m)) = rh(m) - h(m)s, \quad \textit{and} \quad [r,r']h(m) = \delta_1(r,r')m. \tag{2.18}$$

(v) *There exists an (R,S)-bimodule homomorphism k on M such that, for every $(r,s) \in R \times S, r' \in R$ and $m \in M$, we have*

$$\eta(((r,s),m) = rk(m) - k(m)s, \quad \textit{and} \quad k(m)[s,s'] = m\delta_2(s',s).$$

(vi) *The biadditive map* $g : M\times M \to M$ *is an* (R,S)*-bimodule homomorphism in each component, and, for every* $(r,s),(r',s')$ *in* $R\times S$ *and* m,m' *in* M*, we have*

$$[r,r']g(m,m') = 0 = g(m,m')[s,s'].$$

Before proving the theorem, let us mention some trivial facts about a biderivation $d : R \times R \to M$, where R is a ring with identity and M is a unitary R-bimodule:

$$d(x,1) = 0 = d(1,y) \quad \text{and} \quad d(x,0) = 0 = d(0,y) \quad \text{for all} \quad x,y \in R. \tag{2.19}$$

The first and the third identities follow easily from applying d to $(x,1) = (x,1)^2$ and $(x,0) = (x,0)+(x,0)$, respectively, and the two others are proved similarly.

Proof of Theorem 2.9. (i) Applying the biderivation δ to the equation

$$(r,s),(1,0)) = ((r,s),(1,0)^2),$$

we conclude that

$$\delta((r,s),(1,0)) = 0 \quad \text{for all} \quad (r,s) \in R\times S. \tag{2.20}$$

Likewise, for every $(r,s)\in R\times S$, we have

$$\delta((r,s),(0,1)) = \delta((1,0),(r,s)) = \delta((0,1),(r,s)) = 0. \tag{2.21}$$

Now, we claim that there exists a biderivation δ_1 of R such that, for every $r,r'\in R$, we have

$$\delta((r,0),(r',0)) = (\delta_1(r,r'),0). \tag{2.22}$$

Set $\delta((r,0),(r',0)) = (u,v)$. In view of (2.20), we have

$$\begin{aligned}(u,v) &= \delta((r,0),(r',0)) = \delta((r,0),(r',0)(1,0))\\ &= (r',0)\delta((r,0),(1,0)) + (\delta(r,0),(r',0))(1,0)\\ &= (u,v)(1,0) = (u,0),\end{aligned}$$

so that $v = 0$. Therefore, $\delta((r,0),(r',0)) = (u,0) =: (\delta_1(r,r'),0)$. Obviously, δ_1 is biadditive. Let $r_1,r_2,r'\in R$. We have

$$\begin{aligned}(\delta_1(r_1r_2,r),0) &= \delta((r_1r_2,0),(r,0))\\ &= \delta((r_1,0)(r_2,0),(r,0))\\ &= (r_1,0)\delta((r_2,0),(r,0)) + \delta((r_1,0),(r,0))(r_2,0)\\ &= (r_1,0)(\delta_1(r_2,r),0) + (\delta(r_1,r),0)(r_2,0)\\ &= (r_1\delta_1(r_2,r) + \delta_1(r_1,r)r_2,0).\end{aligned}$$

Thus, δ_1 is a derivation in the first coordinate. Similarly, δ_1 is a derivation in the second coordinate, so that δ_1 is a biderivation of R. By an analogue computation one can show that there exists a biderivation δ_2 of S such that, for each pair $s, s' \in S$, we have

$$\delta((0,s),(0,s')) = (0,\delta_2(s,s')). \tag{2.23}$$

Next, we claim that

$$\delta((r,0),(0,s)) = 0 = \delta((0,s),(r,0)) \quad \text{for all } r \in R, s \in S. \tag{2.24}$$

Assume that $\delta((r,0),(0,s)) = (t,z) \in R \times S$. Then, by (2.21), we have

$$\begin{aligned}(t,z) = \delta((r,0),(0,s)) &= \delta((r,0)(1,0),(0,s)) \\ &= (r,0)\delta((1,0),(0,s)) + \delta((r,0),(0,s))(1,0) \\ &= (t,z)(1,0) = (t,0).\end{aligned}$$

Hence $z = 0$. On the other hand, by applying δ to

$$((r,0),(0,s)) = ((r,0),(0,s)(0,1)),$$

we find that $t = 0$, and thus, $\delta((r,0),(0,s)) = 0$. The other identity in (2.24) is proved similarly.

Finally, since δ is biadditive, from (2.22)-(2.24) one obtains (2.15).

(ii) Recalling that $\gamma : (R \times S) \times (R \times S) \to M$ is a biderivation, from (2.19) we find that

$$\gamma((1,0),(1,1)) = \gamma((0,1),(1,1)) = \gamma((1,1),(1,0)) = \gamma((1,1),(1,1)) = 0. \tag{2.25}$$

Assume that $\gamma((1,0),(1,0)) = m^* \in M$. Using(2.25), we obtain

$$\gamma((1,0),(0,1)) = \gamma((0,1),(1,0)) = -m^* \quad \text{and} \quad \gamma((0,1),(0,1)) = m^*. \tag{2.26}$$

Let $r \in R$ be arbitrary, and put $\gamma((r,0),(0,1)) = m$. Applying γ to the identity $((r,0),(1,0)) = ((r,0)(1,0),(1,0))$, and using relations (1.2) and (2.26), we arrive at $m = rm^*$. A similar computation shows also that $\gamma((1,0),(r,0)) = rm^*$. Now, applying γ to the identities

$$\gamma((r,0),(r',0)) = \gamma((r,0),(r',0)(1,0));$$

$$\gamma((r,0),(r',0)) = \gamma((r,0)(1,0),(r',0)),$$

and using (1.2) and (2.26) again, we deduce that, for each pair $r, r' \in R$, we have

$$\gamma((r,0),(r',0)) = rr'm^* = r'rm^*, \quad \text{and} \quad [r,r']m^* = 0. \tag{2.27}$$

Analogue computation shows that, for every $r \in R$ and $s, s' \in S$, we have

$$\gamma((r,0),(0,s)) = -rm^*s = \gamma((0,s),(r,0)); \tag{2.28}$$

$$\gamma((0,s),(0,s')) = m^*ss' = m^*s's, \quad \text{and} \quad m^*[s,s'] = 0. \tag{2.29}$$

Now, using the biadditivity of γ and Equations (2.27)-(2.29), we obtain (2.16) and (2.17). This proves (ii).

(iii) Recall that $f : M \times M \to R \times S$ is an $R \times S$-bimodule homomorphism in both arguments. Let $m, m' \in M$ and put $f(m, m') = (r, s)$. Then, from

$$\begin{aligned}(r, s) &= f(m, m') = f((1, 0)m(0, 1), m') \\ &= (1, 0)f(m, m')(0, 1) \\ &= (1, 0)(r, s)(0, 1) \\ &= (0, 0)\end{aligned}$$

we infer that $f = 0$. Moreover, since $\alpha : (R \times S) \times M \to R \times S$ and $\beta : (R\times S)\times M \to M$ are $R\times S$-bimodule homomorphisms in the second argument, we conclude also that $\alpha = \beta = 0$.
(iv) Recalling that $\beta : (R\times S)\times M \to M$ is a derivation in the first component, using (1.2), for every $r \in R, m \in M$, we have

$$\begin{aligned}\beta((r, 0), m)) &= \beta((r, 0)(1, 0), m) \\ &= (r, 0)\beta((1, 0), m) + \beta((r, 0), m)(1, 0) \\ &= r(\beta(1, 0), m).\end{aligned} \tag{2.30}$$

Similarly, by applying β to the identity $((0, s), m) = ((0, 1)(0, s), m)$, we observe that

$$\beta((0, s), m)) = \beta((0, 1), m)s \quad \text{for all} \quad s \in S, m \in M. \tag{2.31}$$

Set $\beta((1, 0), m) = h(m)$. Since $\beta((1, 1), m) = 0$ (β is a derivation in the first component), we get $\beta((0, 1), m) = -\beta((1, 0), m) = -h(m)$. Now, using (2.30) and (2.31), we conclude that, for every $r \in R, s \in S$ and $m \in M$, we have

$$\beta((r, 0), m) = rh(m), \quad \text{and} \quad \beta((0, s), m) = -h(m)s. \tag{2.32}$$

Since β is additive on $R \times S$, the latter relations imply that

$$\beta((r, s), m) = \beta((r, 0), m) + \beta((0, s), m) = rh(m) - h(m)s,$$

as desired.

Next, we show that h is a left R-homomorphism on M. Let $r \in R$ and $m \in M$. Then, since β is a left $R \times S$-homomorphism on M, in view of (1.2), we have

$$\begin{aligned}h(rm) &= \beta((1, 0), rm) = \beta((1, 0),(r, 0)m) \\ &= (r, 0)\beta((1, 0), m) \\ &= (r, 0)h(m) = rh(m).\end{aligned}$$

A similar argument shows that h is also a right S- homomorphism. To prove the second identity in (2.18), let $r, r' \in R$ and $m \in M$. By (1.2), (2.6), (2.15) and (2.32), we have

$$\begin{aligned} rr'h(m) &= \beta((r,0),r'm) = \beta((r,0),(r',0)m) \\ &= (r',0)\beta((r,0),m) + \delta((r,0),(r',0))m \\ &= r'rh(m) + (\delta_1(r,r'),\delta_2(0,0))m \\ &= r'rh(m) + \delta_1(r,r')m, \end{aligned}$$

so that $[r,r']h(m) = \delta_1(r,r')m$. The proof of (v) is similar, hence supressed.
(vi) Since α and θ are both zero functions, by Theorem 2.6(vii), the mapping $g : M \times M \to M$ is an $R \times S$-bimodule homomorphism in both arguments. Hence, for every $r, r' \in R$ and $m, m' \in M$, we get

$$g(rm, r'm') = rg(m, r'm') = rr'g(m, m');$$

$$g(rm, r'm') = r'g(rm, m') = r'rg(m, m').$$

Thus, $[r, r']g(m, m') = 0$. Similarly, for every $s, s' \in S$ and $m, m' \in M$, we obtain $g(m, m')[s, s'] = 0$. This completes the proof of the theorem. □

Corollary 2.10. *Let d and $T((R \times S), M)$ be as above. Then, for every $X = ((r,s),m), Y = ((r',s'),m')$ in $T((R \times S), M)$, the biderivations D, Γ and G defined in Remark 2.7, can be expressed as*

$$\begin{aligned} D(X,Y) &= ((\delta_1(r,r'),\delta_2(s,s')), rh(m') - h(m')s + r'k(m) - k(m)s'), \\ \Gamma(X,Y) &= (0, rr'm^* + m^*ss' - rm^*s' - r'm^*s), \\ G(X,Y) &= (0, g(m,m')). \end{aligned}$$

Moreover,

$$d = D + \Gamma + G, \quad D(((1,0),0),((1,0),0)) = 0,$$

and Γ is the extremal biderivation of $T((R \times S), M)$ determined by $A = ((0,0), m^*)$.

Proof. By part (iii) of Theorem 2.9, and in view of Remark 2.7, $F = 0$, so that $d = D + \Gamma + G$. The expressions for D and G are obvious, and clearly $D(((1,0),0),((1,0),0)) = 0$. To see that Γ is an extremal biderivation, note that, in view of (2.17), we have $[A,[X,Y]] = 0$, and a simple computation shows that

$$\Gamma(X,Y) = (0, rr'm^* + m^*ss' - rm^*s' - r'm^*s) = [X,[Y,A]].$$

□

Now, we can easily translate the corollary above to describe the biderivations of the triangular ring T (see also [3, Theorem 4.11] and [7, Theorem 2.4]:

Theorem 2.11. *Let d be a biderivation of the triangular ring T. Then d can be decomposed into the sum of three biderivations D, Γ and G such that $D(E_{11}, E_{11}) = 0, \Gamma$ is an extremal biderivation, and G is a special biderivation.*

Remark 2.12. Note that by Corollary 2.10, for every $X = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ and $Y = \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}$ in T, the biderivations D, Γ and G are given by

$$D(X,Y) = \begin{pmatrix} \delta_1(r,r') & rh(m') - h(m')s + r'k(m) - k(m)s' \\ 0 & \delta_2(s,s') \end{pmatrix},$$

$$\Gamma(X,Y) = (rr'm^* + m^*ss' - rm^*s' - r'm^*s)E_{12} = [X,[Y,A]],$$

where $A = m^*E_{12}$, and

$$G(X,Y) = g(m,m')E_{12}.$$

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