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TWO-GEODESIC TRANSITIVE GRAPHS OF PRIME POWER ORDER

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Abstract. In a non-complete graph Γ, a vertex triple (u,v,w) with v adjacent to both u and w is called a 2-*geodesic* if $u \neq w$ and u,w are not adjacent. The graph Γ is said to be 2-*geodesic transitive* if its automorphism group is transitive on arcs, and also on 2-geodesics. We first produce a reduction theorem for the family of 2-geodesic transitive graphs of prime power order. Next, we classify such graphs which are also vertex quasiprimitive.
Keywords: 2-geodesic transitive graph, 2-arc transitive graph, automorphism group.
MSC(2010): Primary: 05E18; Secondary: 20B25.

1. Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set and automorphism group, respectively. A *geodesic* from a vertex u to a vertex v in a graph Γ is one of the shortest paths from u to v, and this geodesic is called an s-geodesic if the distance between u and v is s. Let $G \leq \mathrm{Aut}(\Gamma)$. A non-complete graph Γ is said to be (G,s)-*geodesic transitive* if, for each $i \leq s$, G is transitive on all i-geodesics of Γ. An *arc* is an ordered pair of adjacent vertices. A vertex triple (u,v,w) with v adjacent to both u and w is called a 2-*arc* whenever $u \neq w$. A graph Γ is said to be G-*arc transitive* if G is transitive on arcs of Γ; further, if G is also transitive on 2-arcs of Γ, then it is called a $(G,2)$-*arc transitive graph*. Moreover, in the previous definitions, if $G = \mathrm{Aut}(\Gamma)$, then G is often omitted and we write simply s-*geodesic transitive*, etc. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. Thus the family of non-complete 2-arc transitive graphs is

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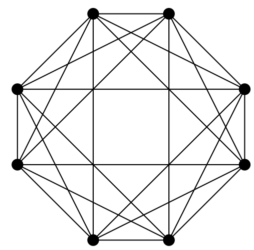

FIGURE 1. $K_{4[2]}$

properly contained in the family of 2-geodesic transitive graphs. The graph in Figure 1 is 2-geodesic transitive but not 2-arc transitive with 8 vertices.

The study of symmetric graphs forms a significant part of current research efforts in algebraic graph theory. The family of 2-arc transitive graphs has been studied intensively, beginning with the seminal result of Tutte [27, 28] that cubic s-arc transitive graphs must have $s \leq 5$; for more work see [1, 2, 15, 18, 20–22, 26, 29, 30].

In [7], Devillers, Li, Praeger and the author determined the structure of $[\Gamma(u)]$ (the induced subgraph on $\Gamma(u)$ of vertices adjacent to vertex u) for any 2-geodesic transitive graph Γ. Later, they classified the tetravalent and prime valency connected 2-geodesic transitive graphs in [8] and [10], respectively. After that, in [9], a reduction theorem for the family of normal 2-geodesic transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. This reduction result reduces the studying of normal 2-geodesic transitive Cayley graphs to finding all examples where automorphism group acts quasiprimitively on vertices and then studying their covers.

A transitive permutation group G is said to be *quasiprimitive*, if every nontrivial normal subgroup of G is transitive. This is a generalization of primitivity as every normal subgroup of a primitive group is transitive, but there exist quasiprimitive groups which are not primitive. For knowledge of quasiprimitive permutation groups, see [22] and [24]. Praeger [22] generalized the O'Nan-Scott Theorem for primitive groups to quasiprimitive groups and showed that a finite quasiprimitive group is one of eight distinct types: Holomorph Affine (HA), Almost Simple (AS), Twisted Wreath product (TW), Product Action (PA), Simple Diagonal (SD), Holomorph Simple (HS), Holomorph Compound (HC) and Compound Diagonal (CD).

Let Γ be a 2-geodesic transitive graph. Note that, if $\mathrm{Aut}(\Gamma)$ acts quasiprimitively of type HA on $V(\Gamma)$, then the socle N of $\mathrm{Aut}(\Gamma)$ is regular on $V(\Gamma)$ and $N \cong \mathbb{Z}_p^r$ for some prime p. Thus Γ has p^r vertices. This observation inspired us to study 2-geodesic transitive graphs with prime power vertices.

For two integers $m \geq 3$ and $b \geq 2$, let $\mathrm{K}_{m[b]}$ denote the *complete multipartite graph*, whose vertex set consisting of m parts of size b, with edges between all pairs of vertices from distinct parts. Let G be a group of permutations acting on the vertex set Ω of a graph Γ. Let N be an intransitive subgroup of G and let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be the set of N-orbits in Ω. Then the *quotient graph* Γ_N of Γ is the graph with vertex set $\mathcal{B}$ such that $\{B_i, B_j\}$ is an edge of Γ_N if and only if there exist $x \in B_i$ and $y \in B_j$ such that $\{x, y\}$ is an edge of Γ. The graph Γ is said to be a *cover* of Γ_N if for each edge $\{B_i, B_j\}$ of Γ_N and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$.

We first produce a reduction theorem.

Theorem 1.1. *Let Γ be a 2-geodesic transitive graph of order p^r where p is a prime. Then one of the following holds.*

(1) *Γ is 2-arc transitive.*

(2) *$\Gamma \cong \mathrm{K}_{p^i[p^j]}$ where $i + j = r$.*

(3) *There exists a nontrivial normal subgroup N of $A := \mathrm{Aut}(\Gamma)$ such that Γ is a cover of Γ_N which is a complete A/N-arc transitive graph with order equal to a power of p.*

(4) *There exists a nontrivial normal subgroup N of $A := \mathrm{Aut}(\Gamma)$ such that Γ is a cover of Γ_N which is $(A/N, 2)$-geodesic transitive of girth 3 with order p-power, and A/N is quasiprimitive on $V(\Gamma_N)$.*

Then 2-geodesic transitive graphs in Theorem 1.1(1) are 2-arc transitive. Such graphs have been studied extensively, see [1, 2, 15, 18, 20, 22, 27, 28, 30]. Theorem 1.1 points out that the study of 2-geodesic transitive but not 2-arc transitive graphs of prime power order reduces to the following three problems: investigating the case that such graphs which are vertex quasiprimitive, studying the 2-geodesic transitive covers of these graphs, and investigating the 2-geodesic transitive covers of complete graphs of prime power order.

We next study 'basic' 2-geodesic transitive graphs with prime power number of vertices, that is we suppose that $\mathrm{Aut}(\Gamma)$ is quasiprimitive on the vertex set. Our second theorem determines all the possible quasiprimitive action types.

For a finite group T, and a subset S of T such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\Gamma := \mathrm{Cay}(T, S)$ of T with respect to S is the graph with vertex set T and edge set $\{\{g, sg\} \mid g \in T, s \in S\}$. In particular, Γ is connected if and only if $T = \langle S \rangle$. The group $R(T) = \{\sigma_t | t \in T\}$ of right translations $\sigma_t : x \mapsto xt$ is a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ and acts regularly on the vertex set. We may identify T with $R(T)$. Godsil [12, Lemma 2.1] observed that $N_{\mathrm{Aut}(\Gamma)}(T) = T : \mathrm{Aut}(T, S)$ where $\mathrm{Aut}(T, S) = \{\sigma \in \mathrm{Aut}(T) | S^\sigma = S\}$. The family of Cayley graphs Γ such that $N_{\mathrm{Aut}(\Gamma)}(T) = \mathrm{Aut}(\Gamma)$ are called *normal Cayley graphs*, and they have been studied under various additional conditions, see [11, 23, 31].

In a later section, a particular well-known graph will play an important role and we define it here. The *Schläfli graph* is the graph on isotropic lines in the $U(4,2)$ geometry and adjacent when disjoint. It is the unique strongly regular graph with parameters $(27,16,10,8)$, and its automorphism group is $U(4,2).\mathbb{Z}_2$. The complement of the Schläfli graph is the collinearity graph of the unique generalized quadrangle $GQ(2,4)$, refer to [3] or [4].

Theorem 1.2. *Let* Γ *be a* 2*-geodesic transitive but not* 2*-arc transitive graph of order* p^r *where* p *is a prime number. Suppose that* $\mathrm{Aut}(\Gamma)$ *acts quasiprimitively on* $V(\Gamma)$ *with a minimal normal subgroup* N*. Then* $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$*,* $N \cong T^i$ *where* $T \cong \mathbb{Z}_p$ *or* T *is listed in Lemma 3.1, and one of the following holds.*

(1) $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$ *of* AS *type and* Γ *is the Schläfli graph or its complement.*

(2) $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$ *of* PA *type and* Γ *is the Hamming graph* $\mathrm{H}(s,p^t)$ *where* $st=r$*.*

(3) $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$ *of* HA *type,* Γ *is a normal Cayley graph* $\mathrm{Cay}(N,S)$*, and* $\langle a\rangle \setminus \{1\} \subset S$ *for each* $a \in S$*.*

We give some examples for Theorem 1.2(3).

Example 1.3. (1) Let $n \geq 2$ and let d be a positive integer. Then the *Hamming graph* $\mathrm{H}(d,n)$ has vertex set $\mathbb{Z}_n^d = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$, seen as a module on the ring $\mathbb{Z}_n = [0, n-1]$, and two vertices u, v are adjacent if and only if $u-v$ has exactly one non-zero entry. Let $\Gamma = \mathrm{H}(d,q)$ where $d \geq 2$, q is a prime power. Then Γ has order q^d, and is locally isomorphic to $d\,\mathrm{K}_{q-1}$. By [17, Proposition 2.2], Γ is 2-geodesic transitive. If $q \in \{3,4\}$, then $\mathrm{Aut}(\Gamma)$ acts primitively of type HA on $V(\Gamma)$.

(2) Let $T = \langle a_1, \ldots, a_d\rangle \cong \mathbb{Z}_3^d$ and $S = \bigcup(\langle a_1\rangle \setminus \{1\})$. Then $\mathrm{H}(d,3) \cong \mathrm{Cay}(T,S)$ is normal $(A,2)$-geodesic transitive where $A = T : \mathrm{Aut}(T,S) \cong S_3 \wr S_d$.

(3) Let $T = \langle a_1, \ldots, a_i, b_1, \ldots, b_i\rangle \cong \mathbb{Z}_2^d$ where $d = 2i$, $i \geq 1$. Let $S = S_a \cup S_b$ where $S_a = \langle a_1, \ldots, a_i\rangle \setminus \{1\}$ and $S_b = \langle b_1, \ldots, b_i\rangle \setminus \{1\}$. Then $\Gamma = \mathrm{Cay}(T,S)$ is a normal $(G,2)$-geodesic transitive Cayley graph where $G = T : \mathrm{Aut}(T,S)$. Further, Γ is a graph of girth 3 and diameter 2, and G acts primitively of type HA on $V(\Gamma)$. In particular, if $i = 2$, then $\Gamma \cong \mathrm{H}(d,4)$.

For a 2-geodesic transitive but not 2-arc transitive graph Γ of prime power order, if $\mathrm{Aut}(\Gamma)$ is quasiprimitive on $V(\Gamma)$, then Theorem 1.2 shows that the quasiprimitive action type is one of AS, PA, or HA. If further $\mathrm{Aut}(\Gamma)$ is quasiprimitive on $V(\Gamma)$ of type AS, then Γ is the Schläfli graph or its complement; if $\mathrm{Aut}(\Gamma)$ is quasiprimitive on $V(\Gamma)$ of type PA, then Γ is a Hamming graph. For Γ with vertex quasiprimitive action type HA, we don't konw much at the moment. To finish the classification of such 'basic' graphs, we pose the following problem.

Problem 1.4. Let Γ be a 2-geodesic transitive graph of prime power order which is not 2-arc transitive. Classify such graphs where $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ of type HA.

2. **Reduction**

In this section, we prove Theorem 1.1, that is, produce a reduction result for the family of 2-geodesic transitive but not 2-arc transitive graphs of prime power order.

A graph Γ is said to be *s-distance transitive* if, for any two pairs of vertices (u_1, v_1) and (u_2, v_2) with the same distance $t \leq s$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that $(u_1, v_1)^g = (u_2, v_2)$. In particular, Γ is s-geodesic transitive implies that it is s-distance transitive.

Lemma 2.1. *Let Γ be a 2-geodesic transitive graph of girth 3 and order p^r where p is a prime number and r is a positive number. Suppose that a nontrivial normal subgroup N of $\mathrm{Aut}(\Gamma)$ is intransitive on $V(\Gamma)$. If there exist vertices u and v in the same N-orbit such that the distance between u and v is 2, then $\Gamma \cong \mathrm{K}_{p^m[p^n]}$ where $p^m \geq 3, p^n \geq 2$, $m + n = r$, and $\Gamma_N \cong \mathrm{K}_{p^m}$.*

Proof. Suppose that N is a nontrivial normal subgroup of $\mathrm{Aut}(\Gamma)$ and is intransitive on $V(\Gamma)$. Let u and v be two vertices of Γ. Suppose that u and v are in the same N-orbit and the distance between u and v is 2. Since Γ is 2-geodesic transitive, Γ is non-complete 2-distance transitive and so [6, Lemma 5.2] holds. Since Γ is arc transitive and N is a normal subgroup of $\mathrm{Aut}(\Gamma)$, it follows that every N-orbit contains no edges of Γ. Note that N is not transitive on $V(\Gamma)$ and Γ has girth 3, N has at least 3 orbits on $V(\Gamma)$. Since the distance between u and v is 2 and u and v lie in the same N-orbit, it follows that only the case (iii) of [6, Lemma 5.2] holds, and so $\Gamma \cong \mathrm{K}_{i[t]}$ for some $i \geq 3, t \geq 2$, and $\Gamma_N \cong \mathrm{K}_i$. Finally, as $|V(\Gamma)| = p^r$ and p is a prime number, it follows that $\Gamma \cong \mathrm{K}_{p^m[p^n]}$ where $p^m \geq 3, p^n \geq 2$, $m + n = r$, and $\Gamma_N \cong \mathrm{K}_{p^m}$. □

Lemma 2.2. *Let Γ be a 2-geodesic transitive graph of girth 3 and order p^r where p is a prime number and r is a positive integer. Let N be a nontrivial normal subgroup of $A := \mathrm{Aut}(\Gamma)$. Suppose that N is intransitive on $V(\Gamma)$ and Γ is a cover of Γ_N. Then Γ_N has girth 3 and order p^i where $i < r$, and either Γ_N is complete A/N-arc transitive or Γ_N is non-complete $(A/N, 2)$-geodesic transitive and N is semiregular on $V(\Gamma)$.*

Proof. Since N is a nontrivial normal subgroup of A, it follows that each N-orbit is a nontrivial block of A of size p^j for some $j < r$. Hence Γ_N has order p^{r-j} and $r - j < r$.

Let $\mathcal{B}$ be the set of N-orbits on $V(\Gamma)$. Since Γ has girth 3 and each N does not contain edges of Γ, it follows that $|\mathcal{B}| \geq 3$. Let (u, v, w, u) be a triangle of Γ. Then u, v and w pairwise lie in distinct N-orbits, and so Γ_N has girth 3.

First suppose that Γ_N is a complete graph. Let (B_0, B_1) and (C_0, C_1) be two arcs of Γ_N. Since Γ is a cover of Γ_N, there exist $x_i \in B_i$ and $y_i \in C_i$ such that (x_0, x_1) and (y_0, y_1) are two arcs of Γ. As Γ is arc transitive, there exists $g \in \mathrm{Aut}(\Gamma)$ such that $(x_0, x_1)^g = (y_0, y_1)$, and hence $(B_0, B_1)^g = (C_0, C_1)$. In particular, $g \in A/N$. Thus Γ_N is A/N-arc transitive.

Next, suppose that Γ_N is non-complete. Then by Lemma 2.1, the distance between any pair of vertices of each N-orbit is greater than 2. Since Γ is 2-geodesic transitive, it is 2-distance transitive, and so [6, Lemma 5.3] is applicable. Since any pair of vertices of each N-orbit is greater than 2, it follows that only the case (iv) of [6, Lemma 5.3] holds, so N is semiregular on $V(\Gamma)$. Let (B_0, B_1, B_2) and (C_0, C_1, C_2) be two 2-geodesics of Γ_N. Since Γ is a cover of Γ_N, there exist $x_i \in B_i$ and $y_i \in C_i$ such that (x_0, x_1, x_2) and (y_0, y_1, y_2) are two 2-geodesics of Γ. As Γ is 2-geodesic transitive, there exists $g \in A$ such that $(x_0, x_1, x_2)^g = (y_0, y_1, y_2)$, and hence $(B_0, B_1, B_2)^g = (C_0, C_1, C_2)$. Note that $g \in A/N$, and so Γ_N is $(A/N, 2)$-geodesic transitive. □

We are ready to prove our first theorem. The *diameter* $\mathrm{diam}(\Gamma)$ of a graph Γ is the maximum distance between two vertices in Γ.

Proof of Theorem 1.1. If Γ is 2-arc transitive, then (1) holds. In the remainder of this proof, we assume that Γ is not 2-arc transitive, and so Γ has girth 3. Suppose that $A := \mathrm{Aut}(\Gamma)$ is not quasiprimitive on $V(\Gamma)$. Then A has a nontrivial normal subgroup N that is intransitive on $V(\Gamma)$. Choosing the maximal such N such that for any $N < M \lhd G$, M is transitive on $V(\Gamma)$.

Since Γ is 2-geodesic transitive, it is 2-distance transitive. It follows that [6, Lemma 5.3] is applicable. Since Γ has girth 3, Γ is not bipartite, so (iii) or (iv) of [6, Lemma 5.3] holds. If [6, Lemma 5.3(iii)] occurs, then $\Gamma \cong \mathrm{K}_{p^m[p^n]}$ for some $p^m \geq 3, p^n \geq 2$, $m + n = r$, and $\Gamma_N \cong \mathrm{K}_{p^m}$. Therefore (2) holds.

Now, suppose that [6, Lemma 5.3(iv)] holds. Then N is semiregular on $V(\Gamma)$, Γ is a cover of Γ_N and $|V(\Gamma_N)| < |V(\Gamma)|$. By Lemma 2.2, Γ_N is $(A/N, s)$-geodesic transitive where $s = \min\{2, \mathrm{diam}(\Gamma_N)\}$. Since for any $N < M \lhd A$, M is transitive on $V(\Gamma)$, it follows that A/N is quasiprimitive on $V(\Gamma_N)$. If Γ_N is complete, then (3) holds.

Finally, suppose that Γ_N is non-complete. Then Γ_N is $(A/N, 2)$-geodesic transitive. Since Γ is arc transitive, it follows that each N-orbit contains no edges of Γ. Since Γ has girth 3, it follows that N has at least 3 orbits on $V(\Gamma)$ and Γ_N has girth 3. Therefore (4) holds. □

3. **Vertex quasiprimitive**

In this section, we study 2-geodesic transitive graphs Γ of prime power order where $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$.

Lemma 3.1 ([14])**.** *Let T be a non-abelian simple group that has a subgroup H of index p^r where p is a prime number. Then one of the following holds.*

(1) $T \cong A_{p^r}$ *and* $H \cong A_{p^r-1}$.
(2) $T \cong PSL(d,q)$*,* H *is a maximal parabolic subgroup of* T*, and* $p^r = (q^d-1)/(q-1)$.
(3) $T \cong PSL(2,11)$*,* $H \cong A_5$ *and* $p^r = 11$.
(4) $T \cong M_{11}$*,* $H \cong M_{10}$ *and* $p^r = 11$.
(5) $T \cong M_{23}$*,* $H \cong M_{22}$ *and* $p^r = 23$.
(6) $T \cong PSU(4,2)$*,* $H \cong \mathbb{Z}_2^4 : A_5$ *and* $p^r = 27$.

Lemma 3.2. *Let* Γ *be a vertex transitive graph of* p^r *vertices where* p *is a prime number. Suppose that* $\mathrm{Aut}(\Gamma)$ *acts quasiprimitively on* $V(\Gamma)$ *with a minimal normal subgroup* M*. Then* $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$ *of type* HA, AS *or* PA*. In particular,* $M \cong T^i$ *where either* $T \cong \mathbb{Z}_p$ *or* T *is listed in Lemma* 3.1.

Proof. Suppose that $A := \mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$. Since $|V(\Gamma)|$ is a prime power, it follows from [19, Theorem 2.2] that A acts primitively on $V(\Gamma)$. Since M is a minimal normal subgroup of A, it follows that $M \cong T^i$ is transitive on $V(\Gamma)$, where T is a simple group. If M is abelian, then M is regular on $V(\Gamma)$ and $T \cong \mathbb{Z}_p$, and so A is primitive on $V(\Gamma)$ of HA type. In the left of this proof, we assume that M is non-abelian. Then T is a non-abelian simple group, and for each vertex u, $|T : T_u| = p^r$ for some positive integer r, so T is one of the groups listed in Lemma 3.1.

If A acts primitively on $V(\Gamma)$ of type TW, HS or HC, then $M = T^i$ with $i \geq 2$ is regular on $V(\Gamma)$ and T is a non-abelian simple group, which contradicts that $|V(\Gamma)| = p^r$. If A acts primitively on $V(\Gamma)$ of type SD or CD, then $|T|$ divides $|V(\Gamma)|$, which contradicts the assumption that $|V(\Gamma)| = p^r$. □

Let G be a permutation group on a set Ω. Then an orbit Δ of G on $\Omega \times \Omega$ is called an *orbital* of G on Ω. The graph $\Gamma(\Delta)$ is called an *orbital graph* of G on Ω if its vertex set is Ω, and (u,v) is an arc of $\Gamma(\Delta)$ if and only if $(u,v) \in \Delta$.

Lemma 3.3. *Let* Γ *be a non-complete graph with prime power number of vertices. If* $\mathrm{Aut}(\Gamma)$ *acts primitively on* $V(\Gamma)$ *of* AS *type, then* Γ *is the Schläfli graph or its complement.*

Proof. Assume that $A := \mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$ of AS type. Let $|V(\Gamma)| = p^r$ where p is a prime number. Then the socle T of A is non-abelian simple and transitive on $V(\Gamma)$, so for each vertex u, $|T : T_u| = p^r$. By [14, Corollary 2], T is either 2-transitive or $T = PSU(4,2)$ with $|V(\Gamma)| = 27$. Since Γ is non-complete, it follows that T is not 2-transitive on $V(\Gamma)$, and so $T = PSU(4,2)$ with $|V(\Gamma)| = 27$, and the action of T on $V(\Gamma)$ is uniquely determined.

By Atlas [5], $A = PSU(4,2).R$ with $R \leq \mathbb{Z}_2$, A has rank 3 and $A_u = \mathbb{Z}_2^4 : A_5$ or $\mathbb{Z}_2^4 : S_5$ for each vertex u. Then it follows from [4] and [13, p. 239, p. 259] that the two orbital graphs are the Schläfli graph and its complement. □

Lemma 3.4. *Both the Schläfli graph and its complement are* 2*-geodesic transitive.*

Proof. Let Γ be the Schläfli graph. Then Γ is strongly regular with parameters $(27, 16, 10, 8)$, and it is also a rank 3 graph. The stabilizer of a vertex u in the automorphism group A of Γ has order $2^7 \cdot 3 \cdot 5$ and acts transitively on the neighbours of u. Thus if $v \in \Gamma(u)$, then the stabilizer of arc (u, v) has order $|A_{u,v}| = 2^3 \cdot 3 \cdot 5$ (in fact $A_{u,v} \cong S_5$). Every element of order 5 in A fixes exactly two vertices and therefore the Sylow 5-subgroup of $A_{u,v}$ has two orbits, both of length 5, on the vertices joined to v but not to u. Thus both Γ and its complement are 2-geodesic transitive. □

Let $\Delta = \{0, 1, 2, \ldots, m-1\}$ and $\Delta^k = \Delta \times \cdots \times \Delta$ where $m, k \geq 2$. Define Γ to be the graph with vertex set Δ^k, and two vertices $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ are adjacent if and only if they have exactly 2 different coordinates.

Lemma 3.5. *Let* Γ *be a graph defined as the above. If* Γ *is* 2*-geodesic transitive, then* Γ *is disconnected.*

Proof. Suppose that Γ is 2-geodesic transitive. If $k = 2$, then vertices $(0, 0)$ and $(1, 0)$ are not in the same connected component, and so Γ is disconnected.

Assume that $k \geq 3$. Let $u = (0, 0, 0, 0^{k-3})$, $v_1 = (1, 1, 0, 0^{k-3})$, $w_1 = (0, 1, 2, 0^{k-3})$, $v_2 = (1, 2, 0, 0^{k-3})$, and $w_2 = (1, 1, 1, 0^{k-3})$. Then (u, v_1, w_1) and (u, v_2, w_2) are two 2-geodesics. Noting that the stabilizer of u in the automorphism group can not map w_1 to w_2, contradicts Γ is 2-geodesic transitive, and hence $m = 2$. However, in this case, vertex $(0, 0, \ldots, 0)$ lies in a connected component with 2^{k-1} vertices, and $(1, 0, \ldots, 0)$ lies in another connected component with also 2^{k-1} vertices, and so Γ is disconnected. □

For every vertex u of Γ, we define $\Gamma \circ \Gamma(u) = \{v \in V(\Gamma) | \Gamma(u) \cap \Gamma(v) \neq \emptyset\}$. Then $\Gamma \circ \Gamma(u) \setminus \Gamma(u) = \Gamma_2(u)$.

Lemma 3.6. *Let* Γ *be a* 2*-geodesic transitive graph of* p^r *vertices where* p *is a prime number. Suppose that* $\mathrm{Aut}(\Gamma)$ *is quasiprimitive on* $V(\Gamma)$ *of* PA *type. Then* Γ *is a Hamming graph* $\mathrm{H}(s, p^t)$ *where* $st = r$.

Proof. Suppose that $A := \mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ of PA type. Then by Lemma 3.2, A acts primitively on $V(\Gamma)$ of PA type. Hence A preserves a Cartesian decomposition $V(\Gamma) = \Delta^k$. Let H be the induced subgroup of A_Δ in Δ. Since A is primitive on $V(\Gamma)$ of PA type, H is primitive on Δ. Let $u \in V(\Gamma)$. Since Γ is 2-geodesic transitive, it follows that both $\Gamma(u)$ and $\Gamma \circ \Gamma(u) \setminus \Gamma(u) = \Gamma_2(u)$ are orbits of A_u in $V(\Gamma) \setminus \{u\}$. It follows from [25, Proposition 2.4] that H is 2-transitive on Δ. Let $v \in \Gamma(u)$. Then by [16], u, v have j distinct coordinates where $j = 1, 2$.

Since H is 2-transitive on Δ, it follows that $v' \in \Gamma(u)$ if and only if u, v' have exactly j distinct coordinates. Since Γ is arc transitive, it follows that any two vertices are adjacent if and only if they have exactly j distinct coordinates. It follows from Lemma 3.5 that $j \neq 2$. If $j = 1$, then Γ is Hamming graph $\mathrm{H}(s, p^t)$ where $st = r$. □

Lemma 3.7. *Let Γ be a 2-geodesic transitive graph of p^r vertices where p is a prime number. Suppose that $\mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$ of HA type with socle N. Then $\Gamma \cong \mathrm{Cay}(N, S)$, for some $S \subseteq N \setminus \{1\}$, is a normal Cayley graph. In particular, if Γ has girth at least 4, then $p = 2$; if Γ has girth 3, then $\langle a \rangle \setminus \{1\} \subset S$ for each $a \in S$.*

Proof. Suppose that $A := \mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$ of HA type. Then $N \cong \mathbb{Z}_p^r$ acts regularly on $V(\Gamma)$, and so Γ is a Cayley graph with respect to N, say $\Gamma = \mathrm{Cay}(N, S)$ for some $S \subseteq N \setminus \{1\}$. Then $N \leq A \leq N : GL(r, p) = N : \mathrm{Aut}(N)$. Since $A = N : A_u$ for $u = 1_N$, it follows that $A_u \leq \mathrm{Aut}(N)$. In particular, $A_u = \mathrm{Aut}(N, S)$ acts on N irreducibly, so Γ is a normal Cayley graph.

If Γ has girth at least 4, then each 2-arc is a 2-geodesic, and so Γ is 2-arc transitive. Hence A_u acts 2-transitively on S. Since $S = S^{-1}$, it follows that $\{a, a^{-1}\}$ is a block of A_u in $\Gamma(u)$ for any $a \in \Gamma(u)$ whenever $o(a) > 2$. This contradicts that A_u is primitive on S. Hence $o(a) = 2$.

Finally, suppose that Γ has girth 3. Since Γ is a 2-geodesic transitive normal Cayley graph, it follows from [9] that $\langle a \rangle \setminus \{1\} \subset S$ for each $a \in S$. □

Proof of Theorem 1.2. Let Γ be a 2-geodesic transitive but not 2-arc transitive graph of order p^r where p is a prime number. Suppose that $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ with a minimal normal subgroup N. Then $N \cong T^i$ for some simple group T. It follows from Lemma 3.2 that $\mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$ of type HA, AS or PA, and either $T \cong \mathbb{Z}_p$ or T is one of the groups listed in Lemma 3.1. If $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ of AS type, then by Lemmas 3.3 and 3.4, Γ is the Schläfli graph or its complement. If $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ of PA type, then by Lemma 3.6, Γ is a Hamming graph $\mathrm{H}(s, p^t)$ with $st = r$. If $\mathrm{Aut}(\Gamma)$ acts quasiprimitively on $V(\Gamma)$ of HA type, then by Lemma 3.7, $\Gamma \cong \mathrm{Cay}(N, S)$ is a normal Cayley graph, and $\langle a \rangle \setminus \{1\} \subset S$ for each $a \in S$. □

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