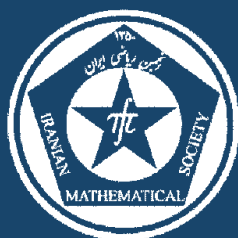


ISSN: 1017-060X (Print)



ISSN: 1735-8515 (Online)

Bulletin of the
Iranian Mathematical Society

Vol. 43 (2017), No. 5, pp. 1529–1530

Title:

A simple proof of Zariski's Lemma

Author(s):

A. Azarang

Published by the Iranian Mathematical Society
<http://bims.ims.ir>

A SIMPLE PROOF OF ZARISKI'S LEMMA

A. AZARANG

(Communicated by Rahim Zaare-Nahandi)

ABSTRACT. We give a simple proof for Zariski's Lemma.

Keywords: Zariski's Lemma.

Keywords: Primary: 12F05; Secondary: 13F10.

1. The result

Our aim in this very short note is to show that the proof of the following well-known fundamental lemma of Zariski follows from an argument similar to the proof of the fact that the field of rational numbers \mathbb{Q} is not a finitely generated \mathbb{Z} -algebra.

Lemma 1.1 (Zariski's Lemma). *Let L be a field extension of a field K . Assume that for some $\alpha_1, \dots, \alpha_n$ in L , $R = K[\alpha_1, \dots, \alpha_n]$ is a field. Then every α_i is algebraic over K .*

In particular, if K is algebraically closed, then $\alpha_i \in K$ for all i . This statement implies the so-called Hilbert's Weak Nullstellensatz, which states that when K is an algebraically closed field, every maximal ideal M of the polynomial ring $K[x_1, \dots, x_n]$ is of the form $M = (x_1 - a_1, \dots, x_n - a_n)$ with $a_i \in K$ for all i .

Usually the proof of Zariski's Lemma depends on two technical lemmas due to Artin-Tate and Zariski, see [3, Proposition 3.2, and its subsequent comment]. Some textbooks on elementary algebraic geometry employ the Noether normalization lemma to prove Zariski's Lemma (see, e.g., [2, Theorem 1.15] and [5], (also see [1] and [4]).

Before giving the proof of the lemma, we recall the following two well-known facts.

Fact 1. If a field F is integral over a subdomain D , then D is a field.

Fact 2. If D is any principal ideal domain (or just a UFD) with infinitely many (non-associate) prime elements, then its field of fractions is not a finitely generated D -algebra.

Proof of the Lemma. We use induction on n for arbitrary fields K and L . For $n = 1$ the assertion is clear. Let us assume that $n > 1$ and the lemma is true for positive integers less than n . Now to show that it is true for n , one may assume that one of α_i 's, say α_1 , is not algebraic over K . Since $K[\alpha_1, \dots, \alpha_n] = K(\alpha_1)[\alpha_2, \dots, \alpha_n]$ is a field, by induction hypothesis, we infer that $\alpha_2, \dots, \alpha_n$ are all algebraic over $K(\alpha_1)$. This implies that there are polynomials $f_2(\alpha_1), \dots, f_n(\alpha_1) \in K[\alpha_1]$ such that all α_i 's are integral over the domain $A = K[\alpha_1][1/f_2(\alpha_1), \dots, 1/f_n(\alpha_1)]$. Since R is integral over A , by Fact 1, A is a field. Consequently, $A = K(\alpha_1)$, which contradicts Fact 2. \square

Acknowledgements

The author would like to thank Professor O.A.S. Karamzadeh for a useful discussion on this note. He would also like to thank the referee for reading the above proof carefully and giving useful suggestions.

REFERENCES

- [1] E. Arrondo, Another elementary proof of the Nullstellensatz, *Amer. Math. Monthly* **113** (2006), no. 2, 169–171.
- [2] K. Hulek, Elementary Algebraic Geometry, Stud. Math. Libr. 20, Amer. Math. Soc. Providence, RI, 2003.
- [3] E. Kunz, Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser Boston, 1985.
- [4] J. McCabe, A Note on Zariski's Lemma, *Amer. Math. Monthly* **83** (1976), no. 7, 560–561.
- [5] M. Reid, Undergraduate Algebraic Geometry, London Math. Soc. Stud. Texts 12, Cambridge Univ. Press, Cambridge, 1988.

(Alborz Azarang) DEPARTMENT OF MATHEMATICS, SHAHID CHAMRAN UNIVERSITY OF AHVAZ, AHVAZ, IRAN.

E-mail address: a_azarang@scu.ac.ir