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ON THE DEFINING NUMBER OF $(2n-2)$-VERTEX COLORINGS OF $K_n \times K_n$

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ABSTRACT. In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G, if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, c)$. In this note we study $d(G = K_n \times K_n, 2n-2)$. We determine an upper bound for $d(G = K_n \times K_n, 2n-2)$ for all n and its exact value for some n.

1. Introduction

A proper c-coloring of a graph G is an assignment of c different colors to the vertices of G such that no two adjacent vertices receive the same color. The (vertex) chromatic number of a graph G is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called a k chromatic graph. In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G, if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum

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cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G,c)$. There are some papers on the defining set of graphs, especially $d(K_n \times K_n,\ \chi = n)$ (the critical set of Latin squares of order n), $d(G_k, \chi = k)$ where G_k is a k-regular graph. Considerable research work is also carried out on the defining set on block designs. The interested reader may see [1, 2, 3, 4, 6, 7, 8, 9, 10 , 11, 12] and their references.

The following material are useful.

Definition 1.1. A graph G with v vertices is called a *uniquely* 2-*list colorable* (U2LC) if there exists $S_1, S_2, ..., S_v$, a list of colors on its vertices, each of size 2, such that there is a unique coloring for G from this list of colors (see [6]).

Lemma 1.2. *A connected graph is U2LC if and only if at least one of its block (a maximal connected subgraph of the graph that has no cut vertex) is not a cycle, a complete graph and a complete bipartite graph (see [6]).*

Definition 1.3. A graph G is $M(2)$ if it is not $U2LC$.

Definition 1.4. A defining set S with an assignment of colors in a graph G, is called a *strong defining set* if there exists an ordering $\{v_1, v_2, ..., v_{n-s}\}$ of the vertices of $G - S$ such that, in the induced list of colors in each of the subgraphs $G - S$, $G - (S \cup \{v_1\})$, $G - (S \cup \{v_1, v_2\})$, ..., $G - (S \cup \{v_1, v_2, ..., v_{n-s}\})$, there exists at least one vertex whose list of colors is of cardinality 1 (see [8]).

Lemma 1.5. *Let G be a k-regular k-vertex coloring graph. Then every cycle in G has a vertex in the defining set of G.*

Proof. Let C be a cycle in G which has no vertex in a defining set. Thus each vertex of C must be forced (uniquely colored), but only k$-$2 colors can be excluded by already colored neighbors. By Lemma 1.2 the cycle C is $M(2)$. This implies there are at least two choices that complete the coloring of the cycle. □

Corollary 1.6. *Let S be a defining set for a k-regular k-vertex coloring graph G, then $G-S$ is a forest.*

Corollary 1.7. *Every defining set of a k-regular k-vertex coloring graph is strong.*

The Corollary 1.7 implies the following result which has been proved in [8].

Theorem 1.8. *Every defining set of a k-regular k-chromatic graph is strong (see [8]).*

Definition 1.9. An $n \times n$ matrix whose entries come from $S = \{1, 2, ..., 2n-1\}$, is called a *silver matrix* if for each $i = 1, ..., n$ the union of the i^{th} row and the i^{th} column contains all elements of S (see [5]).

If n is an even positive integer then the silver matrix is constructed as follows.

Let m_{ij} be the (i, j) entry of $n \times n$ matrix M. We put
(1) $m_{ij} = i + j \pmod{n-1}$ if $i < j < n$
(2) $m_{ij} = 2i \pmod{n-1}$ if $i < j = n$
(3) $m_{ij} = n$ if $i = j$
(4) $m_{ij} = m_{ji} + n \pmod{n-1}$ if $i > j$.

Definition 1.10. A *Latin square of order* n is an $n \times n$ array or matrix with entries taken from the set $\{1, 2, ..., n\}$ with the property that each entry occurs exactly once in each row or column.

2. Lower bound of $d(K_n \times K_n, 2n-2)$

In this section we determine a lower bound for the defining set of $(2n-2)$-colorings of $K_n \times K_n$.

Proposition 2.1. $d(K_n \times K_n, 2n-2) \geq (n-1)^2$.

Proof. Let $G = K_n \times K_n$ and let S be its defining set. By Corollary 1.6, $G-S$ has no cycle. Thus in each row and each column of G at least $n-2$ vertices belong to S. Therefore, $d(K_n \times K_n, 2n-2) \geq n(n-2)$. If $n-2$ elements of each column or each row belong to S, then every vertex v of $G-S$ is a neighbor of a vertex of the same column and a vertex of the same row. Hence the degree of every vertex in S is exactly 2. Thus $G-S$ is a cycle. Therefore, $d(K_n \times K_n, 2n-2) \geq n(n-2)+1 = (n-1)^2$. $\square$

Remark 2.2. Let S be a defining set of $(2n-2)$-colorings of $G = K_n \times K_n$. Let $N = V(G) \setminus S$. Then by Proposition 2.1, N has at most 2 vertices from each given row or column. Therefore, by a permutation of the rows or columns we can assume that N is a subset of the $*$ vertices in the following table:

$\star$	$\star$							
	$\star$	$\star$						
		$\star$	$\star$					
			$\star$	$\star$				
				$\star$	$\star$			
					$\star$	$\star$		
						$\star$	$\star$	
							$\star$	$\star$
								$\star$

Let $L(i,j)$ be the list of colors of the vertex (i,j). In the following, it is shown that, there are no four vertices as $(i,j),(i,j+1),(i+1,j+1)$ and $(i+1,j+2)$ in $N = V(G) \setminus S$ such that $L(i,j) = \{a\}, L(i,j+1) = \{a,b\}, L(i+1,j+1) = \{b,c\}$ and $L(i+1,j+2) = \{*\}$, or there are no four vertices as $(i,j),(i+1,j),(i+1,j+1)$ and $(i+2,j+1)$ in $N = V(G) \setminus S$ such that $L(i,j) = \{*\}, L(i+1,j) = \{b,c\}, L(i+1,j+1) = \{a,b\}$ and $L(i+1,j+2) = \{a\}$. Without loss of generality one can assume that $i = j = 1$.

Lemma 2.3. *Let S be a defining set of $(2n-2)$-colorings of $G = K_n \times K_n$. Let $N = V(G) \setminus S$. Then N has no path on four vertices such that its lists of colors are as follows*:

	1	2	3
1	a	ab	
2		bc	$\star$

Table B

	1	2
1	$\star$	
2	bc	ab
3		a

Table A

Proof. Assume that the color of the vertex $(2,1)$ in the table A is k. Hence the color k will not exist in the first row and in the second column. If the color k appears in the first row then from $2n-1$ vertices of the first row and the first column, two vertices have not been colored and two vertices have color k; i.e. from $2n-2$ colors, at most $2n-4$ colors have been used. Thus there are two choices for coloring of the $(1,1)$ vertex, which is a contradiction. If the color k appears in the second column then from $2n-1$ vertices of the second row and the second column, three vertices have not been colored, and two vertices have color k. So at most $2n-5$ colors are used for coloring $2n-4$ vertices, and hence there are three choices for coloring of the $(2,2)$ vertex which is a contradiction. Therefore, the color k does not exist in the first row and in the second column. Hence the color k must be assigned to the vertex $(1,2)$, thus $a=k$ or $b=k$ which is a contradiction too. Therefore, there is no path on four vertices in N given by the table A. The argument for the table B is similar. □

By Corollary 1.7 every defining set of $(2n-2)$-vertex coloring of $K_n \times K_n$ is strong.

If there exists a path on at least three vertices in $N = V(G) \setminus S$, then the internal vertex of the path has a list of at least two colors. In the following we show that there is no path on five vertices in N.

lemma 2.4. *Let S be a defining set of $(2n-2)$-colorings of $G = K_n \times K_n$ and $N = V(G) \setminus S$. Then the induced subgraph $\langle N \rangle$ has no path on five vertices as a subgraph.*

Proof. Contrarily, assume that $\langle N \rangle$ has a path P on five vertices as follows:

	1	2	3	4
1	$*$	yz		
2		bc	ab	
3			$*$	
4				

By Corollary 1.7 S is strong. Since the internal vertices of P have a list of two colors, at least one of the end vertices of this path has a list of one color. Suppose $L(1,1) = \{x\}$. There are two cases for the list of $L(1,2)$.

Case 1.

$x \notin L(1,2) = \{y,z\}$. Since $P - (1,1)$ is $M(2)$, $L(3,3)$ has only one element, say $L(3,3) = \{a\}$. But the element a has to lie in $L(2,3)$, otherwise the subgraph induced by $\langle(1,2),(2,2),(2,3)\rangle$ will be $M(2)$. Hence $L(2,3) = \{a,b\}$. The subgraph $\langle(1,2),(2,2)\rangle$ is $M(2)$. Thus $L(2,2)$ has to contain b. By (table B) of Lemma 2.3 this case does not arise.

Case 2.

$x \in L(1,2)$ and $L(1,2) = \{x,y\}$.

	1	2	3	4
1	$*$	xy		
2		$bc = zt$	ab	
3			$*$	
4				

There are two subcases:

(i) $y \notin L(2,2) = \{z,t\}$. Since $\langle(2,2),(2,3),(3,3)\rangle$ is $M(2)$, it follows that $L(3,3)$ has one element as a. Since $\langle(2,2),(2,3)\rangle$ is $M(2)$, it is clear that $a \in L(2,3)$ and $L(2,3) = \{a,b\}$. Also $\langle(2,2)\rangle$ is $M(2)$, hence we have $b \in L(2,2)$ and $L(2,2) = \{z,t\} = \{b,c\}$. By Lemma 2.3 (table B) this case can not happen either.

(ii) $y \in L(2,2)$.

	1	2	3	4
1	x	xy		
2		yz	$\star$	
3			$\star$	
4				

By Lemma 2.3 (table A) this case is also impossible. $\square$

Theorem 2.5. *Let S be a defining set to $(2n-2)$-colorings of $G = K_n \times K_n$ and $N = V(G) \setminus S$. Then*

$$|N| \leq \lfloor \frac{8n}{5} \rfloor,$$

and hence

$$d(G, 2n-2) \geq n^2 - \lfloor \frac{8n}{5} \rfloor.$$

Proof. The induced subgraph $\langle N \rangle$ is a subgraph of a path on $2n-1$ vertices. By Lemma 3, $\langle N \rangle$ has no path on five vertices as a subgraph. Thus for any path on five vertices at least one vertex does not belong to N. Hence the number of vertices in N is at most $(2n-1) - \lfloor \frac{2n-1}{5} \rfloor$, i.e.

$$|N| \leq (2n-1) - \lfloor \frac{2n-1}{5} \rfloor = \lfloor \frac{8n}{5} \rfloor.$$

Therefore,

$$d(G, 2n-2) \geq n^2 - \lfloor \frac{8n}{5} \rfloor.$$

$\square$

3. Defining number of $K_n \times K_n$ for some values of n

In this section we show that $d(K_n \times K_n, 2n-2) = n^2 - \lfloor \frac{8n}{5} \rfloor$ for $n = 1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 15, 18, 23, 28$ and for $n = 10m$, where m is a positive integer.

Lemma 3.1. *Let n be a positive integer such that $8n \overset{5}{\equiv} 0, 1, 2$. If $d(K_n \times K_n, 2n-2) = n^2 - \lfloor \frac{8n}{5} \rfloor$ then $d(K_{2n} \times K_{2n}, 2(2n)-2) = (2n)^2 - \lfloor \frac{8(2n)}{5} \rfloor$.*

Proof. Suppose that n is a positive integer such that $8n \overset{5}{\equiv} 0$. We consider a $2n \times 2n$ matrix M as

$\mathcal{A}$	$\mathcal{B}$
$\mathcal{C}$	$\mathcal{A}$

where $\mathcal{A}$ is an $n \times n$ matrix corresponding to $(2n-2)$-colorings of $K_n \times K_n$, $\mathcal{B}$ is an $n \times n$ Latin square with numbers $\{2n-1, 2n, ..., 3n-2\}$ and $\mathcal{C}$ is an $n \times n$ Latin square with numbers $\{3n-1, 3n, ..., 4n-2\}$. It is easy to see that the defining set of $\mathcal{A}$'s, n^2 entries of $\mathcal{B}$ and n^2 entries of $\mathcal{C}$ consist the defining set of $K_{2n} \times K_{2n}$. Therefore its defining number is $n^2 - \lfloor \frac{8n}{5} \rfloor + n^2 - \lfloor \frac{8n}{5} \rfloor + n^2 + n^2 = (2n)^2 - \lfloor \frac{8(2n)}{5} \rfloor$.

For $8n \overset{5}{\equiv} 1, 2$ similar proofs work. □

Remark 3.2. In the following arrays the non-indexed labels denote the colors of the vertices in the defining set of $K_n \times K_n$ and the indexed labels denote the colors of the vertices that are forced (uniquely colored) with respect to the indices.

Theorem 3.3. *For* $n = 1, 2, 3, 4, 5, 6, 8, 9, 13, 15, 18, 23, 28,$

$$d(K_n \times K_n, 2n-2) = n^2 - \lfloor \frac{8n}{5} \rfloor.$$

Proof. We introduce the defining set of size $n^2 - \lfloor \frac{8n}{5} \rfloor$ for $n = 1, 2, 3, 4, 5, 6, 8, 9, 13, 14, 15, 18, 23, 28.$

$n = 1$

1_1

$n = 2$

1_1	2_2
2	1_3

$n = 3$

4_1	2_2	1
2	3	4_3
3	1	2_4

$n = 5$

8_1	6_2	7	1	2
6	7_4	8_3	2	1
3	4	5	8_5	6
5	3	4	6_6	7_8
4	5	3	7	8_7

$n = 6$

1_1	2_4	3	5	4	6
2	7_3	4_2	1	6	5
9	10	8	6_5	3	7
8	9	7	3_8	10_7	2
7	8	10	2	5_6	9
10	3	9	4	1	8_9

$n = 9$

16_1	15_2	14	4	5	10	11	12	13
15	14_4	16_3	5	4	11	12	13	10
1	2	3	16_5	15_6	12	13	10	11
2	3	1	15	16_7	13	10	11	12
3	6	7	8	9	16_8	14	1	2
7	1	8	9	6	14_9	15_{11}	2	3
8	9	2	6	7	15	16_{10}	3	1
9	7	6	14	8	4	5	16_{12}	15_{13}
6	8	9	7	14	5	4	15	16_{14}

$n = 13$

11_1	9_2	10	7	8	1	2	3	12	13	14	15	16
9	10_4	11_3	8	7	2	3	1	13	14	15	16	12
4	5	6	11_5	9_6	3	1	2	14	15	16	12	13
5	6	4	9	10	11_7	8_8	7	15	16	12	13	14
6	4	5	10	11	8	7_{10}	9_9	16	12	13	14	15
17	18	19	20	21	22	23	24	11_{11}	9	4	5	6
18	19	20	21	22	23	24	17	9_{12}	10_{14}	5	6	4
19	20	21	22	23	24	17	18	10	11_{13}	6	4	5
20	21	22	23	24	17	18	19	8	7	11_{15}	9	10
21	22	23	24	17	18	19	20	7	8	9_{16}	10	11
22	23	24	171	18	19	20	21	1	2	3	11_{17}	8
23	24	17	18	19	20	21	22	2	3	1	8_{18}	7_{20}
24	17	18	19	20	21	22	23	3	1	2	7	9_{19}

$n = 14$

26_1	25_2	24	22	8	9	10	23	11	20	21	5	6	7
25	24_4	26_3	5	23	10	11	9	20	21	22	6	7	8
12	13	16	26_5	25	18	19	4	3	2	1	15	14	17
13	12	15	25_6	24_8	19	18	1	4	3	2	16	17	14
14	15	12	24	26_7	16	13	2	1	4	3	17	18	19
3	4	1	20	21	26_9	25_{10}	22	2	23	6	7	8	5
4	1	2	21	22	25	26_{11}	3	23	7	20	8	5	6
15	16	17	6	5	13	14	26_{12}	25	8	7	10	19	18
16	17	18	7	6	14	15	25_{13}	24_{15}	5	8	19	10	13
17	18	19	8	7	15	16	24	26_{14}	6	5	14	13	10
18	19	14	9	10	24	17	11	12	26_{16}	25_{17}	13	16	15
19	14	13	10	11	17	24	12	9	25	26_{18}	18	15	16
1	2	3	23	9	11	12	20	21	22	4	26_{19}	24_{20}	25
2	3	4	11	20	12	9	21	22	1	23	24	25_{22}	26_{21}

$n = 15$

26_1	27_2	28	15	16	8	5	6	13	11	12	9	10	14	7
27	28_4	26_3	16	15	7	8	5	10	13	11	12	9	6	14
17	18	19	26_5	27	20	21	22	1	2	23	24	25	3	4
18	19	17	27_6	28_8	21	22	20	2	3	24	25	23	4	1
19	17	18	28	26_7	22	20	21	3	4	25	23	24	1	2
4	1	2	14	9	26_9	27_{10}	28	15	16	10	11	12	13	3
3	4	1	12	14	27	28_{12}	26_{11}	16	15	9	10	11	2	13
20	21	22	5	6	23	24	25	26_{13}	27	17	18	19	7	8
21	22	20	6	7	24	25	23	27_{14}	28_{16}	18	19	17	8	5
22	20	21	7	8	25	23	24	28	26_{15}	19	17	18	5	6
2	3	4	13	5	6	7	8	14	1	26_{17}	27_{18}	28	15	16
1	2	3	8	13	5	6	7	4	14	17	28_{20}	26_{19}	16	15
23	24	25	9	10	17	18	19	11	12	20	21	22	26_{21}	27
24	25	23	10	11	18	19	17	12	9	21	22	20	27_{22}	28_{24}
25	23	24	11	12	19	17	18	9	10	22	20	21	28	26_{23}

$n = 23$

44_1 40_2 41	21 20 22	13 14 11	12 7 8	9 10 15	16 35 36	43 37 38	39 42
40 41_3 44_3	20 22 21	14 13 10	11 12 7	8 9 16	15 36 35	42 38 37	43 39
17 18 19	44_5 40_6 41	15 16 9	1 11 12	7 8 36	40 14 13	41 35 39	37 38
18 19 17	40 41_8 44_7	16 15 8	9 10 11	12 7 41	39 13 14	40 36 35	38 37
3 2 1	6 5 4	44_{10} 21_9 7	8 9 10	11 12 35	36 37 38	39 40 41	42 43
2 1 6	5 4 3	17 18 44_{11}	40_{12} 41 13	14 15 40	41 39 37	38 19 20	35 36
1 6 5	4 3 2	18 17 40	41_{14} 44_{13} 14	15 13 39	38 41 40	37 20 19	36 35
6 5 4	3 2 1	19 20 16	22 21 44_{15}	40_{16} 41 37	35 38 39	36 41 40	17 18
5 4 3	2 1 6	20 19 21	16 22 40	41_{18} 44_{17} 38	37 40 41	35 39 36	18 17
4 17 23	24 25 26	27 28 29	30 31 32	33 34 44_{19}	40 19 18	5 6 4	2 3
24 3 18	25 26 27	28 29 30	31 32 33	34 23 40_{20}	41_{22} 17 19	4 5 6	1 2
19 25 2	26 27 28	29 30 31	32 33 34	23 24 41	44_{21} 18 17	3 4 5	6 1
26 27 28	1 21 29	30 31 32	33 34 23	24 25 22	20 40 44_{23}	2 3 4	5 6
27 28 29	30 6 20	31 32 33	34 23 24	25 26 21	22 41_{26} 40_{24}	1 2 3	4 5
28 29 30	22 31 5	32 33 34	23 24 25	26 27 20	21 44_{25} 41	6 1 2	3 4
29 31 34	32 33 30	21 22 23	24 25 26	27 28 13	14 15 16	44_{27} 17 18	19 20
30 34 32	33 23 31	22 44 24	25 26 27	28 29 14	13 16 15	21_{28} 18 17	20 21
31 26 25	23 24 32	33 34 12	21 27 28	29 30 11	10 9 8	7 44_{29} 41	16 22
32 24 26	27 28 33	34 23 25	7 16 29	30 31 12	11 10 9	8 41_{30} 40_{32}	22 21
25 33 27	28 29 34	23 24 22	26 8 30	31 32 7	12 11 10	9 40 44_{31}	21 16
34 32 31	29 30 23	24 25 26	27 28 9	13 33 8	7 12 11	10 15 14	44_{33} 41
23 30 33	34 32 24	25 26 27	28 29 31	10 14 9	8 7 12	11 13 15	41_{34} 40_{36}
33 23 24	31 34 25	26 27 28	29 30 15	32 11 10	9 8 7	12 14 13	40 44_{36}

.

For $n = 4, 8, 18, 28$ one can use the Lemma 3.1 and consider the defining numbers for $n = 2, 4, 9, 14$. □

For $n = 10m$, consider a silver matrix of size $2m$ with entries $X = A_{2m}, A_1, A_3, ..., A_{2m-1}, A_{2m+1}, ..., A_{4m-1}$, where

$$X = \begin{array}{|c|c|c|c|c|} \hline 8_1 & 6_2 & 7 & 1 & 2 \\ \hline 6 & 7_4 & 8_3 & 2 & 1 \\ \hline 3 & 4 & 5 & 8_5 & 6 \\ \hline 5 & 3 & 4 & 6_6 & 7_8 \\ \hline 4 & 5 & 3 & 7 & 8_7 \\ \hline \end{array}$$

form its main diagonal entries. Now we replace the entry A_i, $1 \le i \le 2m-1$ by the 5×5 Latin square with $\{5i+4, 5i+5, 5i+6, 5i+7, 5i+8\}$ and for A_i, $2m+1 \le i \le 4m-1$ by the 5×5 Latin square with $\{5i-1, 5i, 5i+1, 5i+2, 5i+3\}$. Then the non indexed labels of the constructed $10m \times 10m$ matrix, illustrate the defining set of $K_{10m} \times K_{10m}$.

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